Structure Preserving Discretizations of Wasserstein Gradient Flows
Part I of IV: The object to discretize

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"A school on gradient flows"
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Where do we need to go today?

Question

What is that alleged interesting structure of a Wasserstein gradient flow that a discretization should preserve?

Let us rewrite Wasserstein as Riemann.

An ODE analogy for gradient flows (1/3)

A function $\boldsymbol{e} : \mathbb{R}^n \to \mathbb{R}$ defines an euclidean gradient flow via

$$\dot{x} = -\nabla \boldsymbol{e}(x).$$

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Solutions $(x_t)_{t \geq 0}$ diminish $\boldsymbol{e}$ "as fast as possible":

$$-\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{e}(x_t) = -\dot{x}_t \cdot \nabla \boldsymbol{e}(x_t) = \|\dot{x}_t\|^2 = \|\nabla \boldsymbol{e}(x_t)\|^2,$$

i.e., $(x_t)_{t \geq 0}$ invests all of its velocity into dissipating $\boldsymbol{e}$.

An ODE analogy for gradient flows (2/3)

On a manifold $\mathcal{M}$ the tangent space $\mathrm{T}_x\mathcal{M}$ at $x \in \mathcal{M}$ is formed by velocities $v \cong \dot{x}_0$ of curves $(x_s)_{s\in(-\sigma,\sigma)}$ with $x_0 = x$.

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A Riemannian tensor assigns to each $x \in \mathcal{M}$ a scalar product $\langle\cdot,\cdot\rangle_x$. It defines a norm on $\mathrm{T}_x\mathcal{M}$ and a distance on $\mathcal{M}$:

$$\|v\|_x^2 = \langle v, v\rangle_x \quad \text{and} \quad \mathbf{d}(x_0, x_1)^2 = \inf_{(x_s)_{s\in[0,1]}} \int_0^1 \|v_s\|_{x_s}^2 \,\mathrm{d}s.$$

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The differential $\mathrm{D}_x \boldsymbol{e}$ of a function $\boldsymbol{e} : \mathcal{M} \to \mathbb{R}$ assigns to each $v \in \mathrm{T}_x\mathcal{M}$ the directional derivative,

$$\mathrm{D}_x \boldsymbol{e}[v] = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \boldsymbol{e}(x_s) \quad \text{with} \quad v \cong \dot{x}_0.$$

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The gradient $\nabla_x\boldsymbol{e}$ is the Riesz representative of $\mathrm{D}_x\boldsymbol{e}$ w.r.t. $\langle\cdot,\cdot\rangle_x$,

$$\langle\nabla_x\boldsymbol{e}, v\rangle_x = \mathrm{D}_x\boldsymbol{e}[v] \quad \text{for all } v \in \mathrm{T}_x\mathcal{M}.$$

An ODE analogy for gradient flows (3/3)

Accordingly, $\boldsymbol{e} : \mathcal{M} \to \mathbb{R}$ defines a Riemannian gradient flow via

$$\dot{x} \cong -\nabla_x \boldsymbol{e}.$$

By definition, solutions $(x_t)_{t \geq 0}$ satisfy

$$-\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{e}(x_t) = -\mathrm{D}_{x_t}\boldsymbol{e}[v_t] = \langle -\nabla_{x_t}\boldsymbol{e}, v_t \rangle_{x_t} = \left\| \nabla_{x_t}\boldsymbol{e} \right\|_{x_t}^2 = \left\| v_t \right\|_{x_t}^2,$$

i.e., all velocity is invested into dissipating $\boldsymbol{e}$.

Otto calculus (1/3)

On the "manifold" of probability densities,

$$\mathcal{P}_2(\mathbb{R}^d) = \left\{ \rho \in L^1(\mathbb{R}^d) \,\middle|\, \rho \geq 0,\ \int \rho \, \mathrm{d}x = 1, \int |x|^2 \rho \, \mathrm{d}x < \infty \right\},$$

the L^2-Wasserstein metric is a distance

$$\mathbb{W}(\rho_0, \rho_1)^2 = \int \left\{ \int |T(x) - x|^2 \rho_0(x) \, \mathrm{d}x \,\middle|\, T\#\rho_0 = \rho_1 \right\}.$$

Can one define a Riemannian metric on $\mathcal{M} = \mathcal{P}_2(\mathbb{R}^d)$ such that $\mathbf{d} = \mathbb{W}$ (…and make the according gradient flow explicit)?

