

## 1. PLANE CURVES

We are interested in polynomials in two variables  $x, y$  with real or complex coefficients. Let  $\mathbb{R}$  and  $\mathbb{C}$  denote the sets of real and complex numbers, respectively.

Some examples of polynomials of the kind we will study are:

$$x^2 + y^2 - 1, xy^{10} - \pi x^4 y^4 + 100x^{12} + \sqrt{2}y^{71}.$$

For non-negative integers  $i, j$  and a complex number  $c$ , a term of the form  $cx^i y^j$  is called a *monomial*. The *degree* of such a monomial is defined to be  $i + j$ . A polynomial is thus simply a sum of monomials. The degree of a polynomial is the degree of the largest degree monomial which appears in it. The degree of the two polynomials above are 2 and 71, respectively.

**Exercise 1.1.** Give a few examples of polynomials and note their degrees.

We are interested in *zeros* or *roots* of polynomials. Let  $f(x, y)$  be a polynomial. An element  $(a, b)$  of  $\mathbb{C}^2$  is called a *zero* of  $f(x, y)$  if  $f(a, b) = 0$ .

For example,  $(0, 1)$  is a zero of the polynomial  $x^2 - 1 = 0$  as well as of  $xy^2 + y - 1$ . On the other hand,  $(0, 1)$  is not a zero of  $x^2 y^3 + 1$ .

**Definition 1.2.** The set of zeros of a polynomial is called a *plane curve*.

Note that plane curves are subsets of  $\mathbb{C}^2$ . If a plane curve  $C$  is the zero set of a polynomial  $f$ , we also say that  $C$  is *defined by*  $f$ . If  $C$  is defined by  $f$ , we also write  $C = V(f)$ .

We have special names for plane curves of small degrees.

**Lines:** A line is the zero set of a polynomial of degree 1.

**Conics:** Zeros of a polynomial of degree 2 form a *conic*.

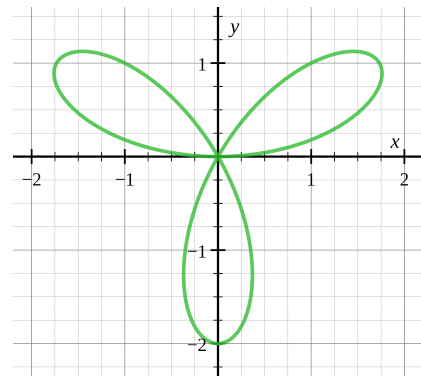
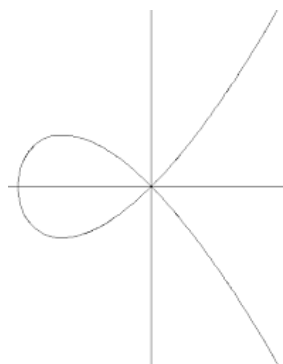
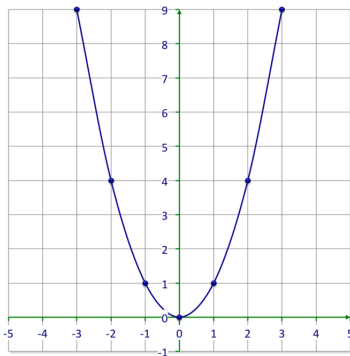
**Cubics:** *Cubics* are zeroes of degree 3 polynomials.

**Curves of degree  $d$ :** A *curve of degree  $d$*  is the zero set of a polynomial of degree  $d$ .

**Exercise 1.3.** Draw the curves in  $\mathbb{R}^2$  defined by the following polynomials.

- (1)  $2x + 6y - 5$ .
- (2)  $x^2 + y^2$ .
- (3)  $x^2 + y^2 - 1$ .
- (4)  $x^2 - y^2$ .
- (5)  $x^2 + y^2 + 1$ .
- (6)  $x^2 - y$ .
- (7)  $\frac{x^2}{4} + \frac{y^2}{9} - 1$ .
- (8)  $y^2 - x^3$ .
- (9)  $y^2 - x^2(x + 1)$ .

Note that  $(0, 0)$  is contained in each of the following three curves:



**Exercise 1.4.** Can you identify the differences in how  $(0, 0)$  sits inside these curves?

**Definition 1.5.** Let  $(a, b)$  is a point in  $\mathbb{C}^2$ . Let  $C$  be a plane curve defined by a polynomial  $f(x, y)$ . Suppose that  $m$  is the largest non-negative integer such that all the partial derivatives of  $f$  of order up to  $m$  vanish at  $(a, b)$ . The *multiplicity* of  $C$  at  $(a, b)$  is defined to be  $m + 1$ .

