

Hydrodynamics of (generalized) hard rods

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Plan of the talk

- Hard rods dynamics

JL Lebowitz, JK Percus, J Sykes, Time evolution of the Total Distribution Function of a One-Dimensional System of Hard Rods, Physical review, 1968

- Euler space time scaling: Euler equations.

Boldrighini, C., Dobrushin, R.L. and Suhov, Yu.M.: One dimensional hard rod caricature of hydrodynamics. J. Stat. Phys. 1983

P.A.Ferrari, DGE Grevino, C. Franceschini, H.Spohn, Generalized hydrodynamics for size inhomogeneous hard rods, 2022

- Euler space time scaling: equilibrium and non-equilibrium fluctuations

H. Spohn, Hydrodynamical Theory for Equilibrium Time Correlation Functions of Hard Rods, Annals of Physics, 1982

C. Boldrighini, W. D. Wick, Fluctuations in a One-Dimensional Mechanical System. I. The Euler Limit, JSP 1988

- Euler space time scaling: *Navier-Stokes* corrections

C. Boldrighini, Yu.M. Suhov, One-Dimensional Hard-Rod Caricature of Hydrodynamics: "Navier-Stokes Correction" for Local Equilibrium Initial States, Commun. Math. Phys. 1997.

F.Hübner, L.Biagetti, J.De Nardis, B.Doyon. Diffusive hydrodynamics of hard rods from microscopics. SciPost Phys. Core, 2026

- Diffusive space-time scaling: equilibrium fluctuations

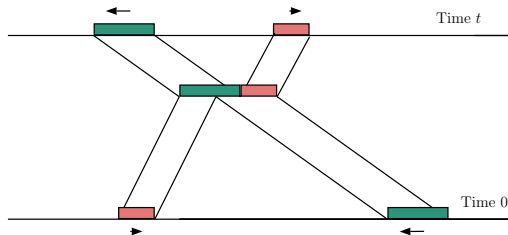
P.A. Ferrari, S. Olla. Macroscopic diffusive fluctuations for generalized hard rods dynamics. Ann. Appl. Probab. 2025.

Rods configuration and dynamics

$$Y = \{(y, v, r) \in \mathbb{R}^2 \times \mathbb{R}_+\}$$

$$\mathcal{Y} := \{Y \subset \mathbb{R}^3 : (y, y+r) \cap (\tilde{y}, \tilde{y}+\tilde{r}) = \emptyset, \text{ for all distinct } (y, v, r), (\tilde{y}, \tilde{v}, \tilde{r}) \in Y\}$$

with the dynamics:



Initial distribution for the Euler equation limit

- $d\mu(v, r)$ a probability measure on $\mathbb{R}^2$.
- $\tilde{\rho}_0(q, v, r)$ a smooth density profile.

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$$\tilde{\sigma}(q, 0) := \iint r \tilde{\rho}_0(q, v, r) d\mu(v, r), \quad \tilde{\pi}(q, 0) := \iint r v \tilde{\rho}_0(q, v, r) d\mu(v, r).$$

assume $0 < \tilde{\sigma}(q, 0) < 1$.

- $\varphi(q, v, r)$ a test function with compact support in q .
- $\mathcal{P}$ a probability distribution on the configuration space $\mathcal{Y}$ such that

$$\varepsilon \sum_{(y, v, r) \in \mathcal{Y}} r \varphi(\varepsilon y, v, r) \xrightarrow[\varepsilon \rightarrow 0]{\text{prob}} \iiint \varphi(q, v, r) r \tilde{\rho}_0(q, v, r) dq d\mu(v, r).$$