Otto calculus (2/3)

Otto's tangent space

The tangent space $\mathrm{T}_\rho \mathcal{P}_2(\mathbb{R}^d)$ is identified with functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ (modulo constants) such that

$$\|\varphi\|_\rho^2 := \int |\nabla \varphi|^2 \rho \, \mathrm{d}x < \infty.$$

The tangent vector $\varphi \in \mathrm{T}_\rho \mathcal{P}_2(\mathbb{R}^d)$ relates to curves $(\rho_s)_{s \in (-\sigma, \sigma)}$ s.t.

$$\dot\rho_s = -\operatorname{div}(\rho_s \nabla \varphi_s) \quad \text{with } \rho_0 = \rho,\ \varphi_0 = \varphi.$$

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Theorem ([Benamou&Brenier'00])

$$\mathbb{W}(\rho_0, \rho_1)^2 = \inf_{\rho_s} \int_0^1 \|\varphi_s\|_{\rho_s}^2 \, \mathrm{d}s.$$

Otto calculus (3/3)

The norm $\|\cdot\|_\rho$ induces the metric tensor

$$\langle \varphi, \psi \rangle_\rho = \int \nabla\varphi \cdot \nabla\psi \, \rho \, \mathrm{d}x.$$

The gradient ψ of $\boldsymbol{E}$ at ρ is thus characterized by

$$\int \nabla\varphi \cdot \nabla\psi \, \rho \, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \boldsymbol{E}(\rho_s) \quad \text{with} \quad \dot{\rho}_s = -\operatorname{div}(\rho\nabla\varphi).$$

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The differential of a functional $\boldsymbol{E} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ at ρ is is implicitly defined via $\delta\boldsymbol{E}/\delta\rho : \mathbb{R}^d \to \mathbb{R}$ such that

$$\int \zeta \, \frac{\delta \boldsymbol{E}}{\delta\rho} \, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \boldsymbol{E}(\rho + s\zeta) \quad \text{for all suitable } \zeta : \mathbb{R}^d \to \mathbb{R}.$$

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Gradient flow equation

$$\partial_t \rho = \operatorname{div}\left(\rho \, \nabla \left(\frac{\delta \boldsymbol{E}}{\delta \rho}\right)\right).$$

Does it get even more abstract?

There are interesting examples.

Standard example 1: Continuity equation

Potential energy

For an external potential $V : \mathbb{R}^d \to \mathbb{R}$ consider

$$\boldsymbol{E}(\rho) = \int V(x)\rho(x)\,\mathrm{d}x.$$

Then $\delta \boldsymbol{E}/\delta\rho = V$, and accordingly,
the $\mathbb{W}$-gradient flow is the continuity equation,

$$\partial_t \rho = \operatorname{div}\big(\rho\,\nabla V\big).$$

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Let $(x_t)_{t\geq 0}$ satisfy

$$\dot{x}_t = -\nabla V(x_t).$$

Then a distributional solution to the continuity equation is

$$\rho_t = \delta_{x_t}.$$

Standard example 2: Nonlocal aggregation equation

Interaction energy

For an even interaction potential $W : \mathbb{R}^d \to \mathbb{R}$ consider

$$\boldsymbol{E}(\rho) = \frac{1}{2}\int \rho\, W * \rho \,\mathrm{d}x = \frac{1}{2}\iint \rho(x) W(x-x')\rho(x')\,\mathrm{d}x\,\mathrm{d}x'.$$

Then $\delta\boldsymbol{E}/\delta\rho = W * \rho$, and accordingly,
the $\mathbb{W}$-gradient flow is the nonlocal aggregation equation,

$$\partial_t \rho = \mathrm{div}\,\big(\rho\,\nabla W * \rho\big).$$

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Then $\delta \boldsymbol{E} / \delta \rho = W * \rho$, and accordingly,
the $\mathbb{W}$-gradient flow is the nonlocal aggregation equation,

$$\partial_t \rho = \operatorname{div} \big(\rho \, \nabla W * \rho \big).$$

Let N particle positions $x_1, \ldots, x_N$ satisfy

$$\dot{x}_k = - \sum_{j \neq k} \nabla W(x_k - x_j).$$

Then a distributional solution to the nonlocal aggregation equation is

$$\rho_t = \frac{1}{N} \sum_k \delta_{x_k(t)}.$$

Standard example 3: Linear diffusion

Boltzmann entropy

Consider the Boltzmann entropy $\boldsymbol{E} := \boldsymbol{H}$, i.e.,

$$\boldsymbol{H}(\rho) = \int \rho \log \rho \, \mathrm{d}x.$$

Then $\delta \boldsymbol{E}/\delta\rho = \log \rho$, and accordingly,
the $\mathbb{W}$-gradient flow is the linear diffusion equation,

$$\partial_t \rho = \mathrm{div}\,\big(\rho \, \nabla \log \rho\big) = \Delta \rho.$$

