Moduli spaces of principal 2-group bundles and a categorification of the Freed–Quinn line bundle

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Let G be a compact Lie group, and fix $\alpha \in H^3(G, U(1))$.

Depending on your background[1], you may have thought of different fun things we could do with this data:

- Chern–Simons theory
- G-equivariant elliptic cohomology
- String structures
- Equivariant gerbes
- Orbifold actions on B-fields
- Higher representation theory
- ...

Focus of today's talk:

- Smooth 2-group $\mathcal{G}(G, U(1), \alpha)$ and its moduli space of principal bundles $\mathrm{Bun}_{\mathcal{G}}(X)$

Vague expectation:
All of the above topics are related to the space $\mathrm{Bun}_{\mathcal{G}}(X)$.

[1] maybe you would like to rename $\alpha = \lambda$

All of the above topics can be hard and/or abstract.

This makes it challenging to relate them.

We focus on the case where the group G is finite.

- We study the space $\mathrm{Bun}_{\mathcal{G}}(X)$ concretely and show it has nice properties.
- This allows us to make explicit connections to the above topics.

Joint work with Daniel Berwick-Evans, Laura Murray, Apurva Nakade, and Emma Phillips.

We fix the following input data for the rest of the talk:

- G a finite group;
- A an abelian Lie group (e.g. $U(1)$);
- $\alpha\colon G^{\times 3} \to A$ a 3-cocycle:

$$d\alpha(g, h, k, l) = \frac{\alpha(h, k, l)\alpha(g, hk, l)\alpha(g, h, k)}{\alpha(gh, k, l)\alpha(g, h, kl)} = 1 \in A$$

Plan for the talk

1. Smooth 2-groups: definitions and examples
2. Principal 2-group bundles: definitions and key properties
3. Relating our original list of topics

Section 1

Smooth 2-groups: definitions and examples

2-groups

Definition: A 2-group $\mathcal{G}$ is a monoidal groupoid in which all objects have tensor inverses.

Example 1: Let H be a group. It defines a groupoid with only identity morphisms, and the group structure makes it into a 2-group:

$$h \otimes k = hk.$$

Example 2: Let A be an abelian group. Consider the category $\mathrm{pt}/\!\!/A$, with one object pt, automorphisms A, and composition given by the group operation in A.

We make $\mathrm{pt}/\!\!/A$ into a monoidal category by setting

$$\mathrm{pt} \otimes \mathrm{pt} = \mathrm{pt};$$
$$a \otimes b = ab \colon \mathrm{pt} \to \mathrm{pt}.$$

This makes $\mathrm{pt}/\!\!/A$ into a 2-group.

Key example: $\mathcal{G} = \mathcal{G}(G, A, \alpha)$

Let $\mathcal{G}$ be the category with objects $g \in G$, and morphisms given by

$$\mathrm{Hom}_{\mathcal{G}}(g, h) = \begin{cases} \varnothing & \text{if } g \neq h; \\ A & \text{if } g = h. \end{cases}$$

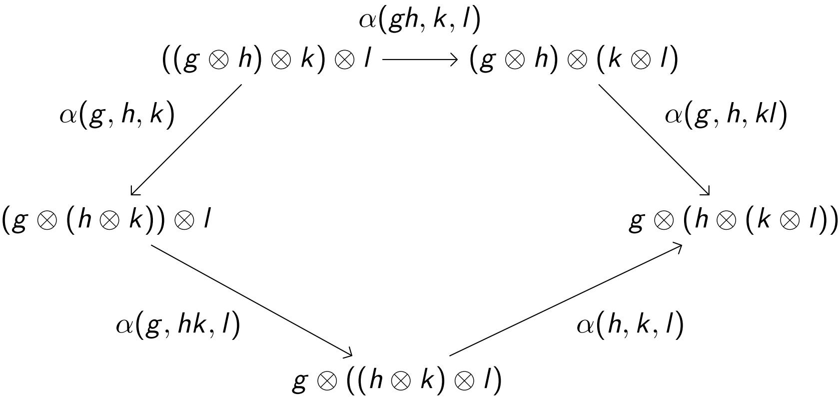
Define $g \otimes h = gh$ for $g, h \in \mathcal{G}$. To make this into a monoidal category, we need to specify associativity morphisms

$$(g \otimes h) \otimes k \to g \otimes (h \otimes k).$$

We define this to be $\alpha(g, h, k) \in A = \mathrm{Hom}_{\mathcal{G}}(ghk, ghk)$.

