Math Circles of India

IISER Bhopal Module

8 July 2022

Who Moved My Cheese?

A mouse named Ashwini wants to steal some cheese and run back home as quickly as possible. She needs to run along the paths between some obstacles, but must always choose the shortest possible route (as shown below).

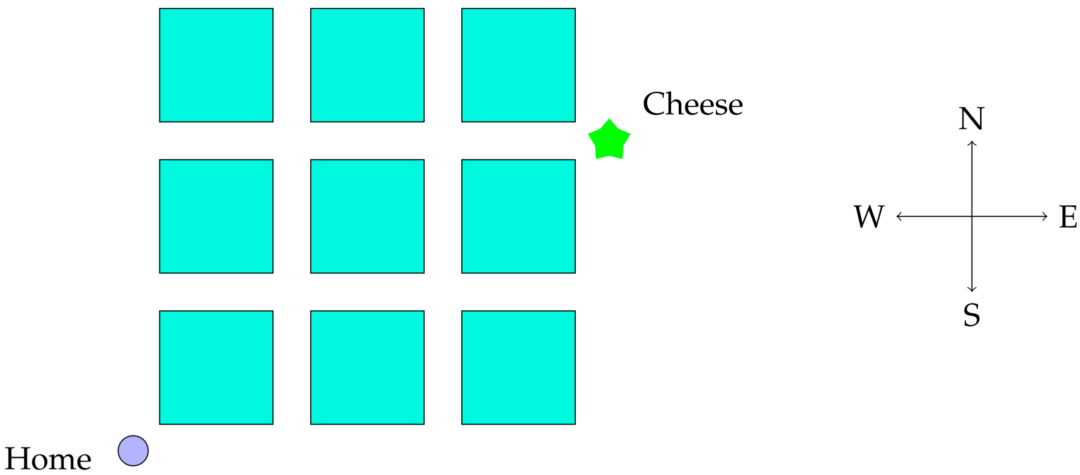


Figure 1: Ashwini Moved my Cheese

There are, however, many possible 'shortest' routes (we think of each block as a step of one unit in that direction).

Example: Two examples are shown below:

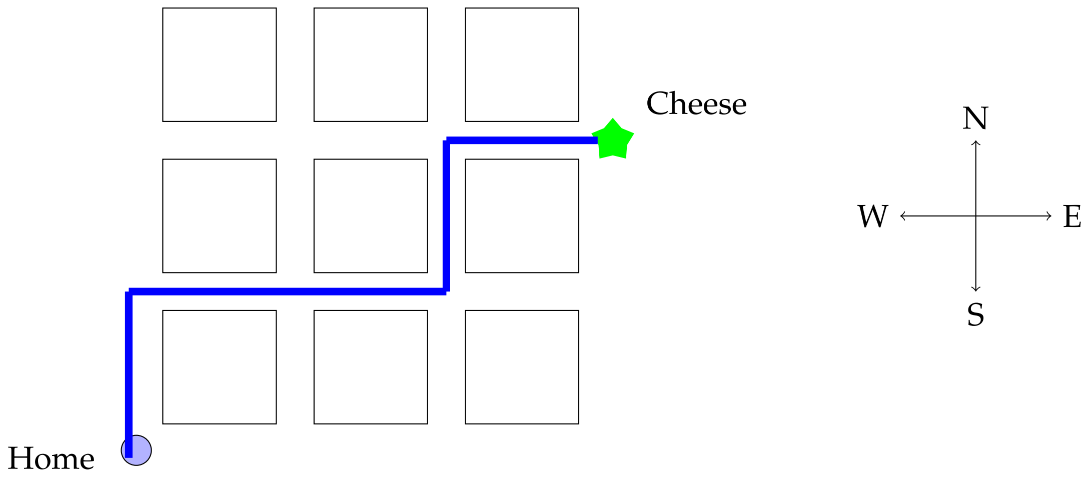


Figure 2: One possible route for Ashwini.

