Twisted Patterson-Sullivan measure and applications to growth problems

Probabilistic Methods in Negative Curvature

Rémi Coulon

March, 2021

CNRS / Université de Rennes 1

General context

Growth of groups

Let G be a group acting properly by isometries on a geodesic space X.

Goal. Measure the "size" of its orbits

Definition (Exponential growth rate)

$$h(G, X) = \limsup_{r \to \infty} \frac{1}{r} \ln \#\{g \in G \;:\; d\left(gx, x\right) \leqslant r\}.$$

Exercise. $h(G, X)$ does not depends on the point x.

Examples

$$h(G, X) = \limsup_{r\to\infty} \frac{1}{r} \ln \#\{g \in G \ : \ d\,(gx, x) \leqslant r\}.$$

Remark and question

Remark

If H is a subgroup of G, then

$$0 \leqslant h(H, X) \leqslant h(G, X).$$

Questions:

- What are the possible values of $h(H, X)$ when H runs over the subgroups of G?
- When do we have $h(H, X) = h(G, X)$?

Main goal of these lectures

Simplified theorem (C.-Dougall-Schapira-Tapie)

Let G be a group acting properly co-compactly by isometries on a Gromov hyperbolic space X. Let $N \lhd G$.

Then $h(N, X) = h(G, X)$ if and only if G/N is amenable.

Last step of a long history.

- Grigorchuk, Cohen. $G = \mathbf{F}_r$.
- Brooks. $G = \pi_1(M)$ with M hyperbolic manifold.
- Burger, Roblin, Tapie, Stadlbauer, Jaerisch, Dougall-Sharp, C.-Dal'bo-Sambusetti, etc.

The case of free groups

Kesten's criterion

Fix S symmetric generating set of $G = \mathbf{F}_r$.

Let μ be the uniform probability measure on S.

Spectral radius of the random walk in G/N:

$$\kappa = \limsup_{n \to \infty} \sqrt[n]{\mu^{*n}(N)}$$

Kesten amenability criterion

The quotient G/N is amenable if and only if $\kappa = 1$.

Kesten vs. Cohen-Grigorchuk

Almost invariants vectors and amenability

Sketch of proof

Sketch of proof

The “easy” direction

Reminder

Simplified theorem (C.-Dougall-Schapira-Tapie)

Let G be a group acting properly co-compactly by isometries on a Gromov hyperbolic space X. Let $N \lhd G$.

Then $h(N, X) = h(G, X)$ if and only if G/N is amenable.

Previously: if G is the free group, then

$$h(N, X) = h(G, X) \implies G/N \text{ amenable.}$$

The “easy” direction

Theorem (Roblin)

Let G be a group acting properly by isometries on a hyperbolic space X. Let $N \lhd G$.

If G/N is amenable, then $h(G, X) = h(N, X)$.

Proof when X is CAT(-1). Write $\bar{X} = X \cup \partial X$ for the visual compactification of X.

Patterson-Sullivan measures

Fix a base point $o \in X$ and a subgroup $H < G$.

Poincaré series

$$\mathcal{P}(s) = \sum_{h \in H} e^{-sd(o,ho)}$$

Converges if $s > h(H,X)$, diverges if $s < h(H,X)$.

Definition

The action of H on X is divergent if $P_H(s)$ diverges at $s = h(H,X)$. It is convergent otherwise.

Examples.

- $G \curvearrowright X$ proper and co-compact $\Longrightarrow$ divergent.

Patterson-Sullivan measures

For every $x \in X$, define a measure $\bar{X}$.

$$\nu_x^s = \frac{1}{P(s)} \sum_{h \in H} e^{-sd(x,ho)} \mathrm{Dirac}(ho)$$

Probability measure if $x = o$.

