Numerical Relativity: Mathematical Formulation

Important Results

General Relativity and differential geometry

- Christoffel symbols
$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc}) \tag{1}$$

- Riemann tensor

  Definition
$$\nabla_a \nabla_b v_c - \nabla_b \nabla_a v_c = R^d_{cba} v_d \tag{2}$$
  Compute from
$$R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc} \tag{3}$$
  Symmetries
$$R_{abcd} = -R_{bacd} \qquad R_{abcd} = -R_{abdc} \qquad R_{abcd} = R_{cdab} \tag{4}$$
$$R_{abcd} + R_{adbc} + R_{acdb} = 0 \tag{5}$$
  (also Bianchi identities.) In n dimensions have $n^2(n^2-1)/12$ independent components.

- Ricci tensor and Ricci scalar
$$R_{ab} \equiv R^c_{acb} \qquad R \equiv g^{ab} R_{ab} \tag{6}$$
  Three ways to compute Ricci
$$R_{ab} = \partial_c \Gamma^c_{ab} - \partial_b \Gamma^c_{ac} + \Gamma^c_{ab}\Gamma^d_{cd} - \Gamma^c_{ad}\Gamma^d_{bc} \tag{7}$$
$$R_{ab} = \frac{1}{2} g^{cd}\left(\partial_a\partial_d g_{cb} + \partial_c\partial_b g_{ad} - \partial_a\partial_b g_{cd} - \partial_c\partial_d g_{ab}\right) + g^{cd}\left(\Gamma^e_{ad}\Gamma_{ecb} - \Gamma^e_{ab}\Gamma_{ecd}\right) \tag{8}$$
$$R_{ab} = -\frac{1}{2} g^{cd}\partial_d\partial_c g_{ab} + g_{c(a}\partial_{b)}\Gamma^c + \Gamma^c\Gamma_{(ab)c} + 2g^{ed}\Gamma^c_{e(a}\Gamma_{b)cd} + g^{cd}\Gamma^e_{ad}\Gamma_{ecb} \tag{9}$$
  where $\Gamma^a \equiv g^{bc}\Gamma^a_{bc}$.

- Einstein tensor
$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R \tag{10}$$

- Einstein's equations
$$\boxed{G_{ab} = 8\pi T_{ab}} \tag{11}$$
  where T_{ab} is stress-energy tensor, and where we assume $\Lambda = 0$.

The $3+1$ Decomposition

- normal vector
$$n^a = -\alpha g^{ab}\nabla_b t \tag{12}$$

- induced or spatial metric
$$\gamma_{ab} = g_{ab} + n_a n_b \tag{13}$$

- Extrinsic curvature
$$K_{ab} \equiv -\gamma_a{}^c \gamma_b{}^d \nabla_c n_d = -\nabla_a n_b - n_a a_b = -\frac{1}{2}\mathcal{L}_{\mathbf{n}}\gamma_{ab} \tag{14}$$
  where $a_a \equiv n^b \nabla_b n_a$ is acceleration of normal observer.

- ADM equations
  - Constraint equations

$$R + K^2 + K_{ij}K^{ij} = 16\pi\rho \qquad \text{Hamiltonian constraint} \tag{15}$$

$$D_i(K^{ij} - \gamma^{ij}K) = 8\pi S^i \qquad \text{momentum constraint} \tag{16}$$

  - Evolution equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \tag{17}$$

$$\begin{aligned} \partial_t K_{ij} &= \alpha(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij}) - D_iD_j\alpha - 8\pi\alpha\left(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)\right) \\ &\quad + \beta^k\partial_k K_{ij} + K_{ik}\partial_j\beta^k + K_{kj}\partial_i\beta^k \qquad (18) \\ & \qquad (19) \end{aligned}$$

Conformal decompositions

- metric

$$\gamma_{ij} = \psi^4\bar{\gamma}_{ij} \tag{20}$$

- extrinsic curvature

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K = \psi^{-2}\bar{A}_{ij} + \frac{1}{3}\psi^4\bar{\gamma}_{ij}K \tag{21}$$

- Hamiltonian constraint

$$\bar{D}^2\psi - \frac{\psi}{8}\bar{R} + \frac{\psi^{-7}}{8}\bar{A}_{ij}\bar{A}^{ij} - \frac{1}{12}\psi^5K^2 = -2\pi\psi^5\rho \tag{22}$$

- momentum constraint

$$\bar{D}_j\bar{A}^{ij} - \frac{2}{3}\psi^6\bar{\gamma}^{ij}\bar{D}_jK = 8\pi\psi^{10}S^i \tag{23}$$