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The solution for initial datum $\bar{\rho}$ is given by

$$\rho_t = H_t * \bar{\rho} \quad \text{with} \quad H_t(z) = (4\pi t)^{-d/2} \exp\left(-\frac{|z|^2}{4t}\right).$$

Standard example 4: Nonlinear Diffusion

Internal energy

For a convex internal energy density $h : \mathbb{R}_{\geq 0} \to \mathbb{R}$, consider

$$\boldsymbol{E}(\rho) = \int h(\rho)\,\mathrm{d}x.$$

Then $\delta\boldsymbol{E}/\delta\rho = h'(\rho)$, and accordingly,
the $\mathbb{W}$-gradient flow is the nonlinear diffusion equation,

$$\partial_t \rho = \Delta P(\rho) \quad \text{with} \quad P(r) = rh'(r) - h(r).$$

Standard example 4: Nonlinear Diffusion

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Of particular interest: power-type energy resp. homogeneous diffusion,

$$\boldsymbol{E}(\rho) = \int \frac{\rho^m}{m-1}\,\mathrm{d}x \quad \rightsquigarrow \quad \partial_t \rho = \Delta(\rho^m).$$

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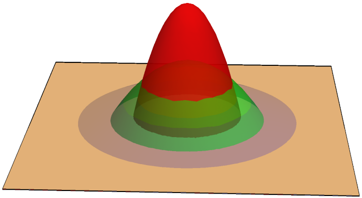

Fourth order example 1: lubrication

Dirichlet energy

Consider the Dirichlet energy

$$\boldsymbol{E}(\rho) = \frac{1}{2}\int |\nabla\rho|^2 \, \mathrm{d}x.$$

Then $\delta\boldsymbol{E}/\delta\rho = -\Delta\rho$, and accordingly,
the $\mathbb{W}$-gradient flow is the thin film equation (a.k.a. Hele-Shaw flow),

$$\partial_t\rho = -\operatorname{div}\big(\rho\,\nabla\Delta\rho\big).$$

Fourth order example 2: quantum diffusion

Fisher information

Consider the Fisher information

$$\boldsymbol{F}(\rho) = 2 \int \left|\nabla \sqrt{\rho}\right|^2 \mathrm{d}x \qquad .$$

Then $\delta \boldsymbol{E}/\delta\rho = -\Delta\sqrt{\rho}/\sqrt{\rho}$, and accordingly,
the $\mathbb{W}$-gradient flow is the DLSS equation (a.k.a. QDD4),

$$\partial_t \rho = -2 \operatorname{div} \left(\rho \, \nabla \left[\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right] \right) \qquad .$$

Fourth order example 2: quantum diffusion

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$$\boldsymbol{F}(\rho) = 2\int \left|\nabla\sqrt{\rho}\right|^2 \mathrm{d}x = \frac{1}{2}\int |\nabla \log\rho|^2 \rho \,\mathrm{d}x.$$

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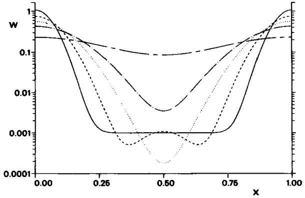


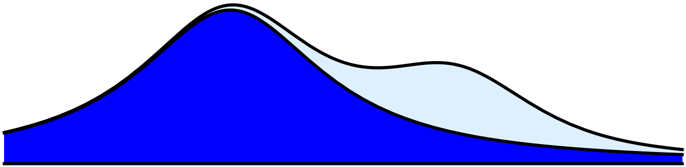
Exotic example 1: Muskat problem

With $a, b > 0$ and $\boldsymbol{\rho} = (\rho_1, \rho_1) \in \mathcal{P}_2(\mathbb{R})^2$, consider

$$\boldsymbol{E}(\boldsymbol{\rho}) = a \int \rho_1^2 \,\mathrm{d}x + b \int (\rho_1 + \rho_2)^2 \,\mathrm{d}x.$$

