# Sketch of a program

Dear participant,

You will find below the sketch of a program (pun intended) for the discussion meeting on Grothendieck-Teichmüller theory at ICTS. This is to be really regarded as a sketch, that is a) not all items will be covered during the lectures – but those which were not discussed or even mentioned may of course become the subject matter of more informal discussions and b) there perhaps will and in fact should be overlaps and repetitions, these being often the source of a deeper understanding.

As for references, given the modern means of access to the mathematical literature, they can easily be retrieved from existing sources. The following far from exhaustive list of metasources may be worth mentioning: the founding papers listed in item 2 below (Grothendieck's famous text as well as its translation into English and much, much more can be found on the site `grothendieckcircle.org`); the papers listed on the homepages of the lecturers, in particular those who have been busy with G.-T. theory essentially from its very beginning; the reference lists of these papers; MathSciNet; etc.

## 1. History in the large

A. Grothendieck's mathematics after 1970 : "Le temps des tâches est révolu".
Forays into the "jungle". More precisely, linear *vs* nonlinear : homotopy *vs* (co)homology, $\pi_1$ as the only "anabelian" invariant in classical algebraic topology – Galois (Grothendieck)-Teichmüller theory, anabelian geometry, the section conjecture.

## 2. Four sources of Grothendieck-Teichmüller theory

A very preliminary and partial survey...

– A. Grothendieck's *Esquisse d'un programme* (1984) : moduli stacks of curves, the two-level principle, etc.

– Y. Ihara : "der Alterstraum", nonabelian Iwasawa theory (Annals, 1986).

– V.G. Drinfel'd : quantum scattering, quantum groups, deforming associativity (Leningrad J. of maths, 1989).

– P. Deligne : *La droite projective privée de trois points* (in *Galois groups over* $\mathbb{Q}$, MSRI, 1990), a turning point, prounipotent completions, rational homotopy theory and mixed Tate motives *vs* Grothendieck's sketch in the *Esquisse*.

## 3. The thrice punctured sphere

Several avatars and characterizations of this central object (Reminder : "le problème des trois points" in [SGA 1]) :

– Topology : pair of pants, $P$, more politely "trinion". Building block of orientable hyperbolic surfaces : $-\chi(P) = 1$.

– Complex and conformal geometries : There exists a *unique* conformal structure on the complex sphere a.k.a $\mathbb{P}^1(\mathbb{C})$ (*not* easy, equivalent to $\mathcal{M}_{0,3}(\mathbb{C}) = (\star)$). The action of $PGL_2(\mathbb{C})$ (Möbius transformations) on the sphere.

– Moduli of curves : $\mathbb{P}^1 \setminus (0, 1, \infty) \simeq \mathcal{M}_{0,4}$ i.e. $\mathbb{P}^1(0, 1, \infty)$ classifies isomorphism classes of spheres with 4 marked points, say $(0, 1, \lambda, \infty)$ w.l.o.g. Proof : cross-ratio.

– Arithmetic geometry : With the classical – grothendieckian – definitions, $\mathbb{P}^1_{\mathbb{Z}} \setminus (0, 1, \infty)$ is *the only smooth hyperbolic curve over* $\mathbb{Z}$. Exercise : prove this (much less well-known) characterization.

## 4. A short reminder on the algebraic fundamental group

Étale covers $Y \to X$, normal schemes, definition of $\pi_1$. $\pi_1(Spec(k)) = Gal(k)$ for a field $k$ ; fixing a separable closure amounts to fixing a geometric basepoint. The fundamental group of $\mathbb{P}^1_k$ minus 0, 1, 2 and 3 points over a separably closed field $k$. Geometric fundamental group, the Galois short exact sequence for normal schemes.

### 5. $\mathbb{P}^1 \setminus (0, 1, \infty)$ again, the Galois action on its geometric fundamental group

The Galois action on the geometric fundamental group of $\mathbb{P}^1_{\mathbb{Q}} \setminus (0, \infty)$, the cyclotomic character $\chi$. Tangential basepoints, a groupoid with 6 such basepoints, the straight path $p$ (Ihara,'s notation) alias (Deligne's *dch* (!)), between the tangential basepoints at 0 and 1. The paths : $x$, $y$, $z$ $(xyz = 1)$ etc. and the automorphism group ($\simeq \mathcal{S}_3$) of $\mathbb{P}^1 \setminus (0, 1, \infty)$. "Coordinates" $f_\sigma(x, y)$ and $\chi(\sigma)$ for $\sigma \in Gal(\mathbb{Q})$. The commutation relation $(p, \sigma) = f_\sigma$. Relations ("equations" rather) I and II. Etc.