Up to passing to a subsequence

$$\nu_x^s \xrightarrow[s \to h_H]{\text{weak}*} \nu_x$$

Patterson-Sullivan density of H: $\quad \nu = (\nu_x)_{x \in X}$

Patterson-Sullivan measures

Main properties

- (Support) ν_x is supported on ∂X (cheated a bit)
- (Normalization) ν_0 is a probability measure
- (Equivariance) $g_*\nu_x = \nu_{gx}$ for every $g \in H$ and $x \in X$.
- (Conformality)

  $$\frac{d\nu_x}{d\nu_y}(\xi) = e^{-h_H \beta_\xi(x,y)}$$

  for every $x, y \in X$ and $\xi \in \partial X$.

Patterson-Sullivan measures

Patterson-Sullivan measures

Patterson-Sullivan measures

Shadows $\mathcal{O}_x(y, r)$.

Patterson-Sullivan measures

Shadow Lemma

Assume that $H \triangleleft G$ is normal. Let $\mu = (\mu_x)$ be an h-conformal, H-invariant density on ∂X.

There exists $C > 0$ and r, such that for every $g \in G$,

$$\frac{1}{C} \|\mu_{go}\| e^{-hd(o,go)} \leqslant \mu_o (\mathcal{O}_o(go, r)) \leqslant C \|\mu_{go}\| e^{-hd(o,go)}$$

Notation. $\|\mu_x\| = \mu_x (\partial X)$.

Patterson-Sullivan measures

Patterson-Sullivan measures

Proof of the “easy” direction

Proposition

Assume that $H \lhd G$ is normal. Let $\mu = (\mu_x)$ be an h-conformal, H-invariant density on ∂X.

The critical exponent of the series

$$\sum_{g \in G} \|\mu_{go}\| e^{-sd(o,go)}$$

is at most h.

Proof of the “easy” direction

Proof of the "easy" direction

Here N is normal in G. Let $Q = G/N$. Assume that Q is amenable.

Goal. $h(N, X) = h(G, X)$.

Averaging Patterson-Sullivan measures.

Fix a Q-invariant mean $M \colon \ell^\infty(Q) \to \mathbb{R}$.

$\nu = (\nu_x)$ Patterson-Sullivan density of N.

Given $f \in C(\partial X)$, and $x \in X$, let

$$\begin{array}{rccc} \psi_{f,x} \colon & G/N & \to & \mathbb{R} \\ & u & \mapsto & \dfrac{1}{\|\nu_{uo}\|} \displaystyle\int f d u_*^{-1} \nu_{ux}. \end{array}$$

Define an h_N-conformal N-equivariant density $\mu = (\mu_x)$ by

$$\int f d\mu_x = M(\psi_{f,x}), \quad \forall f \in C(\partial X).$$

Proof of the "easy" direction

Jensen inequality (exp is convex)

$$\|\mu_{go}\| = M(\exp \circ \ln(\psi_{\mathbb{1},go})) \geqslant \exp\left(M(\ln(\psi_{\mathbb{1},go}))\right)$$

Said differently

$$\|\mu_{go}\| \geqslant e^{\chi(g)},$$

where

$$\begin{array}{rccc} \chi\colon & G & \to & \mathbb{R} \\ & g & \mapsto & M(\ln(\psi_{\mathbb{1},go})). \end{array}$$

is a homomorphism.

Proof of the "easy" direction

$$
\begin{aligned}
\sum_{g \in G} \|\mu_{go}\| \, e^{-sd(o,go)} \geqslant \sum_{g \in G} e^{-sd(o,go)} e^{\chi(g)} &\geqslant \sum_{g \in G} e^{-sd(o,go)} \cosh \circ \chi(g) \\
&\geqslant \sum_{g \in G} e^{-sd(o,go)}
\end{aligned}
$$

Hence $h(G, X) \leqslant h(N, X)$.

The optimal statement

Main goal of these lectures

Simplified theorem (C.-Dougall-Schapira-Tapie)

Let G be a group acting properly co-compactly by isometries on a Gromov hyperbolic space X. Let $N \lhd G$.

Then $h(N, X) = h(G, X)$ if and only if G/N is amenable.

