Math Circles of India

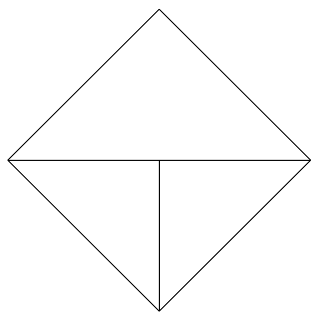
IISER Bhopal Module

22 July 2022

1 Maps and Graphs

You're the king, and you want to divide your kingdom among your children. Moreover, you want to do it in such a way that each state should share some boundary with each other state (and the shared boundary should be more than a point).

Example 1.1. If you had three children, this is what you could do.



1.1 Can you do this with four children?

1.2 Can you do the same with five children?

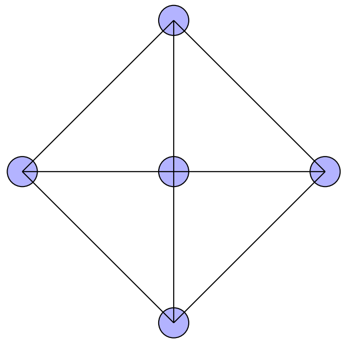
Hint: You can't.

To see why, think of each region as a point on the plane, and each relationship (sharing a boundary) is depicted by a line joining these points. What should this picture look like to answer your question?

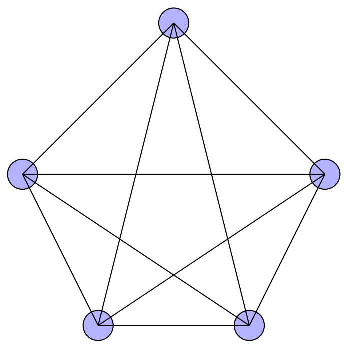
We need a little theory before we start.

Definition 1.2. A *graph* $G = (V, E)$ consists of two sets; a finite set V of vertices, and a set E of edges. Vertices are represented by points, and edges are represented by line segments (or even curves) connecting these points.

Example 1.3. Given below is a graph representing Question 1.1. This is called a *complete* graph with four vertices (every vertex is connected to every other vertex). It is denoted by K_4.



Example 1.4. Given below is the complete graph with five vertices, K_5.

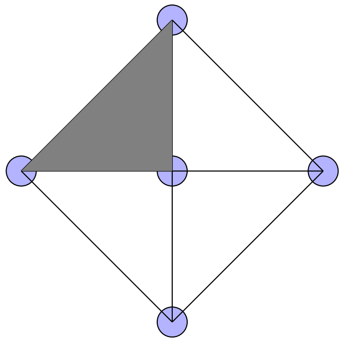


Definition 1.5. A graph G is *planar* if it can be represented on the plane $\mathbb{R}^2$ in such a way that no two edges intersect.

In our examples, K_4 is planar, and we will try to show that K_5 is not (this answers Question 1.2). We now want to determine if a graph is planar or not.

Definition 1.6. Given a planar graph G expressed as a subset of $\mathbb{R}^2$, a *face* of G is the region enclosed by a family of edges, which cannot be further subdivided.

Example 1.7. A face marked out in K_4 is shown below.



Note that

- The set of faces may change depending on how G is represented as a subset of $\mathbb{R}^2$.
- Every graph has at least one face, its *exterior* (If a graph has only one that one face, it is called a *tree*).
- If G is not a tree, then it must have a *cycle*, a list of edges which start and end at the same point.

Definition 1.8. A graph is said to be *connected* if it is in one piece, i.e. every vertex is connected by a *path* to every other vertex.

Let $G = (V, E)$ be a connected planar graph, thought of as a subset of $\mathbb{R}^2$. Write $v =$ the number of vertices, $e =$ number of edges and $f =$ the number of faces in G.

1.3 For a connected planar graph, prove that $v - e + f = 2$.

Hint: First prove this when G is a tree. Now if G has a cycle, then determine what happens to the number of faces if we delete one edge of the cycle.

Definition 1.9. A planar graph G is *maximal* every face has exactly three edges surrounding it.

1.4 For a maximal planar graph with $v \geq 3$, prove that $3f = 2e$.

Hint: Find the perimeter of each face (the number of the edges enclosing it).

1.5 For a connected planar graph, prove that $e \leq 3v - 6$.

Hint: First consider the case when G is maximal planar.

1.6 Now can you answer Question 1.2?

2 Colouring Maps

Consider a map of a country which is divided into states. For instance, you can use the map of India as an example.

Definition 2.1. Given a map M, a *proper colouring* of M is an assignment of colours to the states in such a way that two states that share a boundary have distinct colours.

Naturally, if M has n states, then we may use n colours. However, we want to determine the *minimum* number of colours needed to do this. This is called the *chromatic number* of M.

Example 2.2. Let us try this for the map of India.

Figure 1: How many colours do you need?