Euler equation as hydrodynamic limit

Theorem

$$K_t^\varepsilon \varphi := \varepsilon \sum_{(y,v,r) \in U_{\varepsilon^{-1}t} Y} r \varphi(\varepsilon y, v, r) \xrightarrow[\varepsilon \rightarrow 0]{\text{prob}} \iiint \varphi(q, w, r) r \tilde{\rho}_t(q, w, r) dq d\mu(w, r),$$

where

$$\partial_t \tilde{\rho}_t(q, v, r) + \partial_q (V_{\text{eff}}(q, v, t) \tilde{\rho}_t(q, v, r)) = 0,$$

$$V_{\text{eff}}(q, v, t) = v + \frac{v \tilde{\sigma}(q, t) - \tilde{\pi}(q, t)}{1 - \tilde{\sigma}(q, t)} = \frac{v - \tilde{\pi}(q, t)}{1 - \tilde{\sigma}(q, t)}$$

$$\tilde{\sigma}(q, t) := \iint r \tilde{\rho}_t(q, v, r) d\mu(v, r), \quad \tilde{\pi}(q, t) := \iint r v \tilde{\rho}_t(q, v, r) d\mu(v, r).$$

From rods to free particles

Order the rods according to their positions. To each $(y, v, r) \in Y$ associate the compressed position

$$x = y - \sum_{\substack{(\tilde{y}, \tilde{v}, \tilde{r}) \in Y \\ \tilde{y} < y}} \tilde{r}.$$

remove the space occupied by the rods

The hard-rod configuration is mapped into a configuration of points:

$$(y, v, r) \longmapsto (x, v, r).$$

In the compressed coordinates the interacting rod trajectories are mapped into free trajectories:

$$x(t) = x(0) + vt.$$

Normal coordinates

We now perform exactly the same compression at the macroscopic level.

$$x = \mathcal{X}(q, t) := q - \int_0^q \tilde{\sigma}(q', t) dq' + \int_0^t \tilde{\pi}(0, s) ds.$$

Since

$$\partial_t \tilde{\sigma} + \partial_q \tilde{\pi} = 0,$$

we have

$$\partial_q \mathcal{X} = 1 - \tilde{\sigma}, \quad \partial_t \mathcal{X} = \tilde{\pi}.$$

Hence

$$\partial_t \mathcal{X} + V_{\text{eff}} \partial_q \mathcal{X} = v.$$

Effective trajectories in physical space become free trajectories in compressed space.

Euler equation in normal coordinates

Let

$$q = \mathcal{Z}(x, t) = \mathcal{X}(\cdot, t)^{-1}(x).$$

Since $dx = (1 - \tilde{\sigma}) dq$, define

$$\rho_t(x, v, r) = \frac{\tilde{\rho}_t(\mathcal{Z}(x, t), v, r)}{1 - \tilde{\sigma}(\mathcal{Z}(x, t), t)}.$$

Then the nonlinear Euler equation becomes

$$\partial_t \rho_t(x, v, r) + v \partial_x \rho_t(x, v, r) = 0.$$

Solution of the Euler equation

In normal coordinates,

$$\rho_t(x, v, r) = \rho_0(x - vt, v, r).$$

Therefore the characteristic starting from q with velocity v is

$$y(q, v, t) = \mathcal{Z}(\mathcal{X}(q, 0) + vt, t).$$

Along this trajectory

$$\frac{d}{dt}y(q, v, t) = V_{\text{eff}}(y(q, v, t), v, t).$$

Equivalently,

$$V_{\text{eff}}(q, v, t) = v + \iint r(v - w)\rho_t(\mathcal{X}(q, t), w, r) d\mu(w, r).$$

Normal modes

Pull back the density in normal coordinates to physical space:

$$\rho_t^n(q, v, r) := \rho_t(\mathcal{X}(q, t), v, r) = \frac{\tilde{\rho}_t(q, v, r)}{1 - \tilde{\sigma}(q, t)}.$$

Then

$$\partial_t \rho_t^n(q, v, r) + V_{\text{eff}}(q, v, t) \partial_q \rho_t^n(q, v, r) = 0.$$

The Euler equation is diagonal in the variables $\rho_t^n(q, v, r)$.