Full theorem (C.-Dougall-Schapira-Tapie)

Let G be a group acting properly by isometries on a Gromov hyperbolic space X. Assume that the action of G on X is strongly positively recurrent. Let $H < G$.

Then $h(H, X) = h(G, X)$ if and only if H is co-amenable in G.

Co-amenability

Proposition-Definition

A subgroup $H < G$ is co-amenable in G is one of the following equivalent assertions holds.

1. There exists a G-invariant mean $M\colon \ell^\infty(G/H) \to \mathbb{R}$.
2. The regular representation $\rho\colon G \to U(\mathcal{H})$ where $\mathcal{H} = \ell^2(G/H)$ almost has invariant vectors.
3. The Cheeger constant of the Schreier graph of G/H is zero.

Examples.

- Let $N \lhd G$.
  N is co-amenable in $G \iff G/N$ is amenable.

- $G = \mathbf{F}(a, b)$ $\qquad$ $H = \langle a^n b a^{-n} \ : \ n \in \mathbb{N} \rangle$

Strongly positively recurrent action

Well understood situation.

The action of G is convex co-compact if G acts properly co-compactly on a quasi-convex G-invariant subset $Y \subset X$.

Goal. Measure "how much" the action is not convex co-compact.

Strongly positively recurrent action

Let $K \subset X$ be a compact subset and let

$$G_K = \{g \in G \ : \ \exists x, y \in K, \ [x, gy] \cap GK \subset K \cup gK\}$$

Strongly positively recurrent action

The entropy at infinity of (the action of) G is

$$h_\infty(G, X) = \inf_{\substack{K \subset X \\ \text{compact}}} h(G_K, X)$$

Observation. $h_\infty(G, X) \leqslant h(G, X)$.

Definition (Schapira-Tapie)

The action of G on X is strongly positively recurrent if

$$h_\infty(G, X) < h(G, X)$$

a.k.a. statistically convex co-compact (Yang)

Strongly positively recurrent action

Examples.

- $G \curvearrowright X$ proper and co-compact.
- If G is hyperbolic relative to $\{P_1, \dots, P_m\}$ and $G \curvearrowright X$ is a cusped uniform action, then

$$h_\infty(G, X) = \min_{1 \leqslant i \leqslant m} h(P_i, X).$$

- Ancona type surfaces

Strongly positively recurrent action

Optimality of the main theorem

Full theorem (C.-Dougall-Schapira-Tapie)

Let G be a group acting properly by isometries on a Gromov hyperbolic space X. Assume that the action of G on X is strongly positively recurrent. Let $H < G$.

Then $h(H, X) = h(G, X)$ if and only if H is co-amenable in G.

Optimality of the main theorem

- Negative curvature

- Strongly positively recurrent action

Growth gap

Definition

A group G has Kazhdan Property (T) is for every unitary representation $\rho\colon G \to \mathcal{U}(\mathcal{H})$ in a Hilbert space, if ρ almost has invariant vector, then ρ has a non-zero invariant vector.

Examples.

Growth gap

Original problem:

$$\text{Describe } \{h(H, X) \ : \ H < G\}.$$

Theorem (C.-Dougall-Schapira-Tapie)

Let G be a group acting properly by isometries on a Gromov hyperbolic space X. Assume that the action of G on X is strongly positively recurrent.

If G has Property (T), then there exists $\varepsilon > 0$, such that for every subgroup $H < G$

- either $h(H, X) < h(G, X) - \varepsilon$,
- or $[G : H] < \infty$.

The “hard” direction

Initial data.

- X CAT(-1) space. G acts properly on X by isometries.

- Let $\mathcal{H}$ be a Hilbert space with a partial order $\prec$ compatible with the Hilbert structure, i.e. $\langle \phi_1, \phi_2 \rangle \geqslant 0$, whenever $\phi_1 \succ 0$ and $\phi_2 \succ 0$.
  Example. $\mathcal{H} = L^2(Y)$, where $f_1 \prec f_2 \iff \forall y \in Y,\ f_1(y) \leqslant f_2(y)$.