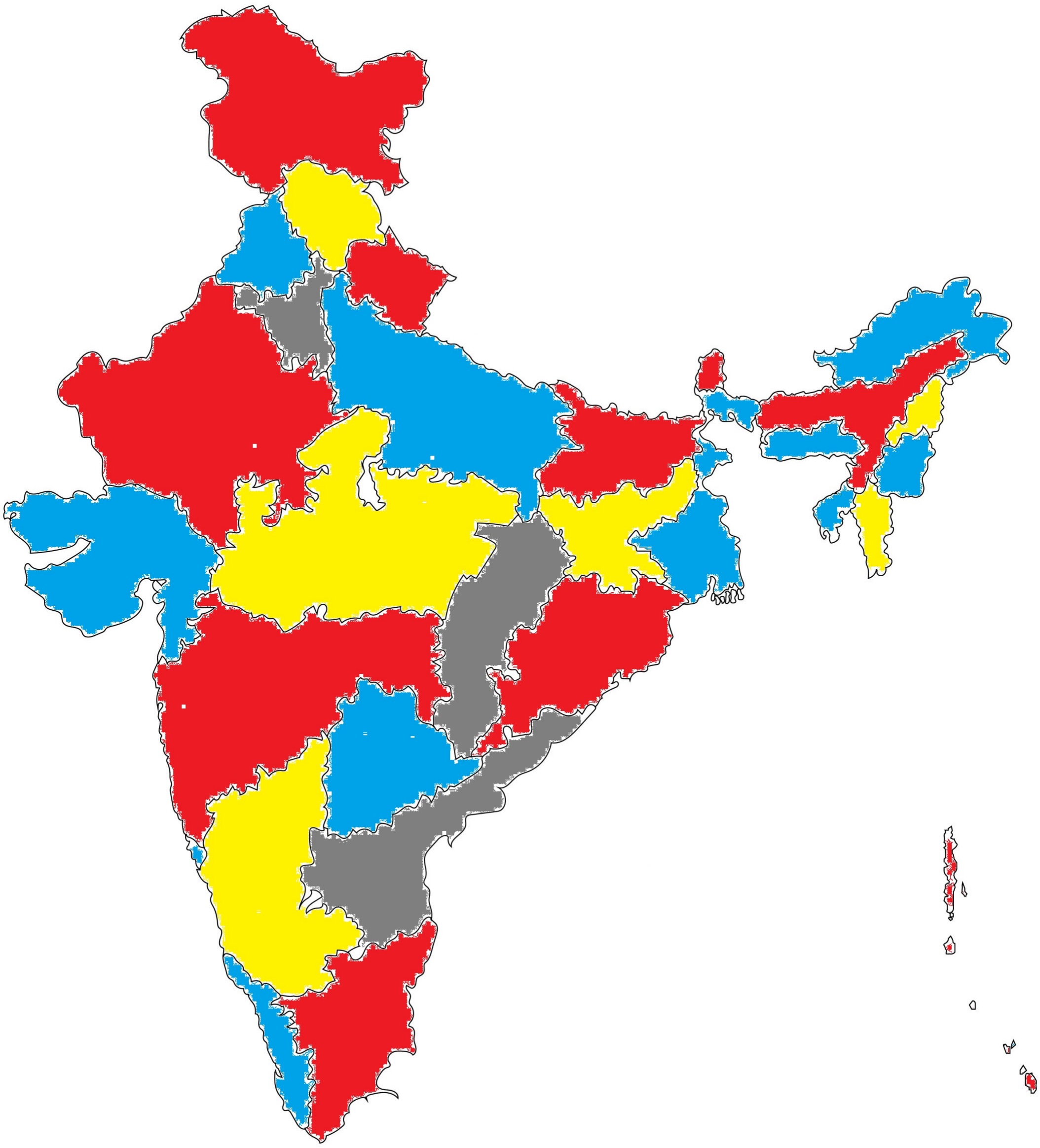


Figure 2: You need four colours. Can you do it with less?

Try this later: To find a proper colouring of the maps of other countries, try https://mathigon.org/course/graph-theory/map-colouring.

Before we return to the question of map colouring, let us examine graphs some more.

Definition 2.3. Two vertices of a graph are said to be *adjacent* if there is an edge connecting them. The *degree* of a vertex v is the number of vertices adjacent to v. It is denoted by $\deg(v)$.

2.1 For any graph, prove that $\sum_{p \in V} \deg(p) = 2e$.

2.2 If G is a connected planar graph, then prove that there is a vertex p such that $\deg(p) \leq 5$.

Hint: Use Question 1.5.

2.3 Prove that every map has a proper colouring with 6 colours.

Hint: Use Question 2.2, and induct on the number of vertices.

Note: This result in Question 2.3 is called the 'Six Colour Theorem'. In fact, one can use a similar argument to prove a 'Five Colour Theorem' (every map has a proper colouring with five colours). For the details, have a look at https://sites.math.rutgers.edu/~sk1233/courses/graphtheory-F11/planar.pdf.

