Plane curves

We are interested in polynomials in two variables x, y with real or complex coefficients. Let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively.

Some examples of polynomials of the kind we will study are:

$$x^2 + y^2 - 1, xy^{10} - \pi x^4 y^4 + 100x^{12} + \sqrt{2}y^{71}.$$

For non-negative integers i, j and a complex number c, a term of the form $cx^i y^j$ is called a *monomial*. The *degree* of such a monomial is defined to be $i + j$. A polynomial is thus simply a sum of monomials. The degree of a polynomial is the degree of the largest degree monomial which appears in it. The degree of the two polynomials above are 2 and 71, respectively.

Exercise: Give a few examples of polynomials and note their degrees.

We are interested in *zeros* or *roots* of polynomials. Let $f(x, y)$ be a polynomial. An element of (a, b) of $\mathbb{C}^2$ is called a *zero* of $f(x, y)$ if $f(a, b) = 0$.

For example, $(0, 1)$ is a zero of the polynomial $x^2 - 1 = 0$ as well as of $xy^2 + y - 1$. On the other hand, $(0, 1)$ is not a zero of $x^2y^3 + 1$.

Definition: The set of zeros of a polynomial is called a *plane curve*.

Note that plane curves are subsets of $\mathbb{C}^2$. If a plane curve C is the zero set of a polynomial f, we also say that C *is defined by* f.

We have special names for plane curves of small degrees.

Lines: A line is the zero set of a polynomial of degree 1.

Conics: Zeros of a polynomial of degree 2 form a *conic*.

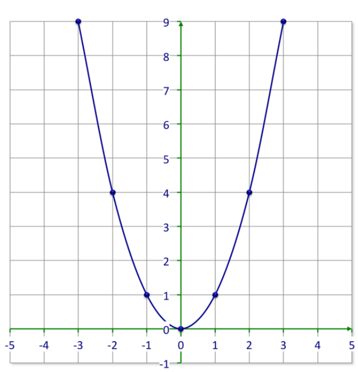
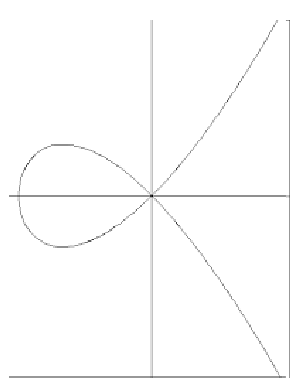
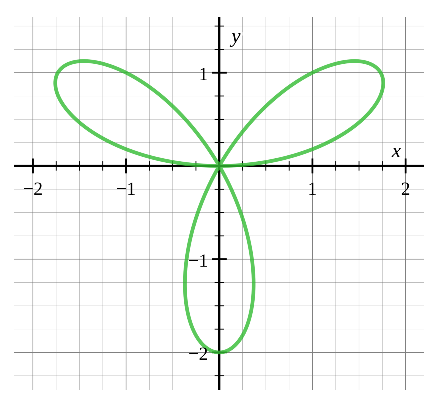
Cubics: *Cubics* are zeroes of degree 3 polynomials.

Curves of degree d: A *curve of degree* d is the zero set of a polynomial of degree d.

Exercise: Draw the curves in $\mathbb{R}^2$ defined by the following polynomials.

- $2x + 6y - 5$.
- $x^2 + y^2$.
- $x^2 + y^2 - 1$.
- $x^2 - y^2$.
- $x^2 + y^2 + 1$.
- $x^2 - y$.
- $\frac{x^2}{4} + \frac{y^2}{9} - 1$.
- $y^2 - x^3$.
- $y^2 - x^2(x + 1)$.

Note that $(0, 0)$ is contained in each of the following three curves:

Exercise: Can you identify the differences in how $(0, 0)$ sits inside these curves?

Definition: Let (a, b) is a point in $\mathbb{C}^2$. Let C be a plane curve defined by a polynomial $f(x, y)$. Suppose that m is the smallest non-negative integer such that all the partial derivatives of f of order up to m vanish at (a, b). The *multiplicity* of C at (a, b) is defined to be $m + 1$.

For example, the multiplicities of the curves defined by $x^2 + y^2 - 1, x^2 + y^2 + 2x, y^2 - x^3, y^9 + x^{20}$ at $(0, 0)$ are $0, 1, 2$ and 9, respectively.

Exercise: Find the multiplicity of the following curves at the indicated points.

- $x + y$ at $(3, 3)$.
- $x + y$ at $(10, -10)$.
- x^{23} at $(0, 0)$.
- x^9y^2 at $(0, 0)$.
- x^9y^2 at $(1, 0)$.
- x^9y^2 at $(0, 1)$.
- $x^2y^{17} + xy^2 - 5x^{10}$ at $(0, 0)$.

Exercise: Let C be a plane curve of degree d. Let p be a point on C and let m be the multiplicity of C at p. Show that $1 \leqslant m \leqslant d$. Give examples to show that both the extreme values can be attained.

Now we will study how to *measure* the set of polynomials of a given degree d. Note that for any $d \geqslant 0$, the set of polynomials of degree d is infinite. However, we can describe these sets with finitely many parameters.

Let $d = 0$. Note that a polynomial of degree 0 is simply an element a of $\mathbb{C}$. So every such polynomial is a complex multiple of 1. So we can say that 1 is enough to describe all the polynomials of degree 0. Since only one monomial (namely, 1) is needed to describe them, we say that the *dimension* of the set of polynomials of degree 0 is 1.

Let $d = 1$. An arbitrary polynomial of degree 1 is of the form $a + bx + cy$ where a, b, c are complex numbers. So we say that $1, x, y$ describe the set of all the polynomials of degree 1. The dimension of the set of polynomials of degree 1 is 3.

Similarly, the monomials $1, x, y, x^2, y^2, xy$ can describe any degree 2 polynomial and the dimension of the set of polynomials of degree 2 is 6.

Exercise: For any non-negative integer d, show that set of all the curves of degree d has dimension $\frac{(d+2)(d+1)}{2}$ and list the monomials which can express any polynomial of degree d.

Now we want to study the following question.

Main Question: Fix r points $p_1, \ldots, p_r$ in $\mathbb{C}^2$. Let $d, m \geqslant 1$ be integers. Is there a curve of degree d which has multiplicity at least m at p_i for each $i = 1, \ldots, r$?

In order to study this, let us look at some specific cases of the above question.

Question 1: Given integers $d, m \geqslant 1$, is there a curve of degree d which has multiplicity at least m at $(0, 0)$?

Next, generalise to any one point p in $\mathbb{C}^2$:

Question 2: Fix a point p in $\mathbb{C}^2$. Let $d, m \geqslant 1$ be integers. Is there a curve of degree d which has multiplicity at least m at p? Can you find some conditions on d and m so that the answer is YES?

Question 3: Suppose $r = 2$. That is, we are given two points p_1, p_2 in $\mathbb{C}^2$. Is there a curve of degree d which has multiplicity at least m at both p_1 and p_2, in the following cases?

$d = 1, m = 1$; $d = 1, m = 2$; $d = 2, m = 2$; $d = 3, m = 2$.

Now the same question, in general, for $r = 2$:

Question 4: Fix 2 points p_1, p_2 in $\mathbb{C}^2$. Let $d, m \geqslant 1$ be integers. Under what conditions on d and m, is there a curve of degree d which has multiplicity at least m at p_i for each $i = 1, 2$?

Question 5: Is there a conic through any given five points of $\mathbb{C}^2$? What about through any given six points?