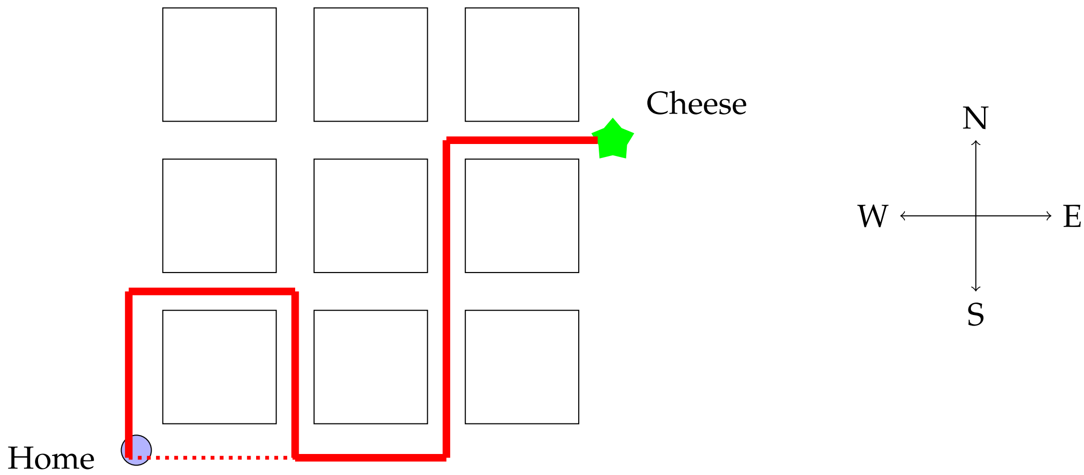


Figure 3: A route Ashwini would not take.

0.1 Question: Determine the number of such routes

Hint: Use the directions shown in the picture.

1 Ashwini's Friends

The next day, Ashwini finds more cheese in a new place. Since there is a lot of it, Ashwini decides to share it with her friends. Now, seven mice (Ashwini, Bhanu, Cyrus, Dhruva, Elango, Fatima, Gaurav) must steal pieces of the cheese and find their way back to their respective homes through a new maze. Once again, each one takes the shortest route possible.

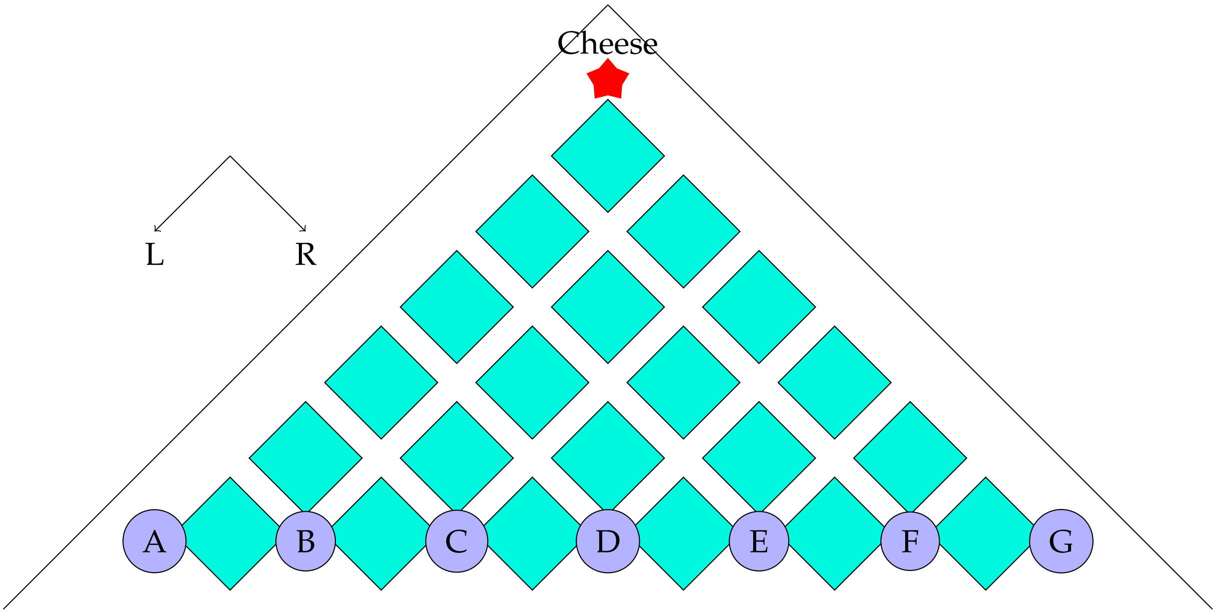


Figure 4: Shortest Route for each of the mice.