- Let $\rho \colon G \to \mathcal{U}(\mathcal{H})$ be a positive unitary representation, i.e. $\rho(g)\phi \succ 0$ for every $g \in G$ and $\phi \succ 0$.
  Example. If G acts on Y, then $\rho \colon G \to \mathcal{U}(L^2(Y))$ regular representation.

Twisted Poincaré series

Twisted Poincaré series.

$$A(s) = \sum_{g \in G} e^{-sd(o,go)} \rho(g).$$

(bounded operator on $\mathcal{H}$).

Convergence. Strong operator topology.

Twisted Poincaré series.

$$A(s) = \sum_{g \in G} e^{-sd(o,go)} \rho(g).$$

Critical exponent. There exists $h_\rho \in [0, h_G]$ such that

- $A(s)$ converges, if $s > h_\rho$
- $A(s)$ diverges, if $s < h_\rho$

Twisted Poincaré series

Lemma.

Let $H < G$. Let $\mathcal{H} = \ell^2(G/H)$ and $\rho\colon G \to \mathcal{U}(\mathcal{H})$ be the regular representation. Then

$$h_H \leqslant h_\rho \leqslant h_G.$$

In particular, $h_H = h_G \implies h_\rho = h_G$.

Twisted Poincaré series

$$A(s) = \sum_{g \in G} e^{-sd(o,go)} \rho(g).$$

Twisted Poincaré series

Exercise.

Let $H < G$. Let $\mathcal{H} = \ell^2(G/H)$ and $\rho\colon G \to \mathcal{U}(\mathcal{H})$ be the regular representation. Let $\phi_u \in \mathcal{H}$ be the Dirac at uH.

For every $u \in G$,

$$\langle A(s)\phi_e, \phi_u \rangle = \sum_{g \in uH} e^{-sd(o,go)}$$

$A(s)$ "combines" the Poincaré series of all H-cosets.

Twisted Poincaré series

Theorem.

Let $\mathcal{H}$ be a Hilbert space with a partial order. Let $\rho\colon G \to \mathcal{U}(\mathcal{H})$ be a positive unitary representation. The following are equivalents

1. $h_\rho = h(G, X)$,
2. ρ almost has invariant vectors.

Exercise. $(2) \implies (1)$

Twisted Patterson-Sullivan measures

Naive attempt.

Consider

$$a_x^s = \frac{1}{\|A(s)\|} \sum_{g \in G} e^{-sd(o,go)} \mathrm{Dirac}(go)\rho(g)$$

Measure on $\bar{X} = X \cup \partial X$ with value in $\mathcal{B}(\mathcal{H})$.

Other point of view. Linear functional $C(\bar{X}) \to \mathcal{B}(\mathcal{H})$.

Take the limit

$$a_x^s \xrightharpoonup[s \to h_\rho]{} a_x$$

Warning

The space of Banach valued measure is not compact!

Ultra-limit of Hilbert spaces.

Let $\omega\colon \mathcal{P}(\mathbb{N}) \to \{0,1\}$ be a *non-principal ultra-filter*, i.e.

1. $\omega(A \sqcup B) = \omega(A) + \omega(B)$ for every disjoint $A, B \subset \mathbb{N}$.
2. $\omega(\mathbb{N}) = 1$,
3. $\omega(A) = 0$, finite $A \subset \mathbb{N}$.

A property P_n holds ω-*almost surely* (ω-as) if

$$\omega\left(\{n \in \mathbb{N} \ : \ P_n \text{ is true}\}\right) = 1.$$

Given a real sequence (u_n) we say that $\lim_\omega u_n = \ell$ if

$$\forall \varepsilon > 0, \quad |u_n - \ell| < \varepsilon \ \omega\text{-as}.$$

Fact. Every real valued bounded sequences admits an ω-limit.

Ultra-limit of Hilbert spaces.

Let $(\mathcal{H}_n)$ be a sequence of Hilbert spaces.