The gradient flow of $\boldsymbol{E}$ in the $\mathbb{W} \times \mathbb{W}$-metric is the Muskat problem,

$$\partial_t \boldsymbol{\rho} = \partial_x \left(\begin{pmatrix} (a+b)\rho_1 & b\rho_1 \\ \rho_2 & \rho_2 \end{pmatrix} \partial_x \boldsymbol{\rho} \right).$$



Exotic example 2: parabolic–parabolic Keller–Segel

Consider

$$\boldsymbol{E}(\rho, c) = \int \left(\rho \log \rho - \rho c + \frac{1}{2}|\nabla c|^2 + \frac{1}{2}c^2\right) \mathrm{d}x.$$

The gradient flow of $\boldsymbol{E}$ in the $\mathbb{W} \times L^2$-metric is the Keller-Segel system,

$$\begin{aligned} \partial_t \rho &= \Delta \rho - \operatorname{div}(\rho \nabla c) \\ \partial_t c &= \Delta c - c + \rho. \end{aligned}$$

Got it . . . but why should I care?

There are benefits from
being a Wasserstein gradient flow.

Benefit 1: Variational form

Morally, the gradient flow is such that, at any $t > 0$,

$$\partial_t \rho_t = \mathrm{div}(\rho_t \nabla \varphi_t) \quad \text{where} \quad \varphi_t \in \operatorname*{argmin}_{\varphi} \left(\frac{1}{2} \|\varphi\|^2_{\rho_t} + \mathrm{D}_{\rho_t} \boldsymbol{E}[\varphi] \right).$$

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Among many things, this implies Hölder continuity:

Hölder estimate

For $t_2 > t_1 > 0$,

$$\frac{\mathbb{W}(\rho_{t_1}, \rho_{t_2})^2}{t_2 - t_1} \leq \int_{t_1}^{t_2} \|\varphi_t\|_{\rho_t}^2 \, \mathrm{d}t \leq \boldsymbol{E}(\rho_{t_1}) - \boldsymbol{E}(\rho_{t_2}).$$

Benefit 2: Dissipation

Morally, the gradient flow balances kinetic energy and dissipation,

$$\|\varphi_t\|_{\rho_t}^2 = |\partial \boldsymbol{E}(\rho_t)|^2 := -\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}(\rho_t) = \int \left|\nabla \frac{\delta \boldsymbol{E}}{\delta \rho}\right|^2 \rho_t \,\mathrm{d}x.$$

This provides a priori estimates:

Dissipation estimate

For $t_2 > t_1 > 0$,

$$\int_{t_1}^{t_2} |\partial \boldsymbol{E}(\rho_t)|^2 \,\mathrm{d}t \leq \boldsymbol{E}(\rho_{t_2}) - \boldsymbol{E}(\rho_{t_1}).$$

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Example: The Boltzmann entropy $\boldsymbol{H}(\rho) = \int \rho \log \rho \, \mathrm{d}x$ has dissipation

$$|\partial \boldsymbol{H}(\rho)|^2 = \int \left|\nabla \log \rho\right|^2 \rho \, \mathrm{d}x = 4 \int |\nabla \sqrt{\rho}|^2 \, \mathrm{d}x = 2\boldsymbol{F}(\rho),$$

i.e., twice the Fisher information. Thus, for $\partial_t \rho = \Delta \rho$,

$$2 \int_{t_1}^{t_2} \boldsymbol{F}(\rho_t) \, \mathrm{d}t \leq \boldsymbol{H}(\rho_{t_1}) - \boldsymbol{H}(\rho_{t_2}).$$

Benefit 3: Positivity

Consider the thin film equation:

$$\partial_t \rho = -\operatorname{div}\big(\rho\, \nabla \Delta \rho\big)$$

From the PDE alone, it is not clear that solutions

- remain non-negative and
- preserve mass.

But the fact that it is formally the $\mathbb{W}$-gradient flow of the Dirichlet energy

$$\boldsymbol{E}(\rho) = \frac{1}{2} \int |\nabla \rho|^2 \,\mathrm{d}x$$

is a strong indication — and actually provides an angle of attack.

There is one more significant thing:

Convexity!