Example: Two examples are shown below:

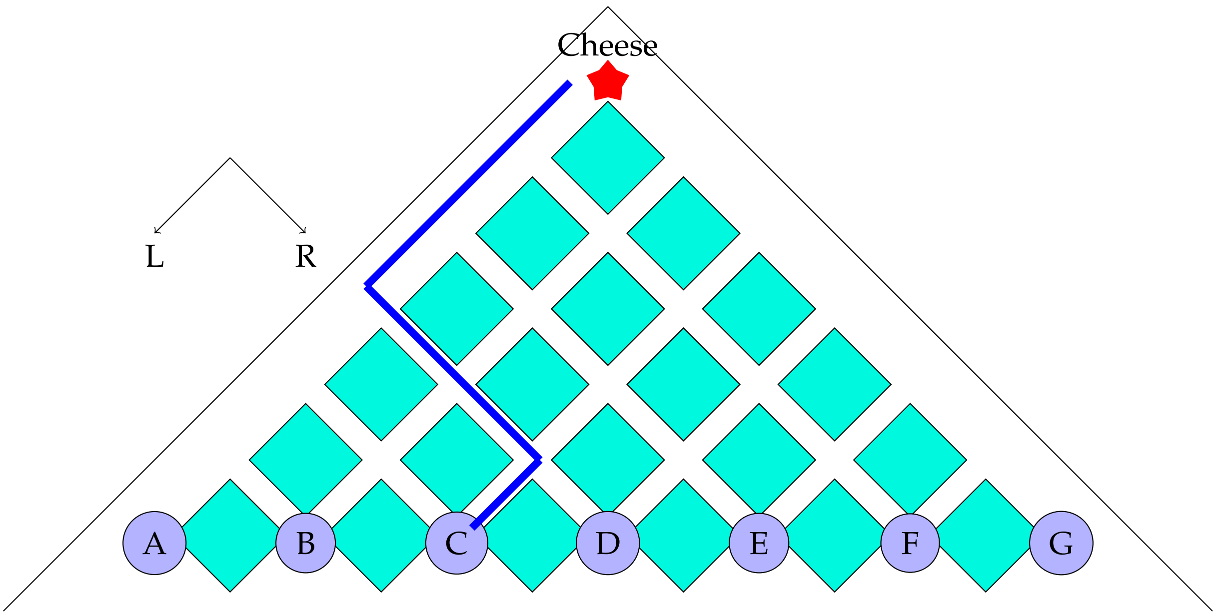


Figure 5: A route that Cyrus could take

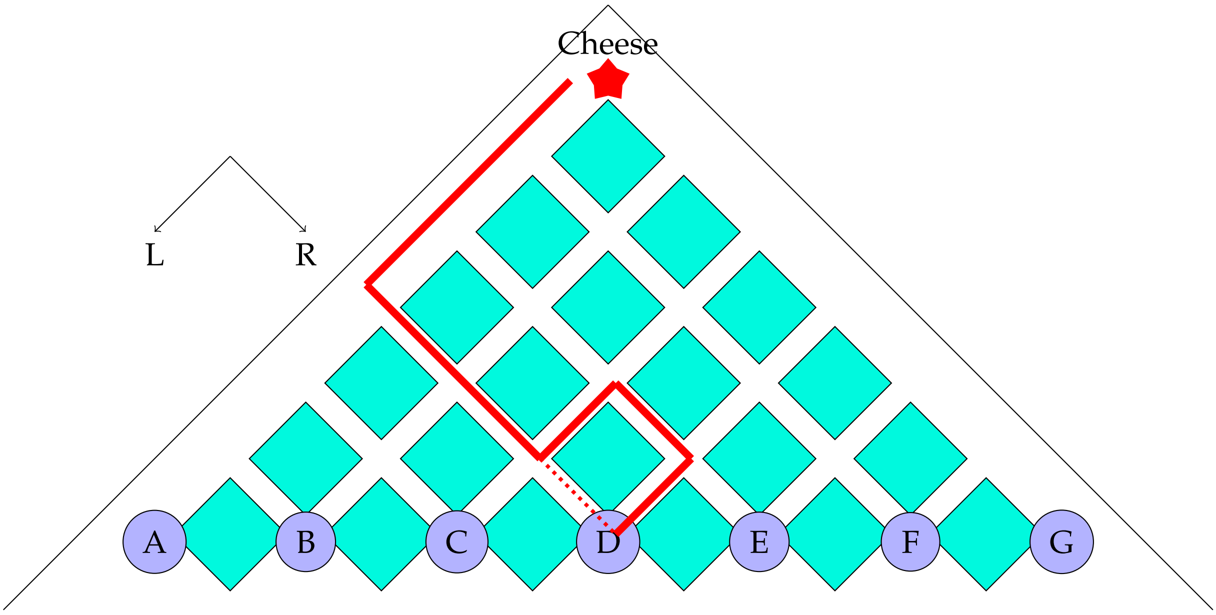


Figure 6: A route that Dhruva would not take

1.1 Question: How many such shortest routes does Dhruva have?

Hint: Look at the directions again!

