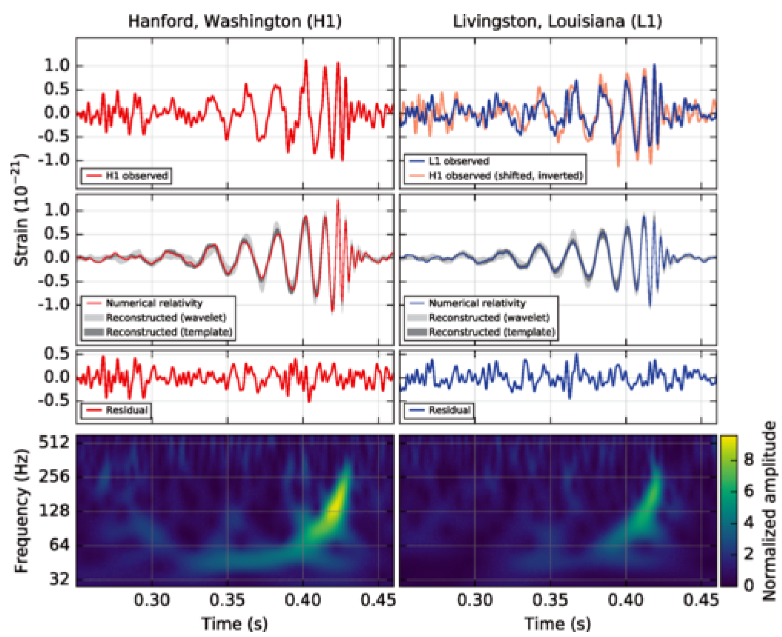
Numerical Relativity: Mathematical Formulation

Course Outline

Numerical relativity has seen dramatic progress in the last decade or two. Numerical simulations of binary black holes, for example, have basically become routine, and have provided theoretical predictions for the gravitational waveforms that were directly detected for the first time in 2015 – in fact, numerical relativity results were prominently featured in Fig. 1 of the discovery paper, reproduced below, highlighting their importance for these observations. Simultaneously, the field has advanced to model more complicated objects, including neutron stars and accretion disks and dynamical processes like tidal disruption and jet formation, often including magnetic fields, radiation transport, and other microphysical effects.

Even after taking a class on general relativity (GR), however, it is probably difficult to follow the literature in numerical relativity. This short course is intended as an introduction to the field, with the goal of developing some fluency in the language of numerical relativity. We will assume some basic knowledge of GR, but will remind ourselves of its most important objects and concepts. We will then discuss how Einstein's field equations of GR can be cast in a form that is suitable for numerical solution. Rather than presenting detailed proofs of every result, the emphasis will be on developing an intuitive understanding of many of the important objects. Towards that end, we will introduce the notion of a "3+1" decomposition between space and time in the more transparent context of electromagnetism before applying the same ideas to GR. We will motivate how such a split results in a set of constraint equations and a set of evolution equations. We will discuss how the constraint equations can be solved to find so-called initial data, and how these initial data can then be integrated forward in time with the help of the evolution equations. This process results in computer simulations of black holes and neutron stars, and allows us to predict the gravitational waves emitted by such exotic objects.



B. P. Abbott *et.al.* (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. **116**, 061102 (2016)

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A tentative outline

- Lecture I: Newton's versus Einstein's gravity

  Newtonian gravity and the gravitational potential, the metric, the covariant derivative, geodesic equation and deviation, the Riemann, Ricci, and Einstein tensors, Einstein's field equations, black holes and the Schwazschild spacetime

- Lectures II and III: The 3+1 decomposition

  Scalar fields, electromagnetism and Maxwell's equations, the Faraday tensor, 3+1 decompositions, the lapse function and shift vector, the extrinsic curvature, the spatial covariant derivative, the Lie derivative, constraint and evolution equations

- Lecture IV: Solving the constraint equations

  Conformal decompositions, elementary solutions to the Hamiltonian constraint, decompositions of the extrinsic curvature, Bowen-York solutions, puncture initial data for black holes

- Lecture V: Solving the evolution equations

  Reformulating Maxwell's and Einstein's equations, slicing and gauge conditions, the moving-puncture method, black hole simulations

Some References

My lectures will mostly be based on

- T. W. Baumgarte & S. L. Shapiro, *Numerical Relativity: Starting from Scratch*, Cambridge University Press

but I also recommend

- M. Alcubierre, *Introduction to 3+1 Numerical Relativity*, Oxford University Press
- T. W. Baumgarte & S. L. Shapiro, *Numerical Relativity: Solving Einstein's Equations on the Computer*, Cambridge University Press
- C. Bona, C. Palenzuela-Luque & C. Bona-Casas, *Elements of Numerical Relativity and Relativistic Hydrodynamics*, Springer
- E. Gourgoulhon, *3+1 Formalism in General Relativity*, Springer
- M. Shibata, *Numerical Relativity*, World Scientific Publishing

Some Conventions

- We will use *geometrized units*, in which $c = G = 1$
- Indices $a, b, c, \ldots, h$ and $o, p, q, \ldots$ run over spacetime indices, while $i, j, k, \ldots, n$ run over spatial indices only ("Fortran" convention)
- We use the Einstein summation convention, by which we sum over repeated indices
- The flat spacetime (or space) metric is denoted by η_{ab} (or η_{ij}) in *any* coordinate system. Only in Cartesian (inertial) coordinates do we have $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$.