A (fake) ODE analogy for convexity (1/2)

Consider the euclidean gradient flow $\dot{x} = -\nabla \boldsymbol{e}(x)$ for a λ-uniformly convex energy $\boldsymbol{e} : \mathbb{R}^n \to \mathbb{R}$, i.e.,

$$\nabla^2 \boldsymbol{e} \geq \lambda \mathbf{1}.$$

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Consider the euclidean gradient flow $\dot{x} = -\nabla \boldsymbol{e}(x)$ for a λ-uniformly convex energy $\boldsymbol{e} : \mathbb{R}^n \to \mathbb{R}$, i.e.,

$$\nabla^2 \boldsymbol{e} \geq \lambda \mathbf{1}.$$

Lemma

If $\lambda > 0$, then:

- $\boldsymbol{e}$ has a unique minimizer $x_\infty \in \mathbb{R}^n$,
- speed, dissipation, energy and distance to x_∞ decay exponentially,

$$\begin{aligned} \left\|\dot{x}_t\right\|^2 &\leq e^{-2\lambda t}\left\|\dot{x}_0\right\|^2, \\ \left\|\nabla \boldsymbol{e}(x_t)\right\|^2 &\leq e^{-2\lambda t}\left\|\nabla \boldsymbol{e}(x_0)\right\|^2, \\ \boldsymbol{e}(x_t) - \boldsymbol{e}(x_\infty) &\leq e^{-2\lambda t}\left(\boldsymbol{e}(x_0) - \boldsymbol{e}(x_\infty)\right), \\ \left\|x_t - x_\infty\right\|^2 &\leq e^{-2\lambda t}\left\|x_0 - x_\infty\right\|^2; \end{aligned}$$

- the flow is λ-uniformly contractive,

$$\left\|\hat{x}_t - \check{x}_t\right\|^2 \leq e^{-2\lambda t}\left\|\hat{x}_0 - \check{x}_0\right\|^2.$$

A (fake) ODE analogy for convexity (2/2)

For $\boldsymbol{e} \in C^2(\mathbb{R}^n)$, λ-uniform convexity is equivalent to

$$\boldsymbol{e}(x_s) \leq (1-s)\boldsymbol{e}(x_0) + s\boldsymbol{e}(x_1) - \frac{\lambda}{2}s(1-s)\|x_1 - x_0\|^2$$

for any choice of $x_0, x_1 \in \mathbb{R}^n$ and $s \in [0,1]$,
where $(x_s)_{s\in[0,1]}$ is the straight line from x_0 to x_1.

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What about simply replacing

- points $x \in \mathbb{R}^n$ by densities $\rho \in \mathcal{P}_2(\mathbb{R}^d)$,
- euclidean norm $\|\cdot\|$ by Wasserstein metric $\mathbb{W}$, and
- straight line $(x_s)_{s\in[0,1]}$ by $\mathbb{W}$-geodesic $(\rho_s)_{s\in[0,1]}$?

Geodesic convexity (1/3)

Definition

$\boldsymbol{E}$ *is* *λ-geodesically convex* *if for any* *$\mathbb{W}$-geodesic* $(\rho_s)_{s\in[0,1]}$*:*

$$\boldsymbol{E}(\rho_s) \leq (1-s)\boldsymbol{E}(\rho_0) + s\boldsymbol{E}(\rho_1) - \frac{\lambda}{2}s(1-s)\mathbb{W}(\rho_0,\rho_1)^2.$$

Geodesic convexity (1/3)

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Examples:

- the Boltzmann entropy $\boldsymbol{H}$ is 0-geodesically convex;
- if $\nabla^2 V \geq \lambda$, then the potential energy is λ-geodesically convex;
- if $\nabla^2 W \geq \lambda$, then the interaction energy is (almost) λ-geodesically convex;
- if $m > (d-1)/d$, then the internal energy with $h(r) = r^m/(m-1)$ is 0-geodesically convex;
- the Dirichlet energy is not λ-geodesically convex for any λ.

Geodesic convexity (2/3)

Back to the gradient flow $\partial_t \rho = \operatorname{div}\left(\rho \nabla \frac{\delta \boldsymbol{E}}{\delta \rho}\right)$.

