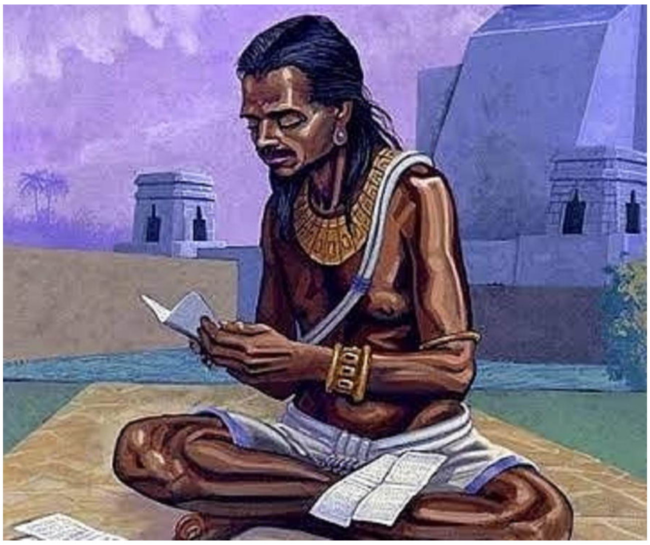
MCI QUESTIONS FOR AUGUST 19, 2022

IISER Bhopal Module

1. Brahmagupta's Triples



(Image courtesy: http : //www.thefamouspeople.com/profiles/brahmagupta − 6842.php)

Brahmagupta was an Indian mathematician born in 598 CE who made significant contributions to mathematics. His book *Brahma Sphuta Siddhanta* is probably the first ancient Indian work in which an entire chapter is devoted to algebra.

Brahmagupta was interested in finding integer solutions (in x and y) equations of the form

$$x^2 - Ny^2 = 1$$

where N is a natural number which is not a perfect square. In trying to solve this problem, he studied the following more general problem of finding solutions of the equation

$$x^2 - Ny^2 = m \tag{1.1}$$

where m is an integer.

Notation 1. *Suppose that integers x_1, y_1 and m_1 are solutions of the equation (1.1). This means that the following equation holds:*

$$x_1^2 - Ny_1^2 = m_1.$$

We denote this solution as a triple $(x_1, y_1; m_1)$. Such triples will be called **Brahmagupta triples** *corresponding to (1.1).*

Example 1.1. *Let $N = 2$ and consider the equation $x^2 - 2y^2 = m$. Then $(3, 2, 1)$ is a Brahmagupta triple of (1.1) since*

$$3^2 - 2(2^2) = 9 - 8 = 1.$$

You can also check that $(17, 12, 1)$ is a Brahmagupta triple.

Problem 1.2. *In this problem, you will learn how to combine two Brahmagupta triples and get a new Brahmagupta triple corresponding to (1.1). Given two triples $(x_1, y_1; m_1)$ and $(x_2, y_2; m_2)$, define $(x_1, y_1; m_1) \odot (x_2, y_2; m_2)$ by*

$$(x_1, y_1; m_1) \odot (x_2, y_2; m_2) := (Ny_1y_2 + x_1x_2, y_1x_2 + y_2x_1; m_1m_2). \tag{1.2}$$

Show that $(x_1, y_1; m_1) \odot (x_2, y_2; m_2)$ as defined above is a Brahmagupta triple for (1.1).

Remark 1.3. *Note that* $(Ny_1y_2 - x_1x_2, y_1x_2 - y_2x_1; m_1m_2)$ *is also a Brahmagupta triple.*

Problem 1.4. *Is the statement of Problem 1.2 true if we take triples of rational numbers?*

Remark 1.5. *If we fix* $m = 1$*, that is, we want to find solutions of the equation*

$$x^2 - Ny^2 = 1,$$

then Problem 1.2 suggest that we can combine two solutions to generate a new solution of the same equation

Problem 1.6. *Consider the equation*

$$x^2 - 92y^2 = 1.$$

We can see that $(1, 0)$ *and* $(-1, 0)$ *are clearly solutions. Can you find any other solutions?*