1.2 Question: For each mouse, determine the number of such routes.

Before we proceed, fill in the each circle of Figure 7 with the number of such routes.

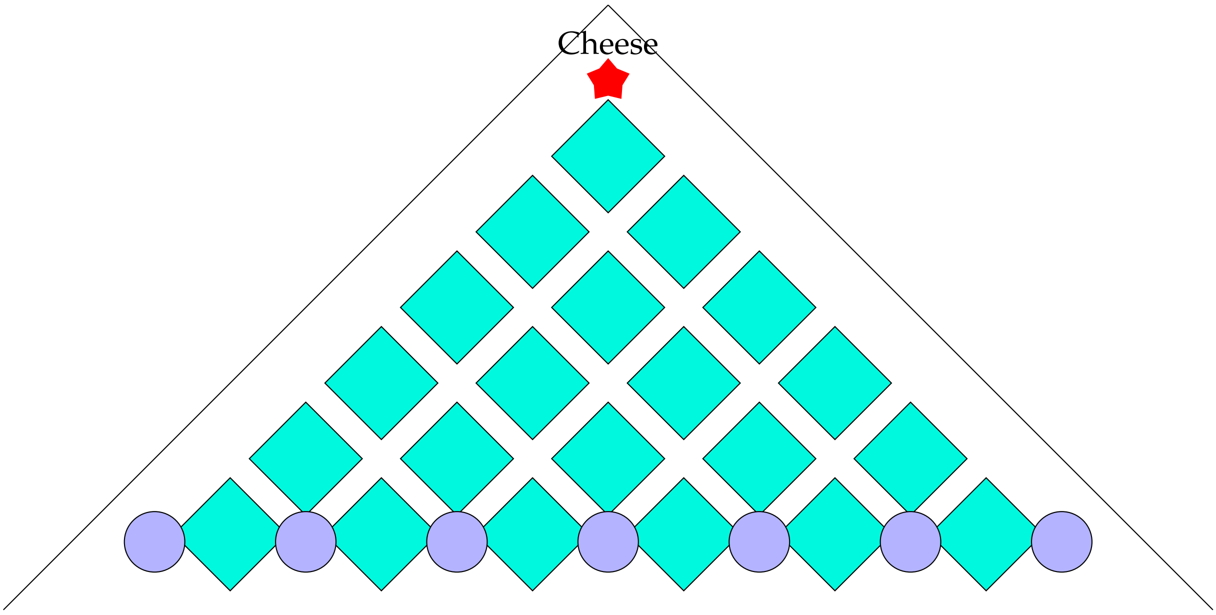


Figure 7: Fill in the number of shortest routes for each mouse

1.3 Question: Do you notice any patterns?

2 More Mice!

Let us add more mice to the problem (because, why not?). Suppose there are six more mice (Hari, Ibrahim, John, Kabir, Lalit and Meena), attempting the same robbery. The homes of these new mice, however, are closer to the prize! (See Figure 8).

2.1 Question: Compute the number of shortest routes for each of the new mice as well.

Using the data from Figure 7, let us now complete the triangle in Figure 9 with these numbers placed at each corner.

