Maths Circle India

TIFR-STCS Maths Circle Team

Session 5: June 24, 2022

1 A Problem of Parity

Credits: Anna Burago, Mathematical Circles Diaries

Arun has a table with two rows and two columns (so that it has a total of four cells). He wants to put one of two numbers, either 0 or 1, in each cell, in such a way that the sum of the numbers in each row, as well as in each column, is even. In how many ways can he do this?

What if he started with a table with three rows and three columns? Four rows and four columns? Six rows and six columns?

2 Another party at Zoya's house

Credits: Shklyarky, Chentsov, and Yaglom, Selected problems and theorems in elementary mathematics (Mir Publishers, Moscow 1979)

After the success of Zoya's first party, Zoya has gotten fond of the idea of throwing parties. This time, she wants to have her party in the little garden at the back of her house and invite 20 of her classmates. Her garden has a round stone table with four stone seats set around it. She wants to ensure that she can find four of her classmates to sit around this table such that each person sitting on one of the stone seats will be friends with their neighbour on boths sides of them. Remembering the difficult time she had figuring out how many people to invite to her last party ((in which she wanted to make sure that the three-seater table in her house was occupied by three classmates who were all friends with each other or by three classmates none of whom were friends with the other two), she starts wondering if 20 is a large enough number to guarantee that she will be able to do this.

But then she remembers that each of these 20 classmates of hers is friends with at least 10 others in this group of 20. And she stops worrying. Why?

3 *"If I speak the refined speech they speak..."*[1]

Credits: Shklyarky, Chentsov, and Yaglom, Selected problems and theorems in elementary mathematics (Mir Publishers, Moscow 1979)

A total of 17 legislative assembly members have been chosen from the assemblies of the four states of Maharashtra, Karnataka, Telangana and Andhra Pradesh to form a legislative tribunal for

[1] This is a rough paraphrase of an excerpt from Vālmīki.

negotiating the sharing of the waters of the Krishna river which originates near Mahabaleshwar and flows through their states on its way to the Bay of Bengal.

Each of the legislators in this tribunal knows at least one of the three languages Marathi, Telugu and Kannada. Due to reasons of political posturing and image, some pairs of legislators are forced to avoid a particular language out of these three when communicating with each other even if both know that language (for example, a Maharashtra legislator may decide for political reasons that he cannot use Kannada to communicate with a particular Karnataka legislator even if they both know Kannada, but he may be willing to communicate in Kannada with his Andhra colleague who also knows Kannada).

Anticipating this problem, the rule for the formation of the tribunal requires that it must be composed in such a way that each pair of legislators shares at least one language (out of these three languages) which they are willing to use to communicate with each other in. This rule is always obeyed during the formation of the tribunal.

According to the rule governing the final report of the tribunal, after three days of negotiations in Mahabaleshwar, the legislators have to constitute a drafting committee of three members. This committee of three has to summarize the decision of the tribunal by writing a report in one of these three languages.

To do this, they must of course have a common language (one of Marathi, Telugu, or Kannada) that they are all willing to use in their communication with each other. When the time to form this committee comes, one of tribunal members calls the press and tells the reporters that according to his information there is no such group of three legislators among them who are all willing to communicate with each other in one of these three languages. Therefore he demands that the tribunal must be dissolved without writing a report.

The reporters have to decide whether to take this legislator seriously. If he is right, it is a good news story and needs to be immediately sent to their newspapers. If not, they would just like to ignore him because the committee will anyway be formed tomorrow and the report written in spite of this political stunt by the legislator.

After thinking about it for a little while, one of them, who likes to solve Maths puzzles as a hobby, tells all the other reporters: If the rule of formation has been obeyed, then what this legislator is saying can never happen. There will always be one group of three that are all willing to use a common language to communicate with each other. Is she right? Explain your answer.

4 Knowing without knowing: 1

This problem was suggested by Varun Narayanan.