(*Hint: Notice that* $10^2 - 92 = 8$*. Now, try to use the previous problems.*)

Problem 1.7.
(1) *Can we always find Brahmagupta triples for the equation (1.1) for any choice of* N*? If not, can you find some counterexamples?*
(2) *If for a given* N*, there do exist Brahmagupta triples, how many such triples are there? Can we find a method for generating all such triples?*

The above questions are contributed by Ajit

1 Colouring Maps

Consider a map M of a country which is divided into states. A *proper colouring* of M is an assignment of colours to the states in such a way that two states that share a boundary have distinct colours.

We want to determine the *minimum* number of colours needed to do this.

1.1 Give a proper colouring of the map of India

Figure 1: How many colours do you need?

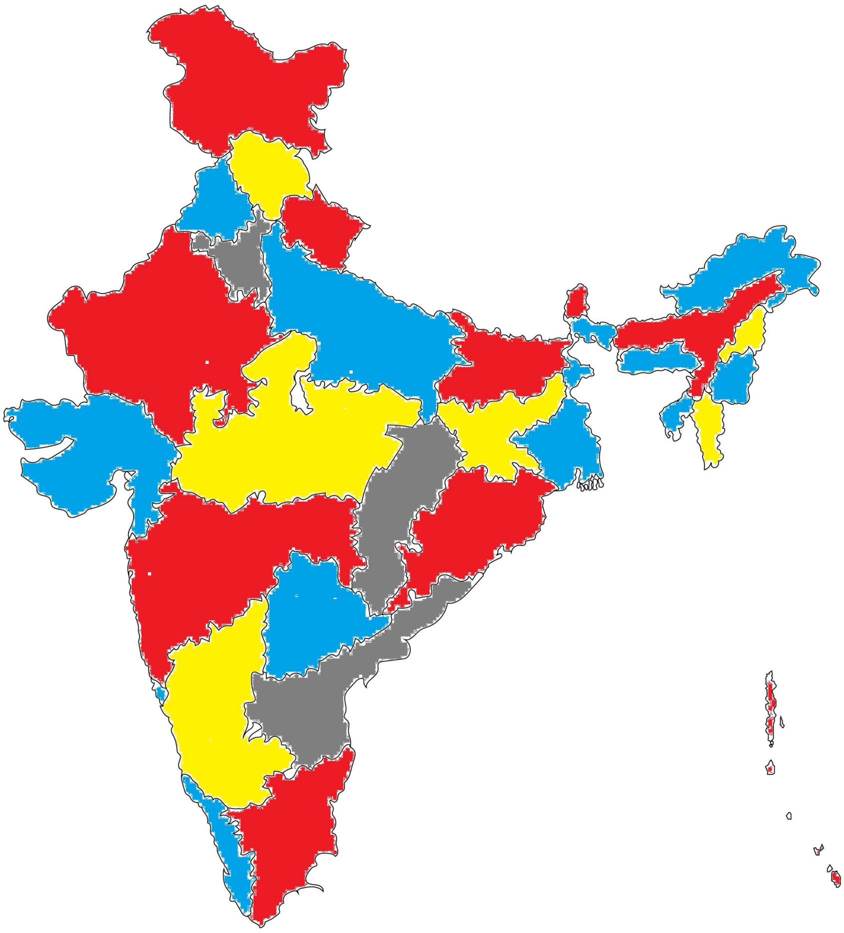


Figure 2: I needed four colours. Can you do it with less?

To find a proper colouring of the maps of other countries, try https://mathigon.org/course/graph-theory/map-colouring.

1.2 The graph of a map

Before we return to the question of map colouring, let us convert the map into something more mathematical. Consider the following recipe:

1. For each state, place a dot on the paper.

2. Draw a line/curve between two dots if the two states share a boundary.

What you will have drawn is a graph! Recall that a graph is a pair $G = (V, E)$ consisting of vertices and edges. Try this for South America.

Figure 3: Draw the graph of South America.