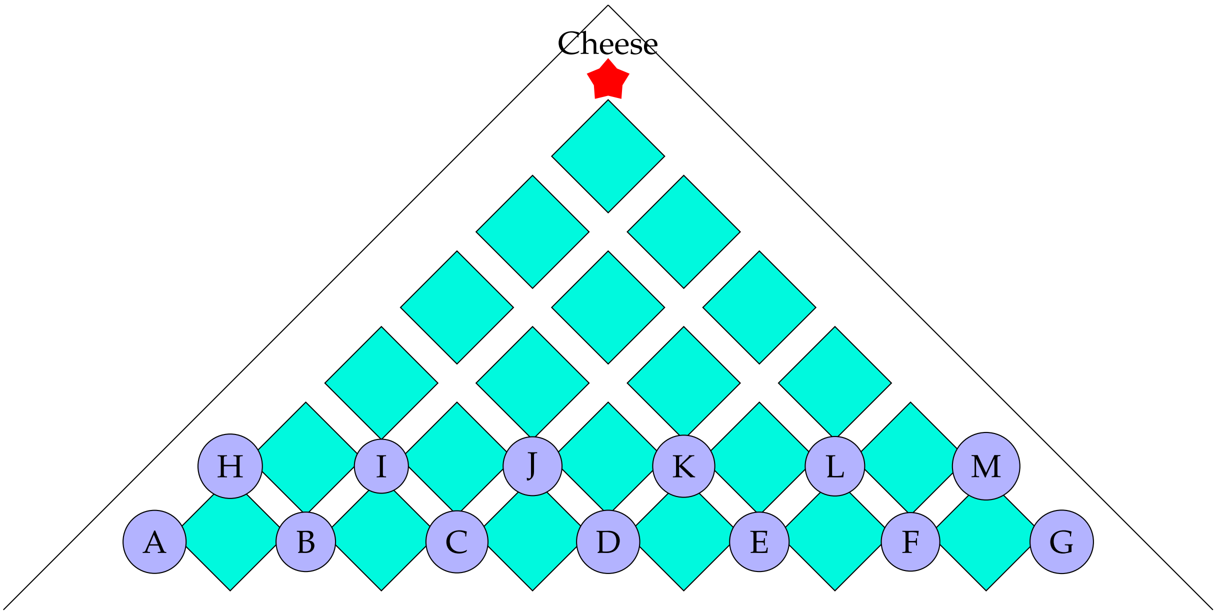


Figure 8: More Mice!

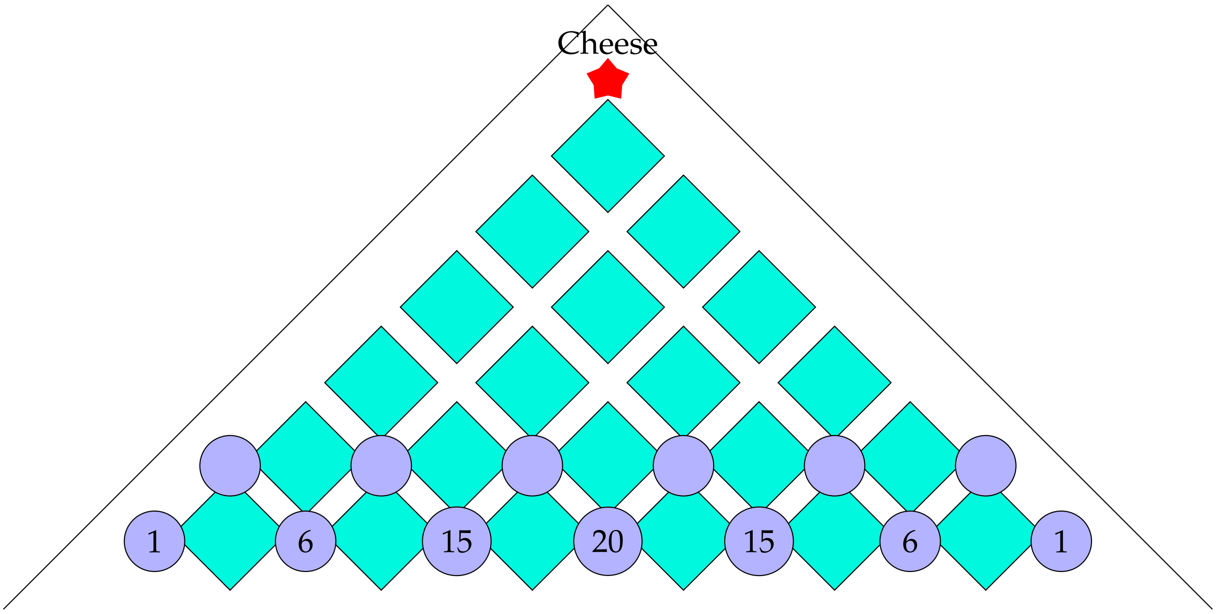


Figure 9: More Mice!

2.2 Question: Do you see a pattern emerging? What do you infer?

3 Mice Everywhere!

Suppose there was a mouse at every corner. First of all, there would (sadly) not be any cheese left. On the bright side, we would have learnt something. Can you fill in all these empty circles?

Indeed, you don't need to count anymore - you may simply apply the rules from Question 2.2 blindly!