The associativity morphisms must satisfy the Pentagon Axiom.

The Pentagon Axiom



$\Rightarrow \alpha(h, k, l)\alpha(g, hk, l)\alpha(g, h, k) = \alpha(g, h, kl)\alpha(gh, k, l).$

- This is exactly equivalent to the 3-cocycle condition on α.

Finite 2-groups

Definition: A 2-group $\mathcal{G}$ is finite if there are finitely many isomorphism classes of objects.

Theorem (Sinh, Baez–Lauda)

Up to equivalence, all finite 2-groups arise in this way, starting from the data of a finite group G, an abelian group A (possibly with a G-action), and a 3-cocycle $\alpha \in Z^3(G; A)$.

$\mathrm{pt}/\!\!/A$-extensions

For $\mathcal{G} = \mathcal{G}(G, A, \alpha)$ as just defined, we have a short exact sequence

$$1 \to \mathrm{pt}/\!\!/A \to \mathcal{G} \to G \to 1,$$

which allows us to view $\mathcal{G}$ as an extension of G by $\mathrm{pt}/\!\!/A$.

The short exact sequence splits $\Leftrightarrow$ the cocycle α is a coboundary.

Recall: A-extensions of G are classified by $H^2(G, A)$.
Similarly, $\mathrm{pt}/\!\!/A$-extensions of G are classified by $H^3(G, A)$.

Example: the string group [Schommer-Pries]

Let α_{str} represent the generator of $H^3(\mathrm{Spin}(n), U(1)) \cong \mathbb{Z}$.

The string group is the 2-group defined by the central extension

$$1 \to \mathsf{pt}/\!\!/ U(1) \to \mathrm{String}(n) \to \mathrm{Spin}(n) \to 1.$$

Finite 2-groups from a representation $\rho_0\colon G_0 \to \mathrm{SO}(n)$

$$
\begin{array}{ccccc}
\mathrm{pt}/\!\!/U(1) & \longrightarrow & \mathcal{G} & \longrightarrow & \mathrm{String}(n) \\
 & & \downarrow & & \downarrow \\
\mathbb{Z}/2 & \longrightarrow & G & \xrightarrow{\rho} & \mathrm{Spin}(n) \\
 & & \downarrow & & \downarrow \\
 & & G_0 & \xrightarrow{\rho_0} & \mathrm{SO}(n)
\end{array}
$$

Here $\mathcal{G}$ is the central extension of G classified by the 3-cocycle $\rho^* \alpha_{\mathsf{str}}$.

Smooth 2-groups

Definition: A smooth 2-group is a group object in the bicategory of smooth stacks.

More concretely, we work in the bicategory with

- Objects: Lie groupoids;
- 1-morphisms: bibundles;
- 2-morphisms: isomorphisms of bibundles.

In particular, all group axioms hold up to specified 2-isomorphisms, which in turn satisfy compatibility conditions.

All of our examples above fit into this framework.

Section 2

Principal 2-group bundles

Fix a smooth 2-group $\mathcal{G}$ and a smooth manifold X.

Goal: define a bicategory of principal 2-group bundles over X.

We can take many approaches, based on analogies with the classical setting. Which is correct?

Our favourite

A principal $\mathcal{G}$-bundle on X is a smooth stack $\mathcal{P}$ over X, equipped with a $\mathcal{G}$-action which is locally trivial:

There exists a surjective submersion $u : Y \to X$ and an isomorphism of $\mathcal{G}$-stacks over Y

$$u^*\mathcal{P} \xrightarrow{\sim} Y \times \mathcal{G}.$$

Understanding this data

We observe that we have a forgetful functor

$$\pi : \mathrm{Bun}_{\mathcal{G}}(X) \to \mathrm{Bun}_G(X).$$

Theorem (BCMNP)

$\pi\colon \mathrm{Bun}_{\mathcal{G}}(X) \to \mathrm{Bun}_G(X)$ *is a torsor over the symmetric monoidal bicategory* $\mathrm{Gerbe}_A(X)$*.*

The proof uses a Čech-style reformulation of the bicategory $\mathrm{Bun}_{\mathcal{G}}(X)$.