Definition 1.1. The *degree* of a vertex p is the number of edges coming out of p. It is denoted by $\deg(p)$.

1.3 For the graph of South America, identify the degree of each vertex.

Given these degrees, answer the following questions:

1. Just by knowing the number of vertices, what is the maximum number of colours you could possibly need?
2. Just by knowing the degrees, what is the maximum number of colours you could possibly need?
3. Do you need that many colours or could you do it with less? For instance, can you ignore the degree of Brazil?

1.4 For any graph, prove that $\sum_{p \in V} \deg(p) = 2e$.

Note: The graph of a map is *planar* in the sense that it can be drawn on a paper and the edges do not intersect each other. Moreover, if we fix one land mass, the graph is *connected*; i.e. it is in one piece.

Recall a fact that we proved last time: For a connected planar graph,

$$e \leq 3v - 6.$$

Here e = the number of edges, and v = the number of vertices.

1.5 If G is a connected planar graph, then prove that there is a vertex p such that $\deg(p) \leq 5$.

1.6 Prove that every map has a proper colouring with 6 colours.

Hint: Use induction on the number of vertices.

Note: This result in Question 1.6 is called the 'Six Colour Theorem'. Oe can use a similar argument to prove a 'Five Colour Theorem' (every map has a proper colouring with five colours). See `https://sites.math.rutgers.edu/~sk1233/courses/graphtheory-F11/planar.pdf`.

Indeed, there is also a Four Colour Theorem. This was raised as a question in 1852 and was finally proved using a 'computer-aided proof' by Appel and Haken in 1976. It was the first major result that was proved this way. To date, there is no proof that does not use a computer.

The above questions are contributed by Prahlad

COUNTING NUMBERS

Recall the following system of numbers:

(i) The numbers $1, 2, 3, \dots$ are called *positive counting numbers*. The positive counting numbers will be denoted by the symbol $\mathbb{N}$.

(ii) The numbers $-1, -2, -3, \dots$ are known as *negative counting numbers*.

(iii) The collection of all positive and negative counting numbers including 0 are known as *integers* and will be denoted by the symbol $\mathbb{Z}$.

(iv) The numbers of the form $\frac{m}{n}$, where m and n are integers such that n is not equal to 0 are known as *fractional numbers*. The collection of all fractional numbers will be denoted by the symbol $\mathbb{Q}$.

Recall that fractional numbers can also be described by their decimal expansion. For example, $\frac{1}{2} = 0.5$ and $\frac{1}{3} = 0.333\dots$.

A decimal expansion is called

(i) *terminating* if there are finitely many digits after the decimal place.

(ii) *non-terminating recurring* if there are infinitely many digits after decimal place with a repetition in equal intervals.

(iii) *non-terminating non-recurring*, otherwise.

The decimal expansion of a fractional number is either terminating or non-terminating recurring. There are numbers having their decimal expansion non-terminating non-recurring. For example, $\pi = 3.14159265358979323\dots$, $e = 2.71828182845904$, and many more. The numbers having their decimal expansion non-terminating non-recurring are called *non-fractional* numbers. The collection of all fractional and non-fractional numbers are called *real numbers* and will be denoted by $\mathbb{R}$.

Consider the empty boxes labeled with positive counting numbers. In particular, we fix the labeling as shown in Figure 1 (The dots in Figure says that if you are at the box with label n, then there always exists a box with label $n + 1$). Such boxes will be called the *counting boxes*.

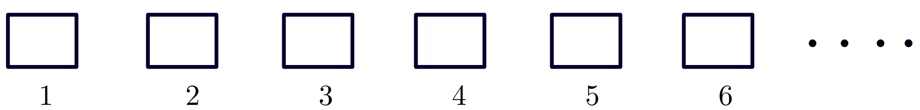


FIGURE 1. The counting boxes.

Let S be a collection of numbers. We say that

(i) S has *size* equals to a positive counting number n, denoted by $|S| = n$, if every number of S can be filled in an unique counting box beginning with the box labeled by 1 and ending with the box labeled by n (rest of them being empty).