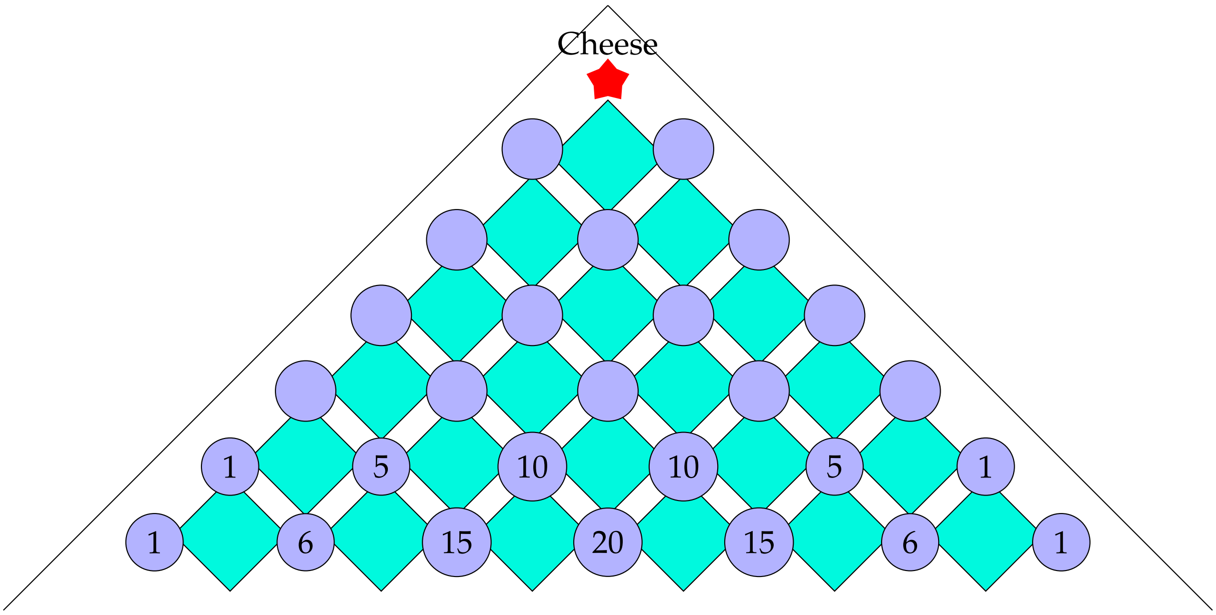


Figure 10: Mice Everywhere!

4 The Conclusion: Pascal's Triangle

The triangle you end up with should look like this (the cheese has been eaten and replaced by the house of a mouse).

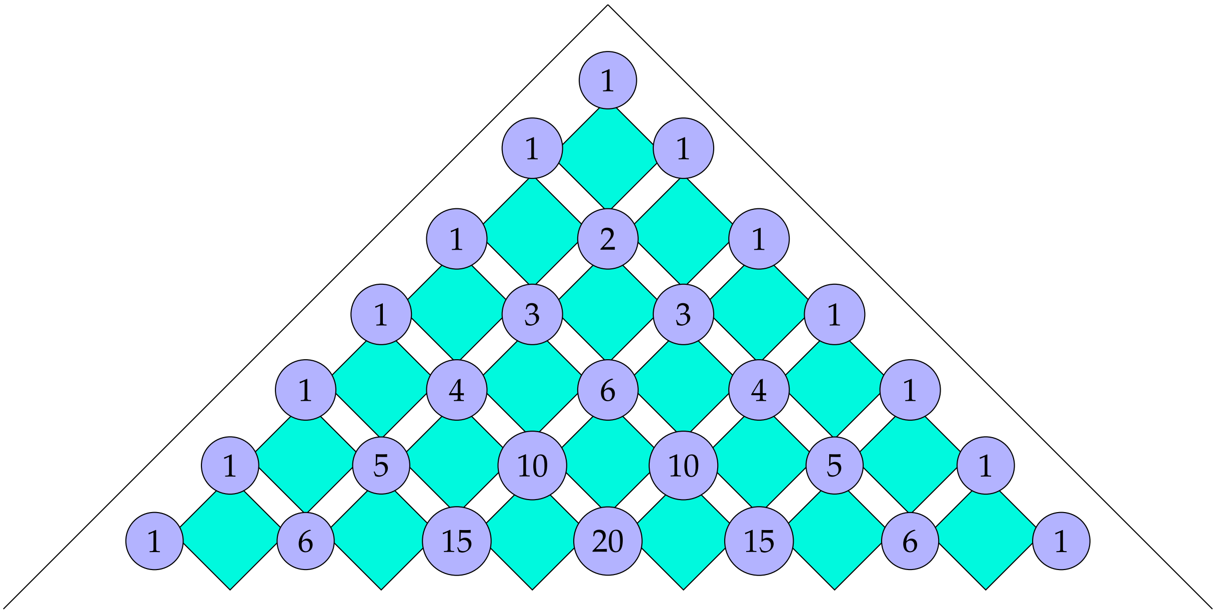


Figure 11: Pascal's Triangle

Some explanation is in order. Let us introduce some terminology first.