### 6. Belyi theorem

Belyi's result and the faithfulness of the Galois action on the geometric fundamental group of $\mathbb{P}^1 \setminus (0, 1, \infty)$. A Pictionary of $SL_2(\mathbb{Z})$ and the "dessins d'enfants".

### 7. A short reminder on Tannakian categories, alias $\otimes$-categories

Why they are an essential feature of "grothendieckian mathematics" : see Galois categories (the nonlinear counterpart) as well as motives and topoi. Why they should be called "Krein-Tannaka-Grothendieck" categories.

Return to the (possibly tangential) basepoints as fiber functors.

### 8. From $\mathbb{P}^1 \setminus (0, 1, \infty)$ to the moduli stacks of curves

$\mathbb{P}^1 \setminus (0, 1, \infty) \simeq \mathcal{M}_{0,4}$, at the crossroads of hyperbolic curves and moduli stacks of curves, as first stressed in this context by Grothendieck himself. It may sound "trivial" but this is a key observation ("une remarque du bon Dieu" !). Moduli schemes $\mathcal{M}_{0,n}$ $(n \geq 3)$ in genus 0, their (geometric) fundamental groups and the connection with braid groups.

### 9. Short reminder on proalgebraic groups

The Levi short exact sequence, the prounipotent maximal ideal, prounipotent groups, the equivalence with their Lie algebras (Quillen's Appendix). Contrast between profinite and proalgebraic groups. The latter are intractable in general : example of the free group on two generators (Tits' theorem).

**10. The second level in genus 0, $\mathcal{M}_{0,5}$**

Everything on $\mathcal{M}_{0,5}$, its fundamental group, the associated Lie algebra. Coordinates, the pentagon relation (relation-equation III). Formal definition of $GT$ in genus 0.

**11. Plane and spherical braid groups**

Their pictorial definitions, configuration spaces (plane and sphere) and their fundamental groups. The Galois action on profinite and prounipotent braid groups, infinitesimal braids, the Lie algebra version of the action, tangential derivations. The first explicit form of the two-level principle : Ihara's result(s) on the characterization of $GT$.

**12. Moduli stacks of curves in higher genus : a very partial introduction**

The $\mathcal{M}_{g,n}$'s, their dimensions and tangent bundles (Riemann-Roch), topological (Knudsen) morphisms, modelling clay surfaces (Grothendieck in Montpellier !), the divisor at infinity. Fibrations $\mathcal{M}_{g,n} \to \mathcal{M}_{g,m}$ $(m < n)$. Configuration spaces *vs* moduli. The relatively easy cases $g = 0, 1, 2$.

The analytic theory, the Teichmüller space $\mathcal{T}_{g,n}$, its contractibility. Fenchel-Nielsen coordinates and ideal triangulations, etc. The (infinite) cover : $\mathcal{T}_{g,n} \to \mathcal{M}_{g,n}$. The groups $\Gamma_{g,n}$ as both $\pi_0$'s (in the analytic setting) and $\pi_1$'s. Their profinite completions and what we ignore e.g. the triviality of $Z(\hat{\Gamma}_g)$ $(g > 2)$. Congruence subgroups and the congruence conjecture. Etc. (Cf. *Panoramas et Synthèses*).

**13. The two-level principle à la Grothendieck**

The original formulation from the *Esquisse* : $\pi_1 \simeq \pi_1^{\infty}$. What it precisely means and how to vindicate/prove the corresponding assertion(s)/theorem(s) .

**14. Curve and other complexes : the Teichmüller lego**

Tensor categories, Maclane's relations in higher genus : curve complexes, etc. The simple connectivity of the maximal curve complex as the fundamental fact behind another formulation of the two-level principle principle. The (profinite) Grothendieck-Teichmüller group in arbitrary genus (i.e. any type $(g, n)$). The "Teichmüller lego" for the action of the Grothendieck-Teichmüller group on the étale geometric fundamental groups of the $\mathcal{M}_{g,n}$'s (i.e. the profinite completions $\hat{\Gamma}_{g,n}$) and on groupoïds based at infinity.