Theorem

If $\boldsymbol{E}$ is λ-geodesically convex with $\lambda > 0$, then

- *$\boldsymbol{E}$ possesses precisely one minimizer ρ_∞;*
- *speed, dissipation, energy excess and distance to ρ_∞ decay exponentially,*

$$\begin{aligned}
\big\|\varphi_t\big\|_{\rho_t}^2 &\leq e^{-2\lambda t}\big\|\varphi_0\big\|_{\rho_0}, \\
\big|\partial \boldsymbol{E}(\rho_t)\big|^2 &\leq e^{-2\lambda t}\big|\partial \boldsymbol{E}(\rho_0)\big|^2, \\
\boldsymbol{E}(\rho_t) - \boldsymbol{E}(\rho_\infty) &\leq e^{-2\lambda t}\big(\boldsymbol{E}(\rho_0) - \boldsymbol{E}(\rho_\infty)\big), \\
\mathbb{W}\big(\rho_t, \rho_\infty\big)^2 &\leq e^{-2\lambda t}\mathbb{W}\big(\rho_0, \rho_\infty\big)^2.
\end{aligned}$$

- *the flow is λ-uniformly contractive,*

$$\mathbb{W}\big(\hat{\rho}_t, \check{\rho}_t\big)^2 \leq e^{-2\lambda t}\mathbb{W}\big(\hat{\rho}_0, \check{\rho}_0\big)^2.$$

Geodesic convexity (3/3)

Applications:

- Concerning decay to equilibrium, you will see a truly bombastic application in a couple of slides.
- Concerning contractivity, one obtains e.g. temporal robustness like: Solutions to $\partial_t \rho = \Delta \rho^m$ with $m > \frac{d-1}{d}$ satisfy

$$\mathbb{W}(\rho_{t+\tau}, \rho_t) \leq \mathbb{W}(\rho_\tau, \rho_0).$$

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Proof: would require a more precise definition of $\mathbb{W}$-gradient flow.

As a weak remedy, we obtain contractivity under a sligthly stronger hypothesis.

Displacement convexity (1/2)

Definition

A segment $(\rho_s)_{s\in[0,1]}$ *w.r.t.* η *is such that*

$$\rho_s = \nabla\big((1-s)\varphi_0 + s\varphi_1\big)\#\eta,$$

and φ_0, φ_1 *are optimal, i.e.,*

$$\mathbb{W}(\rho_i,\eta)^2 = \int |\nabla\varphi_i(x) - x|^2\eta(x)\,\mathrm{d}x \quad \textit{for } i = 0, 1.$$

Note: Segments are more general than geodesics.

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Definition

$\boldsymbol{E}$ *is* λ*-displacement convex if for any segment* $(\rho_s)_{s\in[0,1]}$*:*

$$\boldsymbol{E}(\rho_s) \le (1-s)\boldsymbol{E}(\rho_0) + s\boldsymbol{E}(\rho_1) - \frac{\lambda}{2}s(1-s)\mathbb{W}(\rho_0,\rho_1)^2.$$

Note: λ-displacement convexity implies λ-geodesically convexity. All of the aforementioned λ-geodesically convex functionals happen to be λ-displacement convex as well.

Displacement convexity (2/2)

Definition

$(\rho_t)_{t\geq 0}$ *is an* $(\mathrm{EVI})_\lambda$*-solution if, for all* $\eta \in \mathcal{P}_2(\mathbb{R}^d)$*,*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{W}(\rho_t,\eta)^2 + \frac{\lambda}{2}\mathbb{W}(\rho_t,\eta)^2 \leq \boldsymbol{E}(\eta) - \boldsymbol{E}(\rho_t).$$

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Theorem

If $\boldsymbol{E}$ *is* λ*-geodesically convex,*
then its gradient flow has an $(\mathrm{EVI})_\lambda$*-solution.*

Proof: will be given tomorrow, under the stronger hypothesis of λ-displacement convexity.

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Proof: will be given tomorrow, under the stronger hypothesis of λ-displacement convexity.