- The *level* of a mouse is the horizontal row on which it appears (The *Cheese* is on level 0, and is assigned the value 1). For instance, Hari is on level 5, while Chitra is on level 6. This number is denoted by n.
- The number of *right* turns that a mouse needs to take is denoted by r. For instance, for John, $r = 2$ (See Figure 8).
- Similarly, the number of *left* turns needed by a mouse is denoted by ℓ. Observe that
$$n = \ell + r.$$
- The number of shortest routes from the choose to a mouse is called a *binomial coefficient*, and is denoted by
$$\binom{n}{r}.$$

In Question 1.3, we had seen that

$$\binom{n}{n-r} = \binom{n}{r}.$$

These are the coefficients that appear when you multiply out the polynomial

$$(1+x)^n = 1 + nx + \ldots + nx^{n-1} + x^n$$

- In Question 2.2, you have observed that a given value at level n is obtained by adding the two values at level $(n-1)$ 'adjacent' to it. This is captured by the formula

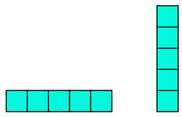
$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Question: Can you obtain this identity *algebraically* using the polynomial $(1+x)^n$?

Note: Pascal's triangle is named after the 17th century French mathematician/philosopher Blaise Pascal (see https://en.wikipedia.org/wiki/Blaise_Pascal).

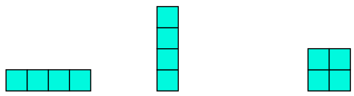
Numbers as Rectangles

Any number of tiles can be arranged as a rectangle, and for most numbers this can be done in many ways. For example, five tiles can be arranged into a rectangle of width 1 and height 5 (a 1×5 rectangle on the left below) or into rectangle of width 5 and height 1 (a 5×1 rectangle on the right below):



Note: For the next few problems we want to count them as two different arrangements.

Example: We can arrange four tiles into three rectangles: 1×4, 4×1, and a 2×2 square (a square is also a rectangle).



Questions:

1. How many rectangular arrangements are there for six tiles?

2. For every number of tiles between 1 and 16, find the number of all possible rectangular arrangements and fill in the table below.

Tiles	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Rectangles																

3. When is the number of rectangular arrangements odd?

4. When is the number of rectangular arrangements exactly two?

5. When is the number of rectangular arrangements exactly three?

6. Given a number n, what is the number of rectangular arrangements equal to?

Creatures of Flatland

The following set of questions is inspired by Edwin Abbott's book titled "Flatland".
The creatures of flatland live in 2-dimensional (2D) space. They do all their activities in 2D only without having a clue about 3D space. So, their world is a 2D space-time unlike ours 3D space-time. Suppose, we have a way of monitoring the creatures and their activities. Seeing that the creatures are intelligent and they know 2D geometry well, we decide to do an experiment to see if they can formulate mathematical ideas about 3D objects, that are easy for us to perceive.

1. In the first experiment, we send a solid sphere through their world. Now, what will the creatures see as the solid sphere passes through their 2D world? You may imagine to be in their world and describe what shape(s) appeared (of course, you do not know about the solid sphere as it does not exist in 2D world).

2. In the second experiment, a tetrahedron passes through their world. What possible shapes the creatures could have seen?

3. How the intelligent creatures could argue that 3D objects passed through their world?

4. Suppose creatures, now, formulate an idea about 3D objects. How can they argue that in the two experiments the 3D objects were distinct geometric figures?

5. Using the knowledge of 2D geometry (which they know very well!), can they predict a 3D object that has not yet passed through their world?

Which side are you on?

1. Can you convince yourself that the *twisted band* shaped object in Figure 12 below can be formed by taking a square shaped sheet of some elastic material as shown in in Figure 12, and gluing together the edges following the identification labels (such that the directions of the arrows on the edges match up)? We call this object a *Möbius band*.