(ii) S has the same *size* as that of $\mathbb{N}$, denoted by $|S| = |\mathbb{N}|$, if every number of S can be filled in an unique counting box.

Example 1. Consider the collection of numbers $S = \{5, -3, e, \frac{1}{2}, \pi\}$. Figure 2 shows that $|S| = 5$.

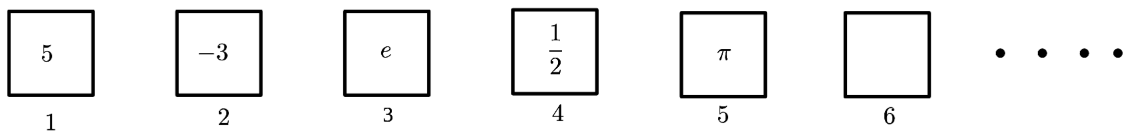


FIGURE 2. An illustration of $|S| = 5$.

Example 2. Let S be the collection of all even positive counting numbers, that is, $S = \{2, 4, 6, \dots\}$. From Figure 3, convince yourself that $|S| = |\mathbb{N}|$.

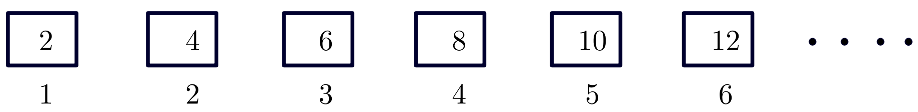


FIGURE 3. An illustration of $|S| = |\mathbb{N}|$.

In Example 1, if we take a sub-collection $S' = \{5, e, \pi\}$ of $S = \{5, -3, e, \frac{1}{2}, \pi\}$, then $|S'| = 3$. Therefore, $|S| \neq |S'|$. Now, consider Example 2. The collection of all even positive counting numbers S is not equal to the collection of all counting numbers $\mathbb{N}$. But, we have $|S| = |\mathbb{N}|$.

Example 3. Let S be the collection of all odd positive counting numbers, that is, $S = \{1, 3, 5, \dots\}$. From Figure 4, convince yourself that $|S| = |\mathbb{N}|$.

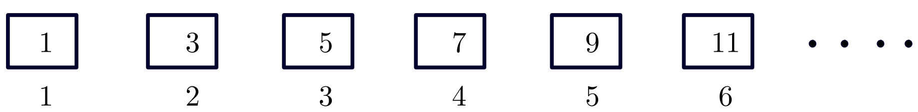


FIGURE 4. An illustration of $|S| = |\mathbb{N}|$.

Problem 1. Show that $|\mathbb{Z}| = |\mathbb{N}|$.

Problem 2. Show that $|\mathbb{Q}| = |\mathbb{N}|$. (*hint*: Any fractional number $\frac{m}{n}$ can be treated as a pair of integers (m, n) in the euclidean plane (see Figure 5).)