Often, Arun and Barun want to decide together where they want to play a game of chess. However, they are afraid that if one of them reveals that he wants to (or wants not to) play, the other might feel pressured to change his choice accordingly. For example, if Barun learns that Arun wants to play, he might decide to play just to humour his friend, even though he might not have wanted to play to begin with.

So they want to make sure that they find out whether *both* of them want to play, in such a way that neither of learn learns anything about the other person's choice until the decision is made. Further, even after the decision has been made, they should not learn anything about the other person's choice that is not already implied by the decision they reached. For example, if the final decision is to *not play*, and Barun's choice was to not play, Barun should not learn whether Arun wanted to play or not (since the decision would be the same in both those cases).

They have convinced themselves that this cannot be done when their friend Kiron, who has been overhearing their discussion, gives them a usual pack of cards and shows them how to do this. (A pack of cards contains 52 cards. One side of each card contains exactly the same design: like a picture of a valley. The other side has a different design on each card: half the cards have red designs, and the other half have black designs).

What did Kiran tell Arun and Barun to do?

Extra Problems

Hope you enjoy thinking about these after the session!

A Knowing without knowing: 2

Credits: C. Radhakrishna Rao, Statistics and Truth (World Scientific Publishing, Singapore 1997)

Arun's school Principal has a problem. She suspects that some of the students in his year cheated in their crucial term one examination that was held online during Covid. Now that everyone is back in school and under her thumb again, she wants to know what fraction cheated out of the two hunderd students that are in Arun's year.

She first thinks of asking a simple yes/no question 'did you cheat?', and telling all the students in his year to write YES or NO on a sheet of paper and drop the sheet in a box. But she knows that the students will be scared that their handwriting will be recognized, so they will not give an honest answer if she did this.

She also knows that the students are all basically good and honest children (except for some of them having copied that one time during the online test during lockdown, an isolated incident that they are already ashamed of and would never repeat). So if there was no fear of being identified, she knows they would honestly answer any questions put to them.

So she comes up with an idea: She hands out a rupee coin (which they don't have to return to her!) in addition to a sheet of paper to each student. Then she tells them to faithfully follow her instructions about i) what to do with the coin and ii) and what to write on the piece of paper before dropping it in a box. They obey her because they are not scared now of being individually identified as cheaters (although each of them has personally handed over to her their sheet of paper with something written on it).

In this way, she finds out (roughly) what fraction of students cheated. Can you think of what she asked the students to do in order to achieve this result?

B Another tall tale of Ravi's?

Credits: Shklyarky, Chentsov, and Yaglom, Selected problems and theorems in elementary mathematics (Mir Publishers, Moscow 1979)

Hasan's friend Ravi (remember him from April?) was quiet for a number of days after Hasan had used Math to catch him telling a tall story about a party that never was. But after a couple of months he has resumed his tales of fun times at parties he got invited to. For example, while talking to Hasan in the lunch recess recently, he claimed he was at a party in which he noticed the following amazing thing about the group of all the people (including him) at the party: If two people at the party knew each other from before, then they had no mutual acquaintances at the party. But any two people at the party who did not know each other from before had exactly two mutual acquaintances at the party. Remembering his earlier tall tales, Hasan is very suspicious that Ravi is making this up.

So he tries to probe further to try and catch his friend red-handed telling another tall tale. He asks: How many people were at the party in total?

To which Ravi replies: 15.

Hasan is immediately excited and asks, just to be absolutely sure before pouncing on his friend: You mean 15 including you, right?

To which Ravi replies: No, 15 apart from me, so 16 in total.

Hasan now concedes to himself that Ravi could be telling the truth. Can you see why?

But he's still a bit suspicious given his earlier experience, so he probes further and asks: Did you by any chance count how many people each person including you knew at the party?

Ravi replies: Yes! I did and I found everyone knew exactly the same number of people, including me.

Now Hasan is sure his friend is telling the truth. And he knows what this number of people is, that each person knows. Do you also know it?