Čech data for a $\mathcal{G}$-bundle $\mathcal{P}$

We can choose a surjective submersion $u\colon Y \to X$ over which $\mathcal{P}$ trivialises.

Fix a trivialisation $d\colon u^*\mathcal{P} \xrightarrow{\sim} Y \times \mathcal{G}$.

Over $Y \times_X Y$ we obtain an automorphism

$$\Phi\colon Y \times_X Y \times \mathcal{G} \to Y \times_X Y \times \mathcal{G},$$

which satisfies a cocycle condition on $Y \times_X Y \times_X Y$:

$$\Gamma\colon p_{12}^*\Phi \circ p_{23}^*\Phi \xrightarrow{\sim} p_{13}\Phi.$$

Γ is a bibundle, and in particular an A-bundle; refining u if necessary, we can also fix a trivialisation of this A-bundle structure. Eventually we find that we can associate with $\mathcal{P}$ a triple

- $u\colon Y \to X$;
- $\rho\colon Y \times_X Y \to G$;
- $\gamma\colon Y \times_X Y \times_X Y \to A$;

satisfying certain cocycle-type conditions.

Definition: An A-gerbe over X is a principal $\mathrm{pt}/\!\!/A$-bundle over X.

In particular, it is determined by a surjective submersion $u\colon Y \to X$ and a Čech 2-cocycle

$$\gamma\colon Y \times_X Y \times_X Y \to A.$$

Definition: An A-2-gerbe over X is determined by a surjective submersion $u\colon Y \to X$ and a Čech 3-cocycle

$$\lambda\colon Y \times_X Y \times_X Y \times_X Y \to A$$

Sketch of the proof

- Given the data (G, α), a principal G-bundle P determines a 2-gerbe on X, unique up to 1-isomorphism:
  - Associate to P Čech data $(u\colon Y \to X, \rho\colon Y \times_X Y \to G)$.
  - Now we obtain a 3-cocycle $\lambda_P = \rho^* \alpha$:

$$\lambda_P(y_1, y_2, y_3, y_4) = \alpha(\rho(y_1, y_2), \rho(y_2, y_3), \rho(y_3, y_4)).$$

- Let $\mathcal{P}$ be a principal $\mathcal{G}$-bundle, with underlying G-bundle $\pi(\mathcal{P}) = P$.
  - Associate to $\mathcal{P}$ the Čech data (u, ρ, γ).
  - The data of γ is equivalent to the data of a trivialisation of the 2-gerbe λ_P.
- The bicategory of trivialisations of a fixed 2-gerbe is a torsor over the symmetric monoidal bicategory of gerbes.

In terms of classifying stacks

Define $B\mathcal{G}$ to be the following homotopy pullback:

$$\begin{array}{ccccc} & & B\mathcal{G} & \longrightarrow & \mathrm{pt} \\ & \nearrow & \downarrow & & \downarrow \\ X & \longrightarrow & BG & \xrightarrow[\alpha]{} & B^3A. \end{array}$$

We expect principal $\mathcal{G}$-bundles to be classified by maps $X \to B\mathcal{G}$;

i.e. by maps $\rho \colon X \to BG$ together with a trivialisation of the composition $\alpha \circ \rho$;

i.e. by principal G-bundles P together with a trivialisation of the 2-gerbe classified by $\rho^*\alpha$.

This is consistent with our definitions.

The special case of flat $\mathcal{G}$-bundles

These are defined to be principal $\mathcal{G}_\delta$-bundles, where $\mathcal{G}_\delta$ is given the discrete topology.

Recall: Flat principal G-bundles are classified by homomorphisms $\pi_1(X) \to G$.

Theorem (BCMNP)

For X with contractible universal cover, flat principal $\mathcal{G}$-bundles $\mathcal{P}$ are classified by homomorphisms of 2-groups $(\rho, \gamma)\colon \pi_1(X) \to \mathcal{G}$.

$\mathrm{Hom}_{\mathrm{bicat}}(\mathrm{pt}/\!\!/\pi_1(X), \mathrm{pt}/\!\!/\mathcal{G})$

Recall that a functor $\mathrm{pt}/\!\!/\pi_1(X) \to \mathrm{pt}/\!\!/G$ is just a homomorphism

- $\rho\colon \pi_1(X) \to G$.