Along the characteristics,

$$\rho_t^n(y(q, v, t), v, r) = \rho_0^n(q, v, r).$$

Fluctuations at the Euler scale

The hydrodynamic limit gives

$$K_t^\varepsilon \varphi \longrightarrow \iiint r \varphi(q, v, r) \tilde{\rho}_t(q, v, r) dq d\mu(v, r).$$

We now consider the fluctuations around this limit:

$$\mathcal{Y}_t^\varepsilon(\varphi) := \frac{1}{\sqrt{\varepsilon}} \left[K_t^\varepsilon \varphi - \iiint r \varphi(q, v, r) \tilde{\rho}_t(q, v, r) dq d\mu(v, r) \right].$$

Assume that at time 0

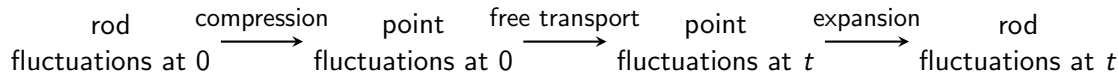
$$\mathcal{Y}_0^\varepsilon \Longrightarrow \mathcal{Y}_0,$$

for some random distribution $\mathcal{Y}_0$.

Then, for $t > 0$, the fluctuations are obtained by transporting $\mathcal{Y}_0$ through the **linearized hard-rod dynamics**.

The idea for the Euler fluctuations

The evolution of the fluctuations can be decomposed into three steps:



At the fluctuation level, compression and expansion have to be **linearized around the macroscopic profile**.

These linear transformations are expressed in terms of **dressing operators**.

Dressing the fluctuations

The compression map depends itself on the density profile:

$$\partial_q \mathcal{X}(q, t) = 1 - \tilde{\sigma}(q, t).$$

Therefore a fluctuation of the rod density produces both

- a fluctuation of the density in normal coordinates,
- a fluctuation of the coordinate transformation $\mathcal{X}$.

Linearizing this transformation defines the dressing operator

$$\text{point fluctuations} \xrightarrow{C_{t,\rho t}} \text{rod fluctuations.}$$

Its inverse $C_{t,\tilde{\rho}t}^{-1}$ transforms rod fluctuations into point fluctuations.

Euler fluctuations: main result

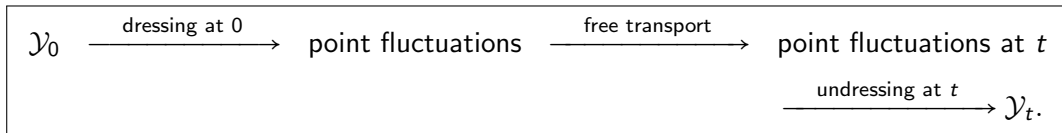
Assume

$$\mathcal{Y}_0^\varepsilon \Longrightarrow \mathcal{Y}_0.$$

Then, for every $t > 0$,

$$\mathcal{Y}_t^\varepsilon \Longrightarrow \mathcal{Y}_t,$$

where $\mathcal{Y}_t$ is obtained from $\mathcal{Y}_0$ by



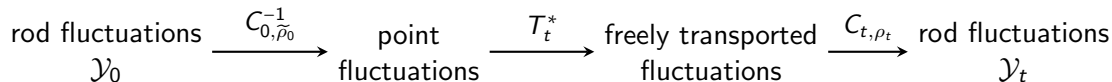
The randomness is entirely contained in the initial field $\mathcal{Y}_0$.

Euler fluctuations: main result

The linearized Euler evolution is

$$U_{t,0} = C_{t,\rho_t} T_t^* C_{0,\tilde{\rho}_0}^{-1}.$$

Thus the evolution is simply



Hence

$$\mathcal{Y}_t = U_{t,0} \mathcal{Y}_0.$$

The linearized Euler equation

Let $\tilde{h}_t$ be a perturbation of the Euler solution $\tilde{\rho}_t$. Then

$$\partial_t \tilde{h}_t + \mathcal{L}_t \tilde{h}_t = 0,$$

where

$$\mathcal{L}_t \tilde{h}(q, v, r) = \partial_q \left[V_{\text{eff}}(q, v, t) \tilde{h}(q, v, r) + \frac{\tilde{\rho}_t(q, v, r)}{1 - \tilde{\sigma}(q, t)} (V_{\text{eff}}(q, v, t) \tilde{\sigma}_{\tilde{h}}(q) - \tilde{\pi}_{\tilde{h}}(q)) \right].$$

with

$$\tilde{\sigma}_{\tilde{h}}(q) = \iint r \tilde{h}(q, w, r') d\mu(w, r'), \quad \tilde{\pi}_{\tilde{h}}(q) = \iint rw \tilde{h}(q, w, r') d\mu(w, r').$$

Its propagator is $U_{t,0} = C_{t,\rho_t} T_t^* C_{0,\tilde{\rho}_0}^{-1}$.