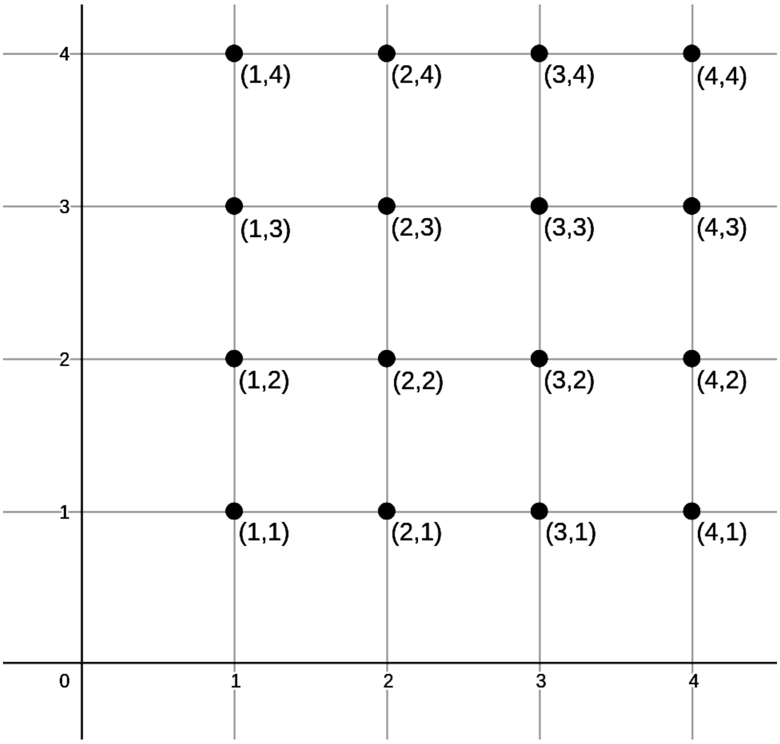


FIGURE 5. Fractional numbers as pair of integers in the euclidean plane.

Problem 3. Show that $|\mathbb{R}| \neq |\mathbb{N}|$.

Hint:

(i) Instead of $\mathbb{R}$, consider real numbers between 0 and 1.

(ii) Assume that a filling of these numbers in the counting boxes is possible (see Figure 6).

(iii) Construct a decimal expansion $0.a_1a_2a_3\ldots$, where a_i are digits between 0 and 9, as follows:

- (a) Choose the first digit a_1 which is different than the first digit after the decimal point of the number in the counting box labeled by 1.
- (b) Now, choose the second digit a_2 which is different than the second digit after the decimal point of the number in the counting box labeled by 2.
- (c) Repeat this process for each successive digits of the decimal expansion $0.a_1a_2a_3\ldots$.

(iv) Now, observe the number constructed above. Is this number belongs to any counting boxes? What can be concluded from this?

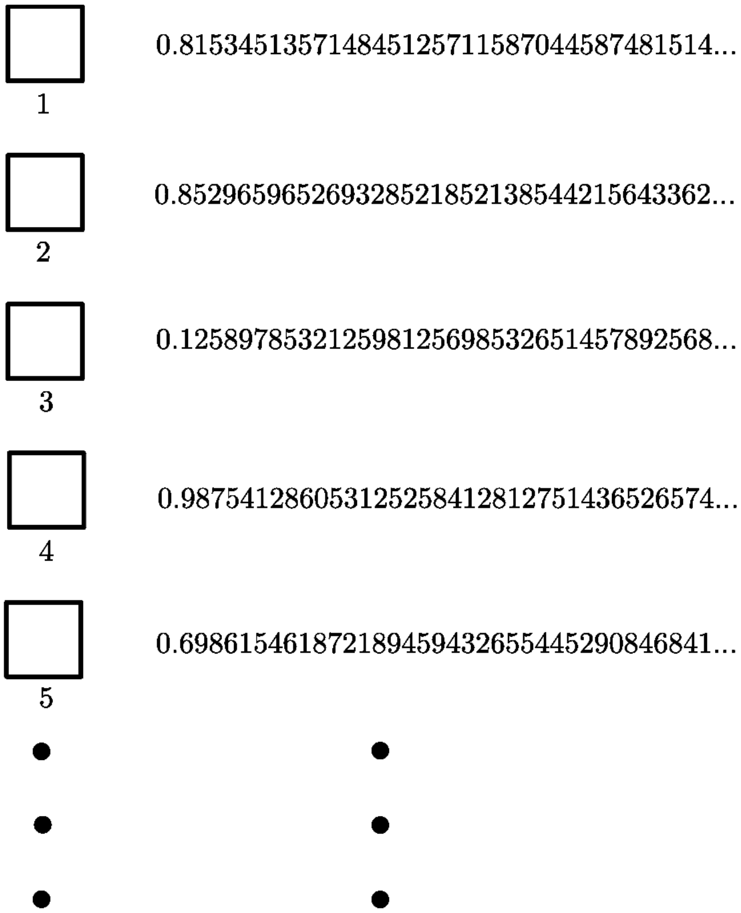


FIGURE 6. A filling of real numbers in counting boxes.