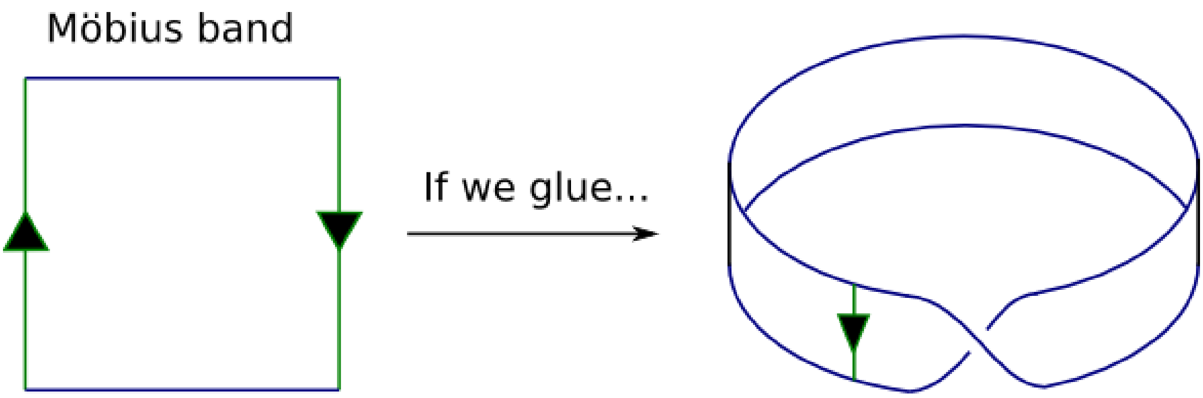


Figure 12: The Möbius band

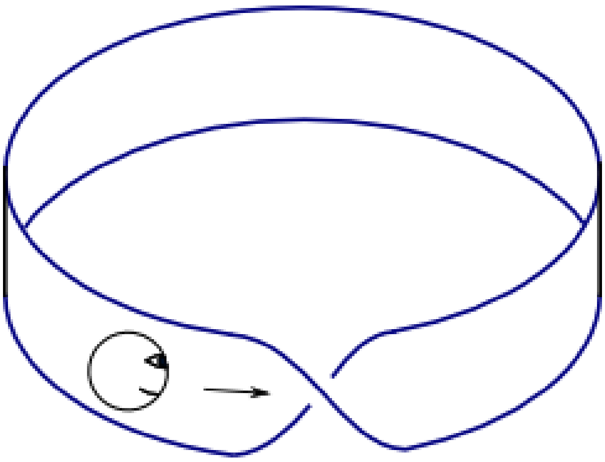


Figure 13: Going around The Möbius band

2. Suppose you start out going right on the Möbius band as in the in Figure 13. What changes as you go around the band once? What happens when you go around the band twice? How many different edges does a Möbius band have?

3. Compare this entire exercise with Problem 3 of Session 3 (where a bagel shaped object was constructed by identifying the opposite edges of a square shaped sheet of some elastic material).

Count carefully..

1. Suppose 18 students are taking part in an online MCI discussion session. During the introduction it turned out that although they come from different parts of the country, each of them knew some of the other participants as they had together attended some of the previously held MCI sessions. Is it possible that 7 of them knew 3 fellow participants each, 5 of them knew 4 fellow participants each and the remaining 6 knew 5 fellow participants each?

2. Can you draw 11 straight line segments on a sheet of paper so that each of them intersect exactly 3 others?

3. Try to find a common pattern from the last two questions to argue if the following statement is true or false:
There are n-points namely, $P_1, P_2, \ldots, P_n$ on a sheet of paper. In the set of points $\{P_1, P_2, \ldots, P_n\}$, let's assume that P_1 is joined to k_1 other points, P_2 to k_2 points, and so on (i.e. P_n is joined to k_n points from the same set of points). If the total number of line segments connecting points in the above set is m. Then $2m = k_1 + k_2 + \ldots + k_n$.

4. Find the number of possibilities (distinguishable) of placing three balls in two bins when
   (a) balls and bins are labelled.
   (b) bins are labelled but balls are unlabelled.

5. Find the number of possibilities of placing r balls in n bins (as considered in problem 4).

6. Let n, k be two natural numbers. Find the number of solutions of the following equation
$$n_1 + \ldots + n_k = n,$$
where $n_1, \ldots, n_k$ are natural numbers.

The problems are contributed by Atreyee, Dheeraj, Kartick and Prahlad.