**15. The prounipotent setting**

Specificity of genera 0, 1, 2. Mention of the Torelli group (Hain's result on its infinitesimal version). The genus 0 case : KZ connection, Associators, the de Rham and

Betti sides. A double torsor : $\underline{GT}$, $Ass$, $\underline{GRT}$. Lie algebras, formality of the genus 0 braid groups (Kohno-Drinfel'd).

**16. Double shuffle and moulds**

Definitions, Properties, Inclusion (Furusho) : $GRT \hookrightarrow DS$ ($DS$, in French $DMR$, ... Betti and de Rham). Possibly the connection with moulds.

**17. Multiple Zeta Values and Polylogarithms**

The two representations, discrete (series) and continuous (iterated integrals ; Écalle, Zagier, Kontsevitch). Possibly more general iterated integrals (Manin, Brown). First properties, double shuffle. Conjectures on the dimensions and the completeness of the double shuffle properties.

Polylogarithms, complex and $\ell$-adic.

**18. Elliptic associators etc.**

The $KZB$ connection (genus 1), elliptic associators, the elliptic versions $GT_{\text{ell}}$ of $GT$ and the corresponding Lie algebras (Enriquez).

**19. From genus 1 to genus 0 and back**

The general elliptic curve ($q = \exp(2i\pi\tau)$) , degeneration in the neighborhood of $q = 0$ ($\tau = i\infty$). The Tate (Mumford) curve, from $KZB$ to $KZ$ (the "Bernouilli map").

**20. Completions of complexes**

Back to the profinite picture in any genus. Completions of the various complexes and the Galois action on the completed versions. Reminder : the rigidity theorems for the discrete complexes (Ivanov). Rigidity of the (completed) maximal curve complex *vs* the automorphism group of the completed curve complex "Profinite associators". How to glue copies of $GT$. $GT$ as the outer automorphism group of completed (symplicial) complexes (indeed graphs).

**21. Anabelian geometry and the section conjecture (generalitiies)**

Post-70 Grothendieck again, the fundamental group again. $K(\pi,1)$ (Eilenberg-Maclane) spaces, the $\mathcal{M}_{g,n}$ are rationally $K(\pi,1)$ (or as orbifolds or as 1-stacks). The letter to Faltings, contrast between curves and AV (non linear *vs* linear once again). The congruence property for $Sp_2g(\mathbb{Z})$ ($g > 1$, Mennicke, Bass-Lazard-Serre), consequence for the arithmetic Galois action on the geometric fundamental groups of AV (ambiguity of the "congruence property", linear – arithmetic groups – and nonlinear – modular – versions).

The anabelian "philosophy" and the section conjecture.

**22. The anabelian geometry of curves**

Dimension 0 : The Neukrich and Ikeda results ; group theoretic characterization of the decomposition groups in the Galois groups of numberfields. Dimension 1 : Nakamura, Tamagawa, Mochizuki. Perhaps a sketch of the proof of Nakamura's early result : how to detect $\lambda$ via the arithmetic Galois action on the fundamental group of $\mathbb{P}^1_{\mathbb{Q}} \setminus (0, 1, \lambda, \infty)$. Usefulness of the "weights".

**23. Birational anabelian geometry**

Bogomolov's idea : the fundamental theorem of projective geometry in an infinite dimensional context. Sketch of results by Pop, Bogomolov-Tchinkel, Topaz. The local theory, etc.

**24. Ihara-Oda-Matsumoto**

The outer automorphism group of the geometric fundamental group functor : $Out(\pi_1^{geom})$. Various rough and more refined statements of "IOM". Results by Pop, Topaz. A kind of "converse" or counterpoint to Grothendieck-Teichmüller : $G_k \simeq Out(\pi_1^{geom})$ with $k$ a numberfield (e.g. $k = \mathbb{Q}$) and on the right a rather large sample (category) of schemes (quasi projective varieties) over $k$. The two groups coincides if one enlarges enough the sample of varieties but then one loses grip on the right-hand group. In the case of $GT$ the r.h.s. coincides with $GT$, one restricts attention to the moduli stacks of curves but equality remains conjectural, indeed perhaps *the* main conjecture in the subject.