To lift to $\mathrm{pt}/\!\!/\mathcal{G}$, we need

- $\gamma\colon \pi_1(X) \times \pi_1(X) \to A$, providing isomorphisms in $\mathcal{G}$:

$$\gamma(a, b)\colon \rho(a)\rho(b) \xrightarrow{\sim} \rho(ab),$$

  and satisfying $d\gamma = 1/\rho^*\alpha$.

Recall that a natural transformation $\rho_1 \to \rho_2$ is given by

- $t \in G$ such that $t\rho_1(a) = \rho_2(a)t$ for all $a \in \pi_1(X)$.

To lift to a natural transformation $(\rho_1, \gamma_1) \to (\rho_2, \gamma_2)$, we need

- $\eta\colon \pi_1(X) \to A$, providing isomorphisms

$$\eta(a)\colon t\rho_1(a) \xrightarrow{\sim} \rho_2(a)t.$$

2-morphisms are given by $\omega \in A$.

We make the following identifications:

$$\begin{array}{ccc}
\mathrm{Bun}^\flat_{\mathcal{G}}(X) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathrm{bicat}}(\mathrm{pt}/\!\!/\pi_1(X), \mathrm{pt}/\!\!/\mathcal{G}) & & \\
\pi \downarrow & & \downarrow \pi & & \\
\mathrm{Bun}^\flat_{G}(X) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathrm{cat}}(\mathrm{pt}/\!\!/\pi_1(X), \mathrm{pt}/\!\!/G) & \simeq & \mathrm{Hom}_{\mathrm{grp}}(\pi_1(X), G)/\!\!/G
\end{array}$$

Theorem (BCMNP)

The action of G on $\mathrm{Hom}_{grp}(\pi_1(X), G)$ lifts to an action of G on the bicategory $\mathrm{Hom}_{bicat}(pt/\!\!/\pi_1(X), pt/\!\!/\mathcal{G})$.

This gives π the structure of a cloven 2-fibration.

A bit about the action of G

For each $g \in G$, we define an autoequivalence

$$F_g \colon \mathrm{Hom}_{\mathrm{bicat}}(\mathrm{pt}/\!\!/\pi_1(X), \mathrm{pt}/\!\!/\mathcal{G}) \to \mathrm{Hom}_{\mathrm{bicat}}(\mathrm{pt}/\!\!/\pi_1(X), \mathrm{pt}/\!\!/\mathcal{G}).$$

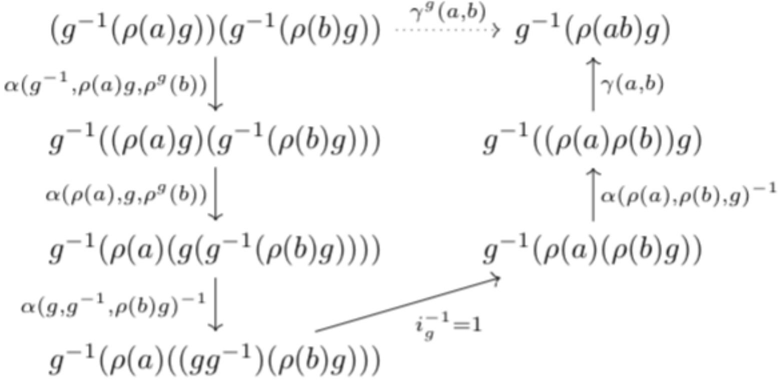
For example, on objects, $F_g(\rho, \gamma) = (\rho^g, \gamma^g)$, where

$$\begin{aligned} \rho^g \colon \pi_1(X) &\to G \\ a &\mapsto g^{-1}(\rho(a)g). \end{aligned}$$

and $\gamma^g(a, b) \in A$ is the natural isomorphism

$$(g^{-1}(\rho(a)g))(g^{-1}(\rho(b)g)) \xrightarrow{\sim} g^{-1}(\rho(ab)g)$$

given on the next slide.



$$\Rightarrow \gamma^g(a,b) = \gamma(a,b)\frac{\alpha(g^{-1},\rho(a)g,\rho^g(b))\alpha(\rho(a),g,\rho^g(b))}{\alpha(g,g^{-1},\rho(b)g)\alpha(\rho(a),\rho(b),g)}$$

We provide natural isomorphisms $F_g \circ F_h \xrightarrow{\sim} F_{hg}$, and higher isomorphisms giving associativity.