Theorem

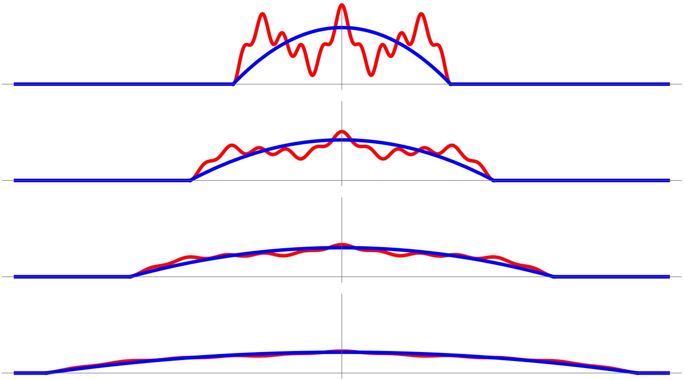
$(\mathrm{EVI})_\lambda$*-solutions are* λ*-contractive.*

...see computations on board ...

Just one more thing!

(It's about atomic bombs!)

A “real” application (1/2)



A “real” application (2/2)

The porous medium equation

$$\partial_t u = \Delta u^m \qquad (\mathsf{PME}_m)$$

has the self-similar solution

$$u_t^*(x) = t^{-d\alpha} U_m(t^{-\alpha} x) \quad \text{with} \quad U_m(z) = \left(B_m - \frac{m-1}{2m} |z|^2 \right)_+^{\frac{1}{m-1}} .$$

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Theorem ([Otto'01])

$$\|u_t - u_t^*\|_{L^1} \leq C t^{-\alpha}.$$

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Proof: Use rescaling to rewrite (PME$_m$) as

$$\partial_t \rho = \Delta \rho^m + \operatorname{div}(x\rho),$$

which the $\mathbb{W}$-gradient flow of the geodesically 1-convex functional

$$\boldsymbol{H}'_m(\rho) = \int \left(\frac{\rho^m}{m-1} + \frac{1}{2}|x|^2 \rho \right) \mathrm{d}x.$$

And here is a peaceful spin-off (1/2)

An analogous result holds for the DLSS equation

$$\partial_t u = -2\,\mathrm{div}\left(u\,\nabla\frac{\Delta\sqrt{u}}{\sqrt{u}}\right), \tag{DLSS}$$

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$$u_t^*(x) = t^{-d/4}U_1(t^{-1/4}x) \quad \text{with} \quad U_1(z) = (4\pi)^{-d/2}\exp\left(-\frac{1}{2}|z|^2\right).$$

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Theorem ([M'+McCann+Savaré'09])

$$\|u_t - u_t^*\|_{L^1} \leq Ct^{-1/4}.$$

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$$\partial_t\rho = -2\,\mathrm{div}\left(\rho\,\nabla\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) + \mathrm{div}(x\rho),$$

which the $\mathbb{W}$-gradient flow of

$$\boldsymbol{F}'(\rho) = \boldsymbol{F}(\rho) + \frac{1}{2}\int |x|^2\rho\,\mathrm{d}x.$$

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And here is a peaceful spin-off (2/2)

Theory guanrantees 1-displacement convexity of the relative entropy,

$$\boldsymbol{H}'(\rho) = \int \rho \log \rho \, \mathrm{d}x + \frac{1}{2} \int |x|^2 \rho \, \mathrm{d}x.$$

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$$\boldsymbol{H}'(\rho) = \int \rho \log \rho \, \mathrm{d}x + \frac{1}{2} \int |x|^2 \rho \, \mathrm{d}x.$$

There is an intimate connection to the relative Fisher information:

$$\begin{aligned}
\left|\partial \boldsymbol{H}'(\rho)\right|^2 &= \int \left|\nabla \log \rho + x\right|^2 \rho \, \mathrm{d}x \\
&= \int \left(|\nabla \log \rho|^2 \rho + |x|^2 \rho + 2 \nabla \rho \cdot x\right) \mathrm{d}x \\
&= 2\boldsymbol{F}(\rho) + \int |x|^2 \rho \, \mathrm{d}x - 2d \int \rho \, \mathrm{d}x \\
&= 2\boldsymbol{F}'(\rho) - 2d.
\end{aligned}$$

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Theorem

$$\begin{aligned} \boldsymbol{H}'(\rho_t) - \boldsymbol{H}'(\rho_\infty) \leq e^{-2t}\big(\boldsymbol{H}'(\rho_0) - \boldsymbol{H}'(\rho_\infty)\big), \\ \boldsymbol{F}'(\rho_t) - \boldsymbol{F}'(\rho_\infty) \leq e^{-2t}\big(\boldsymbol{F}'(\rho_0) - \boldsymbol{F}'(\rho_\infty)\big). \end{aligned}$$

That's all for today

Thank you, and see you tomorrow!