Indeed, there is also a Four Colour Theorem. This was raised as a question in 1852 and was finally proved using a 'computer-aided proof' by Appel and Haken in 1976. It was the first major result that was proved this way. To date, there is no proof that does not use a computer.

3 Reduce it to the previous one!

1. (a) Let's start with constructing the following polygons.
      i. A triangle with three acute angles.
      ii. A quadrilateral with three acute angles.
      iii. A pentagon with three acute angles.
      iv. A hexagon with three acute angles.

   Using the above constructions try finding a pattern and check if the following statement is true or false:
   For any natural number n that is bigger than 3, one can construct a

convex n-gon with exactly three acute angles.

(b) Can you use a similar method of argument as above to prove that *the number* $n^2 + n + 41$ *is a prime for any natural number* n?

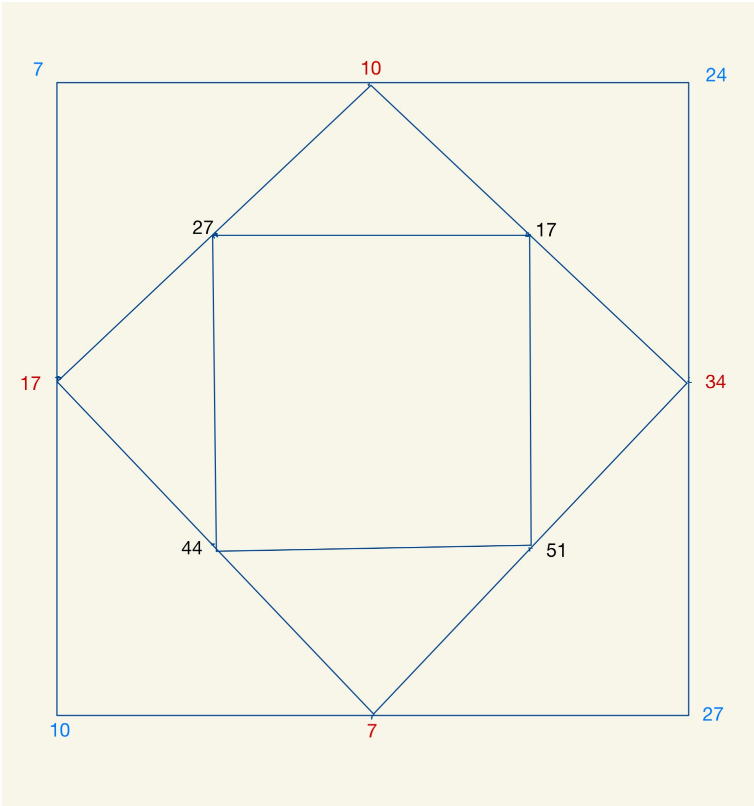
4 Fold and Unfold

1. Imagine that a ladybug sitting on one vertex of a wooden cube is trying to get to the opposite vertex. Can you describe the shortest path that it can follow?

2. Assume that a fly sitting on the outer surface of a cylindrical drinking glass crawls to another point on the inner surface of the glass. Can you find the shortest route possible for the fly? (Please disregard the thickness of the glass.)

5 Sequences and Patterns

1. Start with any four non-negative integers arranged in a given cycle. In the next step, generate a new cycle of four non-negative integers by taking differences of consecutive pairs, always subtracting the smaller number from the larger (or equal) number. For example, if the cycle that we start with is $\{27, 17, 51, 44\}$, then the first iteration yields the following:
   $\{(27-17), (51-17), (51-44), (44-27)\} = \{10, 34, 7, 17\}$.
   Continuing in this manner, we can check that the next iteration gives us the cycle $\{24, 27, 10, 7\}$. These iterations are pictorially depicted in the following diagram.



Starting with a cycle of any four non-negative integers, if you repeat this procedure enough number of times, will you always end up with a cycle with four equal numbers?

2. Anwesha likes to play with numbers and generate patterns. During the summer vacation, she developed a simple procedure for generating a sequence of numbers, starting with a given number. Each number was generated by calculating the sum of squares of the digits of the previous number. More specifically, she started her investigation with the number 76. In the first iteration, summing the squares of digits gave her $7^2 + 6^2 = 49 + 36 = 85$. Next, she repeated the same process with the number 85. She then got $8^2 + 5^2 = 64 + 25 = 89$. Applying the same procedure to 89, she got $8^2 + 9^2 = 145$ in the next iteration. Anwesha thus had the following sequence:

$$76 \rightarrow 85 \rightarrow 89 \rightarrow 145 \rightarrow \dots$$

(a) If you start with any two digit natural number and build the sequence according to Anwesha's procedure, can you say what happen eventually? Does the sequence eventually stop at a certain number?After experimenting with several two-digit natural numbers, Anwesha came to the conclusion that there are only two possibilities. Can you figure out what they are?

(b) What happens if you start with a three digit number and repeat the same process? Do you get the same set of possibilities? What about higher number of digits?

The problems for this session are contributed by Ajit, Atreyee and Prahlad.