The dressing operator

For a point fluctuation h , define

$$\sigma_h(x, t) = \iint r h(x, v, r) d\mu(v, r), \quad \pi_h(x, t) = \iint rv h(x, v, r) d\mu(v, r),$$

and

$$\mathcal{Z}_h(x, t) = \int_0^x \sigma_h(x', t) dx' - \int_0^t \pi_h(0, s) ds.$$

The dressing operator is explicitly

$$C_{t, \rho_t}[h](\mathcal{Z}(x, t), v, r) = \frac{1}{1 + \sigma(x, t)} \left[h(x, v, r) - \partial_x \left(\frac{\rho_t(x, v, r)}{1 + \sigma(x, t)} \mathcal{Z}_h(x, t) \right) \right].$$

This is the linearization of the expansion from points to rods.

In the homogeneous case, the coordinate-fluctuation term simplifies and C_{t, ρ_t} reduces to the usual **algebraic dressing operator**.

The inverse dressing operator

For a rod fluctuation $\tilde{h}$, let

$$\mathcal{X}_{\tilde{h}}(q, t) = - \int_0^q \tilde{\sigma}_{\tilde{h}}(q', t) dq' + \int_0^t \tilde{\pi}_{\tilde{h}}(0, s) ds.$$

Then

$$C_{t, \tilde{\rho}_t}^{-1}[\tilde{h}](\mathcal{X}(q, t), v, r) = \frac{1}{1 - \tilde{\sigma}(q, t)} \left[\tilde{h}(q, v, r) - \partial_q \left(\frac{\tilde{\rho}_t(q, v, r)}{1 - \tilde{\sigma}(q, t)} \mathcal{X}_{\tilde{h}}(q, t) \right) \right].$$

Compression and expansion are explicitly invertible also at the linearized level.

Generation of long-range correlations

Suppose initially the fluctuations are locally correlated:

$$R_{0,0}(q, v, r; q', v', r') = \delta(q - q') \Gamma_0(q; v, r; v', r').$$

Under the linearized Euler evolution,

$$R_{t,t} = U_{t,0} R_{0,0} U_{t,0}^*.$$

For an inhomogeneous background, the dressing operator is **non-local in space**. Consequently the linearized evolution generically generates an off-diagonal contribution:

$$R_{t,t} = R_{t,t}^{\delta} + R_{t,t}^{\text{lr}}, \quad R_{t,t}^{\text{lr}}(q, q') \neq 0 \quad \text{for } q \neq q'.$$

Local correlations are not preserved by inhomogeneous Euler evolution.

Local GGE are not dynamically stable under inhomogeneous Euler evolution.

Navier–Stokes correction

We now look at the first correction to the Euler limit:

$$\mathbb{E}K_t^\varepsilon = \tilde{\rho}_t + \varepsilon \tilde{u}_t + o(\varepsilon).$$

More precisely,

$$\varepsilon^{-1} \left[\mathbb{E}(K_t^\varepsilon \varphi) - \iiint r \varphi(q, v, r) \tilde{\rho}_t(q, v, r) dq d\mu(v, r) \right] \longrightarrow \iiint r \varphi \tilde{u}_t dq d\mu.$$

We assume here that the initial fluctuation covariance $R_{0,0}$ is **regular** in the spatial variables.

Unlike a closed diffusive correction, $\tilde{u}_t$ depends on the two-point correlations of the Euler fluctuations.

The correction $\tilde{u}_t$ is generated by the **quadratic effect of the Euler fluctuations**.

