Geodesic Divergence in Certain Riemannian Manifolds

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Background

Attempts to generalise the Gromov boundary

Ever since Gromov defined his notion of hyperbolicity in [Gromov] and topologised the space of geodesic rays in the Gromov boundary, there has been various generalisations of the Gromov boundary to a broader class of spaces such as the Morse boundary [Cordes] or the Bowditch boundary for relatively hyperbolic groups by [Bowditch]. One of these generalisations is the quasi-redirecting boundary by Qing and Rafi [Qing-Rafi] which is well-defined for a much broader class of metric spaces and is invariant under quasi-isometries. It is quite well-behaved in the sense that it coincides with the Gromov boundary for a δ-hyperbolic space, with the Bowditch boundary if G is relatively hyperbolic and is a point when G exhibits linear divergence.

Quasi-Redirection

Definitions

- **Quasi-geodesic** - A map $f : I \to X$ where $I \subseteq \mathbb{R}$ is said to be a quasi-geodesic if there exists (q, Q) such that $\frac{|x-y|}{q} - Q \leq d(f(x), f(y)) \leq q|x-y| + Q$.
- **Quasi-redirection** - Let $\alpha, \beta : [0, \infty) \to X$ be two quasi-geodesic rays. We say that α redirects to β (denoted by $\alpha \preceq \beta$) if for every $r > 0$, there is a (q, Q)-quasigeodesic ray γ_r such that $\gamma_r \equiv \alpha$ in $B(a(0), r)$ and γ_r eventually coincides with β. We say that $\alpha \sim \beta$ if $\alpha \preceq \beta$ and $\beta \preceq \alpha$. Denote by $P(X)$ the resulting equivalence classes of quasi-geodesic rays.

Some assumptions on X to make it "well-behaved"

Assumptions

- **Assumption 0**- X is a proper geodesic metric space. Furthermore, there exists a (q_0, Q_0) such that every $x \in X$ lies on a $(q_0, Q_0)-$ray.
- **Assumption 1**- For (q_0, Q_0) as in the previous assumption, every $\mathfrak{a} \in P(X)$ contains a (q_0, Q_0) ray. Call that ray the central element and denote it as $\mathbf{a} \in \mathfrak{a}$.
- **Assumption 2**- For every $\mathfrak{a} \in P(X)$, there is a function $f_{\mathfrak{a}} : [1, \infty) \times [0, \infty) \to [1, \infty) \times [0, \infty)$ such that if $\mathfrak{b} \preceq \mathfrak{a}$, then any (q, Q)-ray $\beta \in \mathfrak{b}$ can be $f_{\mathfrak{a}}((q, Q))$-quasiredirected to $\mathbf{a}$.

Topologising the boundary

Defining a topology

Recall that points in $P(X)$ are equivalence classes of quasi-geodesic rays. To unify the treatment, define $\mathbf{x} \in X$ as follows. $\mathbf{x} = \{$ quasi-geodesic rays in X which pass through $x\}$. We can then define $\mathcal{U}(\mathfrak{a}, r) = \{\mathfrak{b} \in X \cup P(X)$: every q-ray in $\mathfrak{b}$ can be $F_{\mathfrak{a}}(q)$ redirected to $\mathbf{a}$ at radius $r\}$ and these along with open sets in X comprise the basis of open sets in X.

Uniformatization

Uniformatization

Let (M, g) be a complete, simply connected Riemannian manifold of dimension 2, then (M, g) is conformally equivalent/biholomorphic to -

- $\mathbb{S}^2$
- $\mathbb{R}^2$
- $\mathbb{H}^2$

From [Grigoryan], we see that Brownian motion a simply connected Riemannian surface is transient if and only if it is conformally equivalent to $\mathbb{H}^2$.

Some prelimanaries in Riemannian geometry

Preliminaries

- Laplace-Beltrami operator - $\Delta_M f = Tr(Hess(f))$
- Green's function- Let $p(t,x,y) : (0,\infty) \times M \times M \to \mathbb{R}$ be the fundamental solution of $\frac{dp}{dt} - \frac{1}{2}\Delta_M(p) = 0$. Then we define $G(x,y) = \int_0^\infty p(t,x,y)dt$ to be the Green's function. We note that $G(x,y)$ is finite for some $x,y \in M$ if and only if it is finite for all $x,y \in M$ if and only if it is transient.
- Bounded geometry- A Riemannian manifold (M,g) is said to exhibit bounded geometry if there exists $(\kappa,\chi) \in [0,\infty) \times (0,\infty)$ such that at all points $p \in M$, $-\kappa \leq \mathrm{sec}_p(\Pi) \leq \kappa$ for all two-dimensional subspaces of T_pM and $\mathrm{inj}(p) > \chi$.

Triangulating planes with bounded geometry

Theorem -[Boissonant, Dyer, Ghosh]

Let (M, g) be a plane with bounded geometry. Then there exists a triangulation τ of M with its path metric ρ such that τ quasi-isometrically embeds into M.

As (M, g) is a Riemannian plane exhibiting bounded geometry, we get a triangulation (τ, ρ) with bounded degree whose path metric makes it quasi-isometric to (M, g). Furthermore, as (τ, ρ) is a one-ended graph, we can construct a rotationally symmetric plane (N, h) with its pole at o such that there exists a $p \in V(\tau)$
$V_N(B(o, n)) = |B(p, n)|$ and
$l(\partial B(p, n)) = |w \in V(\tau) : d(w, v) = n|$ for all $n \in \mathbb{N}$. We see that (N, h) is quasi-isometric to (M, g) and that they have the same conformal type.