The multiplicity of a curve  $C$  at a point  $p$  is denoted by  $\text{mult}_p C$ .

**Example 1.6.** The multiplicities of the curves defined by  $x^2 + y^2 - 1, x^2 + y^2 + 2x, y^2 - x^3, y^9 + x^{20}$  at  $(0, 0)$  are 0, 1, 2 and 9, respectively.

**Exercise 1.7.** Find the multiplicity of the following curves at the indicated points.

- (1)  $x + y$  at  $(3, 3)$ .
- (2)  $x + y$  at  $(10, -10)$ .
- (3)  $x^{23}$  at  $(0, 0)$ .
- (4)  $x^9 y^2$  at  $(0, 0)$ .
- (5)  $x^9 y^2$  at  $(1, 0)$ .
- (6)  $x^9 y^2$  at  $(0, 1)$ .
- (7)  $x^2 y^{17} + xy^2 - 5x^{10}$  at  $(0, 0)$ .

**Exercise 1.8.** Let  $C$  be a plane curve of degree  $d$ . Let  $p$  be a point on  $C$  and let  $m$  be the multiplicity of  $C$  at  $p$ . Show that  $1 \leq m \leq d$ . Give examples to show that both the extreme values can be attained.

Now we will study how to *measure* the set of polynomials of a given degree  $d$ . Note that for any  $d \geq 0$ , the set of polynomials of degree  $d$  is infinite. However, we can describe these sets with finitely many parameters.

Let  $d = 0$ . Note that a polynomial of degree 0 is simply an element  $a$  of  $\mathbb{C}$ . So every such polynomial is a complex multiple of 1. So we can say that 1 is enough to describe all the polynomials of degree 0. Since only one monomial (namely, 1) is needed to describe them, we say that the *dimension* of the set of polynomials of degree 0 is 1.

Let  $d = 1$ . An arbitrary polynomial of degree 1 is of the form  $a + bx + cy$  where  $a, b, c$  are complex numbers. So we say that  $1, x, y$  describe the set of all the polynomials of degree 1. The dimension of the set of polynomials of degree 1 is 3.

Similarly, the monomials  $1, x, y, x^2, y^2, xy$  can describe any degree 2 polynomial and the dimension of the set of polynomials of degree 2 is 6.

**Exercise 1.9.** For any non-negative integer  $d$ , show that set of all the curves of degree  $d$  has dimension  $\frac{(d+2)(d+1)}{2}$  and list the monomials which can express any polynomial of degree  $d$ .

Now we want to study the following question.

**Exercise 1.10 (Main Exercise).** Fix  $r$  points  $p_1, \dots, p_r$  in  $\mathbb{C}^2$ . Let  $d, m \geq 1$  be integers. Is there a curve of degree  $d$  which has multiplicity at least  $m$  at  $p_i$  for each  $i = 1, \dots, r$ ?

In order to study this, let us look at some specific cases of the above question.

**Exercise 1.11.** Given integers  $d, m \geq 1$ , is there a curve of degree  $d$  which has multiplicity at least  $m$  at  $(0, 0)$ ?

Next, generalise to any one point  $p$  in  $\mathbb{C}^2$ :

**Exercise 1.12.** Fix a point  $p$  in  $\mathbb{C}^2$ . Let  $d, m \geq 1$  be integers. Is there a curve of degree  $d$  which has multiplicity at least  $m$  at  $p$ ? Can you find some conditions on  $d$  and  $m$  so that the answer is YES?

**Exercise 1.13.** Suppose  $r = 2$ . That is, we are given two points  $p_1, p_2$  in  $\mathbb{C}^2$ . Is there a curve of degree  $d$  which has multiplicity at least  $m$  at both  $p_1$  and  $p_2$ , in the following cases?

$d = 1, m = 1$ ;  $d = 1, m = 2$ ;  $d = 2, m = 2$ ;  $d = 3, m = 2$ .

Now the same question, in general, for  $r = 2$ :

**Exercise 1.14.** Fix 2 points  $p_1, p_2$  in  $\mathbb{C}^2$ . Let  $d, m \geq 1$  be integers. Under what conditions on  $d$  and  $m$ , is there a curve of degree  $d$  which has multiplicity at least  $m$  at  $p_i$  for each  $i = 1, 2$ ?