Navier–Stokes correction and correlations

Define

$$\mathcal{D}_t^n(q, v, r) := \iint (v - v') R_{t,t}^n(q, v, r; q, v', r') r' d\mu(v', r').$$

Then

$$(\partial_t + \mathcal{L}_t) \tilde{u}_t = -\partial_q [(1 - \tilde{\sigma}) \mathcal{D}_t^n], \quad \tilde{u}_0 = 0.$$

The normal-mode covariance evolves according to

$$\partial_t R_{t,t}^n + (\mathcal{L}_t^n \otimes I + I \otimes \mathcal{L}_t^n) R_{t,t}^n = 0,$$

where

$$\mathcal{L}_t^n f = V_{\text{eff}} \partial_q f + (\partial_q \rho_t^n) \Gamma[f].$$

$(\tilde{\rho}_t, \tilde{u}_t, R_{t,t}^n)$ form a closed system of first-order equations.

Equilibrium and the Poisson structure

We now restrict to a homogeneous stationary state:

$$\tilde{\rho}(q, v, r) = \tilde{\rho}(v, r).$$

In normal coordinates, the equilibrium configuration is a marked Poisson point process

$$X^\varepsilon = \{(x, v, r)\}$$

with intensity

$$\varepsilon^{-1} \rho \, dx \, d\mu(v, r).$$

Define

$$\sigma = \rho \iint r \, d\mu(v, r), \quad \pi = \rho \iint rv \, d\mu(v, r).$$

Expanding the Poisson points into rods gives the stationary hard-rod equilibrium, with

$$\tilde{\rho}(v, r) = \frac{\rho}{1 + \sigma}.$$

Equilibrium hard rods are Poisson points in normal coordinates.

Poisson fluctuations

For a test function φ ,

$$\xi^{X,\varepsilon}(\varphi) := \varepsilon^{-1/2} \left[\varepsilon \sum_{(x,v,r) \in X^\varepsilon} r \varphi(x, v, r) - \rho \iiint r \varphi(x, v, r) dx d\mu \right].$$

Then

$$\xi^{X,\varepsilon} \Longrightarrow \xi^X,$$

where ξ^X is a centered Gaussian white noise with

$$\mathbb{E}[\xi^X(\varphi)\xi^X(\psi)] = \rho \iiint r^2 \varphi(x, v, r) \psi(x, v, r) dx d\mu(v, r).$$

The diffusive randomness will come from this Poisson structure.

From Euler to diffusive scale

At equilibrium,

$$V_{\text{eff}}(v) = \frac{v - \tilde{\pi}}{1 - \tilde{\sigma}}$$

is constant.

On the Euler scale, a rod with velocity v therefore follows

$$y(t) \simeq y(0) + V_{\text{eff}}(v)t.$$

To see the next scale, we observe the rod at time $\varepsilon^{-2}t$ and subtract its ballistic motion:

$$\boxed{\varepsilon \left[y(\varepsilon^{-2}t) - y(0) - V_{\text{eff}}(v)\varepsilon^{-2}t \right].}$$

The fluctuations of the number and lengths of crossed rods are of order ε^{-1} and survive under this scaling.

Two spatial scales

At the Euler scale there is only one spatial scale:

$$x \sim \varepsilon^{-1}.$$

At the diffusive time scale $t \sim \varepsilon^{-2}$, two spatial scales coexist:

fluctuation we observe

$$x \sim \varepsilon^{-1}$$

environment explored by the dynamics

$$x \sim \varepsilon^{-2}.$$

The fluctuation on the first scale is transported through an environment whose Poisson fluctuations are seen on the larger scale.

The larger-scale fluctuations become the noise driving the motion of the smaller-scale fluctuation.

Diffusive motion of a tagged rod

After subtracting the Euler motion,

$$\varepsilon \left[y(\varepsilon^{-2}t) - y(0) - V_{\text{eff}}(v)\varepsilon^{-2}t \right] \Longrightarrow B_t(v).$$

The family $\{B_t(v)\}_v$ is Gaussian, with

$$\mathbb{E}[B_t(v)B_s(v')] = (t \wedge s) \Gamma(v, v').$$

where

$$\Gamma(v, v') = \frac{\rho}{2} \iint r^2 (|v - w| + |v' - w| - |v - v'|) d\mu(w, r).$$

In particular,

$$\mathbb{E}[B_t(v)^2] = t \mathcal{D}(v), \quad \mathcal{D}(v) = \rho \iint r^2 |v - w| d\mu(w, r).$$

Two scales of the same Poisson field

The same Poisson configuration X^ε is observed on two scales.