Quasi-Redirection on Riemann surfaces

Statement

Let (M, g) be a simply connected Riemannian 2-manifold with bounded geometry. Then

- If (M, g) is conformally equivalent to S^2, then quasi-redirecting boundary of (M, g) is empty
- If (M, g) is conformally equivalent to $\mathbb{R}^2$ with the Euclidean metric, then the quasi-redirecting boundary of (M, g) is a single point.
- If (M, g) is conformally equivalent to $\mathbb{H}^2$, then the quasi-redirecting boundary of (M, g) is homeomorphic to S^1.

When (M, g) is conformally equivalent to $\mathbb{R}^2$

Proof

As quasi-redirection is a quasi-isometry invariant and as compact metric spaces lack quasi-geodesic rays, we can restrict our attention to rotationally symmetric planes (M, g) with its pole at o where $g = dr^2 + f(r)^2 d\theta^2$. Recall that (M, g) is conformally equivalent to $\mathbb{R}^2$ if and only if $\int_0^\infty \frac{dr}{2\pi f(r)} = \infty$ (from [Grigor'yan]) which in turn implies that $\limsup_{r\to\infty} \frac{f(r)}{r} < \infty$ which implies that geodesic rays in (M, g) diverge linearly. Therefore, if (M, g) is conformally equivalent to $\mathbb{R}^2$, then ∂M has a single point.

When (M, g) is conformally equivalent to $\mathbb{H}^2$

Proof

Just like the previous slide, we do not lose any generality if we suppose that (M, g) is a rotationally symmetric plane with its pole at p and $g = dr^2 + f(r)^2 d\theta^2$. As (M, g) is conformally equivalent to $\mathbb{H}^2$, we get that $\int_0^\infty \frac{dr}{f(r)} < \infty$ (see [Grigor'yan]). From bounded geometry, we get that there exists a superlinear function κ such that $0 < \liminf_{r\to\infty} \frac{f(r)}{\kappa(r)} < \infty$ so we can make (M, g) to be quasi-isometric (and conformally equivalent) to a Cartan-Hadamard manifold such that $\lim_{r\to\infty} \frac{l(\partial B(o,r)}{\kappa(r)} \in (0, \infty)$. As geodesic balls in Cartan-Hadamard manifolds are convex, geodesic rays diverge superlinearly and therefore if α, β are two distinct geodesic rays with $\alpha(o) = \beta(0) = o$, then $\alpha \not\prec \beta$ and $\beta \not\prec \alpha$. Therefore, we get that $\partial M = \mathbb{S}^1$.

Martin boundary

Let (M, g) be a complete Riemannian manifold and assume that Brownian motion on it is transient. We recall that this is equivalent to Δ_g admitting a Green's function $G : M \times M \to (0, \infty)$. Choose $x_o \in M$ and consider the function $K(\dot{,}x) = \frac{G(\dot{,}x)}{G(x,x_o}$ which are harmonic on $M \setminus \{x\}$ and is 1 at x_0. Consider all sequences $\{x_n\}_{n\in\mathbb{N}}$ with $d(x_o, x_n) \to \infty$ such that $K(\dot{,}x_n)$ also converge to a positive harmonic function and deem two sequences going off to infinity equivalent if their limits coincide. The space of such equivalence classes $\partial_\Delta M$ is called the Martin boundary of (M, g) and the resulting compactification of (M, g) is called the Martin compactification. If Brownian motion is recurrent on (M, g), then Δ_g does not admit a Green's function and therefore we can deem all sequences going off to infinity to get a one-compactification of (M, g) if non-compact or leaving (M, g) as it is if it is compact.

Martin Boundary and the Quasi-Redirecting Boundary

As positive harmonic functions are preserved under conformal mappings, we can just consider the case when (M, g) is conformally equivalent to $\mathbb{H}^2$. As conformal mappings $f : M \to N$ induce a homeomorphism of their Martin boundaries and quasi-isometries $h : M \to N$ induce a homeomorphism between quasi-redirecting boundaries, we can once again work with rotationally symmetric planes (M, g) without losing any generality. As geodesic rays do not redirect to each other, we get that quasi-geodesic rays converge to a unique point in the geodesic boundary (i.e $\lim_{r\to\infty} \frac{\exp_p^{-1}(\alpha(r))}{d(p,\alpha(r))}$ exists in UT_pM) and Theorem 6 of [Ancona] which ensures that geodesic rays diverge to a point in the Martin boundary, we get that the quasi-redirecting boundary is indeed the Martin boundary for rotationally symmetric planes.

Homogenous spaces

We say that a Riemannian n-manifold (M, g) is homogenous if $\mathrm{Isom}(M)$ acts transitively on M. It is clear that every homogenous manifold is quasi-isometric to a Lie group G and from [Cornulier],we see that M has an unique asymptotic cone (denote it as $\mathrm{Con}_\omega(M)$) . We further see that $\mathrm{Con}_\omega(M)$ is also a homogenous space and therefore, either every point is a cut-point or none is. If the former holds true, then M is quasi-isometric to a non-compact rank 1 symmetric space with $\partial M = \mathbb{S}^{n-1}$. If the latter holds true then M exhibits linear divergence and therefore ∂M has a single point.

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