**Exercise 1.15.** Is there a conic through any given five points of  $\mathbb{C}^2$ ? What about through any given six points?

The answer to Main Exercise 1.10 is given by the following.

**Exercise 1.16.** Let  $p_1, \dots, p_r \in \mathbb{C}^2$  be distinct points. Let  $d > 0, m_1, \dots, m_r \geq 0$  be integers.

If  $\frac{(d+2)(d+1)}{2} - \sum_{i=1}^r \frac{(m_i+1)m_i}{2} > 0$ , show that there exists a degree  $d$  curve passing through  $p_i$  with multiplicity at least  $m_i$  for  $i = 1, 2, \dots, r$ .

Using Exercise 1.16, show the following.

**Exercise 1.17.**

- (1) There is a conic through any given 5 points in  $\mathbb{C}^2$ .
- (2) There is a cubic through any given 9 points in  $\mathbb{C}^2$ .
- (3) Let  $p_1, p_2, \dots, p_7$  be distinct points in  $\mathbb{C}^2$ . There exists a cubic passing through  $p_1$  with multiplicity 2 and passing through other points  $p_2, \dots, p_7$ .

**1.1. Infimum and supremum.** Let  $S$  be a non-empty set of real numbers. We say that  $S$  is *bounded above* if there exists an integer  $N$  such that  $s \leq N$  for every  $s \in S$ . Similarly,  $S$  is *bounded below* if there exists an integer  $M$  such that  $s \geq M$  for every  $s \in S$ .

Suppose that  $S$  is bounded below. Then a real number  $x$  is called a *lower bound* of  $S$  if  $s \geq x$  for all  $s \in S$ . The *greatest lower bound* or *infimum* of  $S$  is a real number  $x$  satisfying the following conditions:

- $x$  is a lower bound of  $S$ , and
- if  $y$  is a lower bound of  $S$  then  $x \geq y$ .

If  $S$  is bounded above, the *least upper bound* or *supremum* of  $S$  is defined similarly.

It is a fact that every bounded above set of real numbers has a supremum and every bounded below set of real numbers has an infimum. The infimum and supremum of  $S$  are denoted by  $\inf S$  and  $\sup S$ , respectively.

**Exercise 1.18.** Let  $S$  be a bounded below set. If  $x, y$  are both infimums of  $S$ , show that  $x = y$ . Similarly, the supremum of a bounded above set is unique.

**Exercise 1.19.** Find the infimum and supremum of the following sets, when applicable.

- (1)  $S_1 = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ .
- (2)  $S_2 = \{x \in \mathbb{R} \mid 0 < x < 1\}$ .
- (3)  $S_3 = \{x \in \mathbb{Q} \mid 0 < x < 1\}$ .
- (4)  $S_4 = \{\frac{a}{b} \in \mathbb{Q} \mid a, b \text{ are positive integers such that } a + b \leq 4\}$ .
- (5)  $S_5 = \{x \in \mathbb{Q} \mid x^2 \leq 2\}$ .

Note that the infimum or the supremum of a set  $S$  need not belong to  $S$ .

**1.2. An invariant.**

**Definition 1.20.** Let  $p_1, \dots, p_r$  in  $\mathbb{C}^2$  be distinct points. Then we define the following number:

$$A(p_1, p_2, \dots, p_r) = \inf \frac{d}{m},$$

where the infimum is taken over all ratios  $d/m$  such that there is a degree  $d$  curve passing through each  $p_i$  with multiplicity at least  $m$ .

**Exercise 1.21.**

- (1) Show that  $A(p) = 1$  for any point  $p$  in  $\mathbb{C}^2$ .
- (2) Show that  $A(p_1, p_2) = 1$  for any distinct points  $p_1, p_2$  in  $\mathbb{C}^2$ .
- (3) Let  $p_1 = (1, 0), p_2 = (2, 0), p_3 = (3, 0)$  and  $q = (1, 1)$ . Show that  $A(p_1, p_2, p_3) = 1$  and  $A(p_1, p_2, q) \leq \frac{3}{2}$ .

We now give some upper bounds for  $A(p_1, \dots, p_r)$ .