Section 3

Connections/applications

The Freed–Quinn line bundle

As before fix G a finite group. Assume that $A = U(1)$ and let α be a 3-cocycle.

- **Dijkgraaf–Witten theory:** Chern–Simons theory for this finite data.

Freed & Quinn construct a line bundle $\mathcal{L}$ on the moduli space of principal G-bundles over Riemann surfaces.

For X a fixed Riemann surface, we obtain a line bundle $\mathcal{L}_X$ on $\mathrm{Bun}_G(X)$ such that the vector space that Chern–Simons theory assigns to X is $\Gamma(X, \mathcal{L}_X)$.

Restricting to the moduli space of principal G-bundles over elliptic curves, we obtain a line bundle $\mathcal{L}_1$. Ganter shows that this line bundle $\mathcal{L}_1$ is the home of twisted G-equivariant elliptic cohomology.

Theorem (BCMNP)

The 2-fibration $\pi : \mathrm{Bun}^{\flat}_{\mathcal{G}}(X) \to \mathrm{Bun}_G(X)$ *categorifies the Freed–Quinn line bundle.*

- Recall that π is a fibration with fibres $\mathrm{Gerbe}_{U(1)}(X)$.
- Isomorphism classes of gerbes are given by $H^2(X, U(1)) \cong U(1)$.
- So taking isomorphism classes along the fibres of π, we obtain a principal $U(1)$-bundle on $\mathrm{Bun}_G(X)$.
- The corresponding line bundle is isomorphic to $\mathcal{L}_X$.

String structures

Assume that $\mathcal{G}$ is a 2-group arising from a representation as in our earlier example:

$$\begin{array}{ccccc}
\mathrm{pt}/\!\!/U(1) & \longrightarrow & \mathcal{G} & \to & \mathrm{String}(n) \\
 & & \downarrow & & \downarrow \\
\mathbb{Z}/2 & \longrightarrow & G & \xrightarrow{\rho} & \mathrm{Spin}(n) \\
 & & \downarrow & & \downarrow \\
 & & G_0 & \xrightarrow{\rho_0} & \mathrm{SO}(n)
\end{array}$$

Let $P_0 \to X$ be an oriented vector bundle with structure group G_0.

A spin structure on P_0 is a lift to a G-bundle P.

Analogously, a string structure on P is a lift to a $\mathcal{G}$-bundle $\mathcal{P}$.

Comparison: Waldorf's string structures

Before having access to Schommer-Pries's convenient construction of $\mathrm{String}(n)$ as a 2-group, different authors took other approaches to defining string structures.

Let $P \to X$ be a principal spin G-bundle.

Waldorf: A string structure on P is a choice of trivialisation of the 2-gerbe λ_P determined by P and the 3-cocycle α.

Theorem (BCMNP)

The two definitions of string structures coincide.

- This is immediate from our results about the 2-fibration π. Indeed, recall that the fibre of $\mathrm{Bun}_{\mathcal{G}}(X)$ over P is equivalent to the bicategory of trivialisations of the 2-gerbe λ_P.

Comparison: Stolz–Teichner's string structures

Let $P \to X$ be a principal flat G-bundle and consider Chern–Simons theory for P, CS_P.

Stolz–Teichner: A geometric string structure on P is a trivialisation of CS_P.

- In particular, for suitable $f : M^2 \to X$, $CS_P(f)$ is a line and we require a non-zero point in this line.
- But we have proven that this is equivalent to an isomorphism class of flat $\mathcal{G}$-bundle over f^*P.
- This is given by $f^*\mathcal{P}$ for $\mathcal{P}$ a flat lift of P.

String structures and trivialisations of Chern–Simons theory

With our results, it is easy to compare

(part of the data of the trivialisation of CS_P)

to

(part of the data of $\mathcal{P} \in \pi^{-1}(P) \subset \mathrm{Bun}_{\mathcal{G}}(X)$).

Work in progress: understand the relationships between the rest of the data

More precisely:

(trivialisation of CS_P for all dimensions)

vs

(bicategory structure truncated to produce Freed–Quinn)

Thank you!