DTP-Math-Circle: Session 4—Probability, Inequalities and Quantum Mechanics

Oct 14 2022

1 The Gabbar probabilities

To appreciate this problem, please see the iconic scene from the Hindi film Sholay: https://www.youtube.com/watch?v=chi9hsfYcDE

Note the following sequence of events:

- Gabbar takes a revolver from one of his henchmen. It has 6 bullets. The way a revolver works is the following. There is a cylinder, which has 6 compartments where bullets are loaded. These compartments are arranged symmetrically, one in each 60-degree sector of the cylinder. After every bullet is fired, the cylinder rotates by 60 degrees, so that the next compartment is ready to be fired from.
- He then shoots 3 bullets in air. So bullets have been fired from three consecutive compartments, and the remaining 3 compartments are loaded.
- He spins the cylinder fast and randomly, so that no one knows where the loaded compartments are.
- He shoots at his henchmen (call them A, B, and Kalia) one by one, to see if they survive (the compartment is empty) or die (the compartment is loaded). These are the three shots that we shall always refer to in this problem.
- It turns out that all three henchmen survive.
- Gabbar kills them anyway by firing the next three shots from the loaded compartments. These shots will not be a part of our problem.

Now let us calculate the probabilities related to the following:

(a) Before Gabbar shoots at A, what is the probability that A would survive? that B would survive ? that Kalia would survive ?

(b) Gabbar shoots at A, but A survives. Now what is the probability that B would survive ? that Kalia would survive ? Should Kalia ask Gabbar to shoot him before B?

(c) Gabbar shoots at B, but B survives. Now what is the probability that Kalia would survive ?

(d) If Gabbar had rotated the revolver at random after shooting A and B, what would be the probabilities of survival of A, B, and Kalia ?

2 A quantum revolver

There are many possible outcomes of Gabbar's shooting of the three henchmen with his bullets. All three survived (SSS), but all three may have died (DDD).

(a) What are the other possible outcomes ? Are there any outcomes that are not possible at all ?

The outcomes that do not seem possible (by common sense) can be possible if we have a "quantum revolver". This means that, once a bullet is fired from a compartment, the remaining bullets distribute themselves in all remaining compartments so that the probability of any of the compartments having a bullet is equal. This looks strange, but such situations are possible with quantum mechanics.

(b) With a quantum revolver as described above (such revolver has not been constructed yet as far as we know), what are the probabilities of survival of A, B, and Kalia ? If Kalia, the seniormost henchman, is given a choice of his position in the execution queue, which position should he choose to maximize his probability of survival?

Samba is an external observer whose life is not in danger, so he can indulge in other activities like calculating probabilities. He denotes a "hit" (dead henchman) by 1 and a "survival" by 0. So possible outcomes are 000, 001, etc.

(c) What are the "classical" probabilities of these 8 outcomes ?

(d) What are the "quantum" probabilities of these 8 outcomes ?

(e) Samba wants to construct a function of the outcomes of the three shots such that it gives 1 for a possible "classical" outcome (without quantum mechanics), and 0 for an outcome with quantum mechanics. Help him construct this function using Boolean operators AND, OR, NOT (There can be multiple answers.)

3 Boole's inequality

Consider a group of children made up of your schoolmates. Let the probability of being good in sports be $P(S)$ where S denotes the set of children who are good in sports, and let the probability of being good in music be $P(M)$, where M denotes the set of children who are good at music. Then what does $P(S \bigcup M)$ stand for? Can you say something about its value relative to the sum of probabilities $P(S) + P(M)$?

Now if we have the probabilities $P(A_i)$ for $i = 1, 2, 3...$, can you say something about the value of $P(A_1 \bigcup A_2 ... \bigcup A_n)$ relative to the sum $P(A_1)+P(A_2)+ ...P(A_n)$?

4 Random splashes of red on a blue ball

A blue ball gets splashed randomly with red color so that 10% of its surface gets colored red. Your friend challenges you to find a position for an inscribed cube (inscribed here means that all vertices of the cube lie on the surface of the ball) such that each vertex of that cube lies in a blue patch. Should you take the challenge?

What if the random splashing of red ends up coloring 15% of the surface red?

Now, what if your friend challenges you to inscribe a regular tetrahedron with the same condition on the position of the vertices? Up to what red fraction would you be comfortable taking up the challenge?

5 Double-slit experiment

In the world of electrons and atoms, which obey quantum mechanics, probabilities for different alternatives cannot always simply be added. And inequalities that are provably true in our usual mathematical theory of probability can end up being false. We will now try to give a very rough illustration of such strange behaviour.

By way of warm-up, first consider a situation in which Gabbar has an electron gun with many bullets, and he shoots a large number of bullets towards a pair of vertical slit-like opening in a wall. These openings are close enough to each other that when Gabbar shoots, he is equally likely to hit either of them. Behind this wall is a thick wooden barrier that stops the bullets that strike it. The embedded bullets make a pattern. This pattern gives us an idea of what the probability of an electron hitting any given area of this barrier is.

(a) Initially block the first slit and keep the second open. What would the pattern of embedded bullets look like?

    Now block the second slit and keep the first one open. How does the pattern change?

(b) What would one expect to happen if both the slits are open? See

    https://www.youtube.com/watch?v=ZqS8Jjkk1HI

    to see what actually happens.

6 Leggett-Garg inequality

You find a gadget which has three displays and a button. When you press the button, you see that the first display lights up and shows a number, either +1 or −1, then after a little while, the second lights up and shows another number, again either +1 or −1, and finally after another pause, the third display does the same thing. And after all three numbers are displayed, the gadget announces that it has wiped its memory clean and reset itself to its original state. You are now free to press the button again. Each time you press the button, a fresh sequence of three numbers seems to appear in the three displays.

Let us call the three numbers that appear in the three displays A, B, and C respectively. You repeat this "experiment" of pressing the button a large number of times, say N. And you measure

$$E_{AB} = \frac{1}{N}\sum_{i=1}^{N} A_i B_i$$

where A_i and B_i are the numbers that appear in the first and second display in the i^{th} run of this "experiment". And we measure E_{BC} and E_{AC} similarly.

Having measured these quantities, you look at the combination

$$K = E_{AB} + E_{BC} - E_{AC}$$

Since there doesn't seem to be a fixed outcome, you start thinking in terms of a probability $P(A, B, C)$ for getting each possible sequence of numbers A, B, C. In terms of $P(A, B, C)$, what is your prediction for E_{AB}, E_{BC}, and E_{AC} when the number of measurements N you make is very large?

Using this, can you place bounds on the range of K?

7 Quantum mechanics

The bounds on K deduced by you in the previous problem are true when the box behaves in a way that obeys the familiar rules of mathematical probability. But in the world of electrons and atoms, these rules are not obeyed, and seemingly impossible things can happen.

This year's Nobel Prize in Physics was actually given to scientists who carefully measured such "impossible" effects.

See https://www.nobelprize.org/prizes/physics/

Closer home, "impossible" effects (that violate the predicted range of a quantity broadly similar to K defined above) were also measured in Bengaluru recently.

See https://www.indiatimes.com/technology/science-and-future/experiment-by-indian-scientist-disproves-einsteins-understanding-of-reality-560623.html

To truly appreciate this, you can choose to learn Physics in college even if your first love is Math!