**Exercise 1.22.** Let  $r$  be a positive integer. Show that  $A(p_1, \dots, p_r) \leq \sqrt{r}$ , for all points  $p_1, \dots, p_r \in \mathbb{C}^2$ . Use Exercise 1.16.

**Hints for Exercise 1.22.** By Exercise 1.16, there exists a curve of degree  $d$  passing through  $r$  points with multiplicity at least  $m$  if

$$\frac{(d+2)(d+1)}{2} - r \frac{(m+1)m}{2} > 0.$$

This is equivalent to  $(d+2)(d+1) - r(m+1)m > 0$ . Now write  $t = d/m$ , or  $d = tm$ . Then the above condition is equivalent to

$$(1.1) \quad (t^2 - r)m^2 + (3t - r)m + 2 > 0.$$

For which values of  $t$  is this inequality valid?

Suppose that  $t^2 > r$ . For  $m$  large enough, show that (1.1) is valid by following the argument below:

Consider the graph of the **quadratic** function in the variable  $m$  on the left hand side of (1.1). It goes to infinity as  $m$  goes to infinity, provided the leading constant is positive.

We say that the term  $(t^2 - r)m^2$  *dominates* left hand side of (1.1).

So for any  $d/m = t > \sqrt{r}$ , there exists a curve of degree  $d$  passing through  $p_1, p_2, \dots, p_r$  with multiplicity at least  $m$ . Thus  $A(p_1, \dots, p_r) \leq \sqrt{r}$ .

We are now able to give the following definition.

**Definition 1.23.** Let  $r$  be a positive integer. Then we define

$$A_r = \sup A(p_1, \dots, p_r),$$

where the supremum is taken over all sets of distinct points  $p_1, \dots, p_r$  in  $\mathbb{C}^2$ .

## 2. BÉZOUT'S THEOREM

Let  $C, D$  be plane curves defined by polynomials  $f$  and  $g$ , respectively. We are interested in the intersection  $C \cap D$ , i.e., the set of common points of  $C$  and  $D$ . So  $C \cap D$  consists of points  $p$  in  $\mathbb{C}^2$  such that  $f(p) = g(p) = 0$ . In general,  $C \cap D$  can be finite or infinite.

**Definition 2.1.** We say that  $C$  and  $D$  have *proper* intersection if their intersection is a finite set. If the intersection is not proper, we say  $C$  and  $D$  have *improper* intersection.

**Exercise 2.2.** In each of the following cases determine if the intersection of the two curves is proper or improper.

- (1)  $C = V(x)$  and  $D = V(y)$ .
- (2)  $C = V(xy)$  and  $D = V(y)$ .
- (3)  $C = V(y^2 - x^3)$  and  $D = V(x)$ .
- (4)  $C = V(y^2 - x^3)$  and  $D = V(y^2 - x^3 - x^2)$ .
- (5)  $C = V(y^2 - x(x-2)(x+1))$  and  $D = V(y^2 + x^2 - 2x)$ .
- (6)  $C = V(x^2 + y^2 + x^3 + y^3)$  and  $D = V(x^3 + y^3 - 2xy)$ .
- (7)  $C = V(y^5 - x(y^2 - x)^2)$  and  $D = V(y^4 + y^3 - x^2)$ .
- (8)  $C = V((x^2 + y^2)^2 + 3x^2y - y^3)$  and  $D = V((x^2 + y^2)^3 - 4x^2y^2)$ .

The following is a well-known classical result about plane curves.

**Theorem 2.3** (Bézout's theorem<sup>1</sup>). Let  $C, D$  be plane curves which intersect properly. Then

$$(\deg C)(\deg D) \geq \sum_{p \in C \cap D} (\text{mult}_p C)(\text{mult}_p D).$$

<sup>1</sup>The statement given here suffices for our purposes, but it is significantly weaker than the classical Bézout's theorem. The full and much stronger version requires the notion of *intersection multiplicity* of two plane curves.

**Exercise 2.4.** In each of the problems in Exercise 2.2 for which the intersection is proper, verify that Theorem 2.3 is true.

We now use Bézout's theorem to compute the value of  $A_r$  for small  $r$ .

**Exercise 2.5.** Show the following.

$$A_1 = A_2 = 1;$$

$$A_3 = 3/2;$$

$$A_4 = A_5 = 2;$$

$$A_6 = 12/5;$$

$$A_7 = 21/8;$$

$$A_8 = 48/17; \text{ and}$$

$$A_9 = 3.$$