On the scale $x \sim \varepsilon^{-1}$:

$$\xi^\varepsilon(\varphi) = \varepsilon^{1/2} \left[\sum_{(x,v,r) \in X^\varepsilon} r \varphi(x, v, r) - \varepsilon^{-1} \rho \iiint r \varphi dx d\mu \right].$$

On the larger scale $x \sim \varepsilon^{-2}$:

$$\zeta^\varepsilon(\varphi) = \varepsilon \left[\sum_{(x,v,r) \in X^\varepsilon} r \varphi(\varepsilon x, v, r) - \varepsilon^{-2} \rho \iiint r \varphi dx d\mu \right].$$

Both converge to Gaussian white noises with the same covariance, but

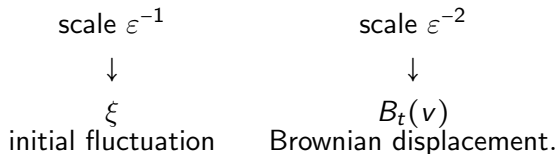
$$\mathbb{E}[\xi^\varepsilon(\varphi)\zeta^\varepsilon(\psi)] = O(\sqrt{\varepsilon}) \longrightarrow 0.$$

Hence

$$(\xi^\varepsilon, \zeta^\varepsilon) \Longrightarrow (\xi, \zeta), \quad \xi \text{ and } \zeta \text{ independent.}$$

Where does the dynamical noise come from?

The two limiting random objects have different origins:



Scale separation implies

ξ is independent of $\{B_t(v)\}_v$.

Therefore, at the diffusive scale, the initial fluctuation field is transported by an **independent random displacement**.

The dynamical noise comes from larger-scale fluctuations of the initial Poisson configuration.

Diffusive evolution of the fluctuations

Let $\Xi_t^{Y,\varepsilon}$ be the equilibrium fluctuation field, observed at time $\varepsilon^{-2}t$ and recentered along the Euler characteristics.

Then

$$\Xi_t^{Y,\varepsilon}(\varphi) \Longrightarrow \Xi_t^Y(\varphi) = \Xi_0^Y(\varphi_{\mathbb{B}_t}),$$

where

$$\varphi_{\mathbb{B}_t}(y, v, r) = \varphi(y + B_t(v), v, r).$$

The Brownian field $\mathbb{B}_t = \{B_t(v)\}_{v \in \mathbb{R}}$ satisfies

$$\mathbb{E}[B_t(v)B_s(w)] = (t \wedge s)\Gamma(v, w),$$

and is independent of the initial field Ξ_0^Y .

Diffusive evolution is a random translation of the initial fluctuation field.

Diffusive fluctuations: multiplicative noise

Equivalently, the limiting field satisfies

$$d\Xi_t^Y(\varphi) = \frac{1}{2} \Xi_t^Y(\mathcal{D} \partial_y^2 \varphi) dt - \Xi_t^Y(\partial_y \varphi) dB_t.$$

The diffusion term is accompanied by a noise which acts by **transporting the existing fluctuation field**.

integrable dynamics: multiplicative transport noise

rather than additive space–time noise.

Some conclusions

- At the Euler scale, hard-rod hydrodynamics is conjugate to **free transport** in normal coordinates.
- No local equilibration mechanism is needed at this scale: the initial profile and its fluctuations are simply transported by the dynamics.
- In particular, in an inhomogeneous state, initially local correlations are not preserved: **long-range correlations are dynamically generated**.
- A genuinely new source of randomness appears only at the diffusive scale:

large-scale fluctuations $\longrightarrow$ Brownian dynamical noise.

- This may be viewed as a form of **thermalization from infinity**: the local fluctuation is randomized by particles arriving from a much larger spatial scale.