Let

$$\prod_{\omega} \mathcal{H}_n = \left\{ (\phi_n) \in \prod_{n \in \mathbb{N}} \mathcal{H}_n \ : \ \|\phi_n\| \text{ is bounded} \right\}.$$

Pseudo-norm. $\|(\phi_n)\| = \lim_\omega \|\phi_n\|$.

$$W = \left\{ (\phi_n) \in \prod_{\omega} \mathcal{H}_n \ : \ \|(\phi_n)\| = 0 \right\} \text{ is a vector space.}$$

Proposition-Definition.

$$\mathcal{H}_\omega := \prod_{\omega} \mathcal{H}_n / W \quad \text{is a Hilbert space.}$$

Notations. $\lim_\omega \phi_n$: image of (ϕ_n) in $\mathcal{H}_\omega$.

Ultra-limit of Hilbert spaces.

For our purpose

- $(\mathcal{H}_n)$ constant sequence equal to $\mathcal{H}$.
- $\mathcal{H}_\omega$ is endowed with a partial order $\prec$ coming from the one on $\mathcal{H}$.
- If (B_n) is a bounded sequence of operators on $\mathcal{H}$ one defines $B_\omega = \lim_\omega B_n$ by

  $$B_\omega \left(\lim_\omega \phi_n \right) = \lim_\omega \left(B_n \phi_n \right).$$

  B_ω belongs to $\mathcal{B}(\mathcal{H}_\omega)$.
- In particular, $\rho \colon G \to \mathcal{U}(\mathcal{H})$ induces a positive unitary representation $\rho_\omega \colon G \to \mathcal{U}(\mathcal{H}_\omega)$

Exercise.

ρ almost have invariant vectors $\iff$ ρ_ω has a non-zero invariant vector.

Twisted Patterson-Sullivan measures

Second attempt.

Recall
$$a_x^s = \frac{1}{\|A(s)\|} \sum_{g \in G} e^{-sd(o,go)} \mathrm{Dirac}(go) \rho(g)$$

Fix a sequence $s_n > h_\rho$, converging to h_ρ.

Define a_x as follows.

$$\begin{aligned} \int f da_x &:= \lim_\omega \int f da_x^{s_n} \\ &= \lim_\omega \frac{1}{\|A(s_n)\|} \sum_{g \in G} e^{-s_n d(o,go)} f(go) \rho(g), \quad \forall f \in C(\bar{X}). \end{aligned}$$

Measure on $\bar{X}$ with values in $\mathcal{B}(\mathcal{H}_\omega)$.

Twisted Patterson-Sullivan measures

Main properties

- (Support) a_x is supported on ∂X (cheated a bit)
- (Normalization) $\int \mathbb{1} da_o$ has norm 1.
- (Twisted equivariance) $g_* a_x = \rho_\omega(g)^{-1} a_{gx}$ for every $g \in G$ and $x \in X$.
- (Conformality)
$$\frac{da_x}{da_y}(\xi) = e^{-h_\rho \beta_\xi(x,y)} \mathrm{Id}$$
for every $x, y \in X$ and $\xi \in \partial X$.

The “hard” direction (continued)

Reminder

Twisted Poincaré series.

$$A(s) = \sum_{g \in G} e^{-sd(o,go)} \rho(g) \quad \text{(bounded operator on } \mathcal{H}\text{).}$$

Critical exponent: $h_\rho \in [0, h_G]$

Theorem.

Let $\mathcal{H}$ be a Hilbert space with a partial order. Let $\rho \colon G \to \mathcal{U}(\mathcal{H})$ be a positive unitary representation. The following are equivalents

1. $h_\rho = h_G$,
2. ρ almost has invariant vectors.

Reminder

Let

$$a_x^s = \frac{1}{\|A(s)\|} \sum_{g \in G} e^{-sd(o,go)} \mathrm{Dirac}(go) \rho(g)$$

Fix a sequence $s_n > h_\rho$, converging to h_ρ.

Fix a non-principal ultra-filter ω.

Define a_x as follows.

$$\int f da_x = \lim_\omega \int f da_x^{s_n}$$

Measure on $\bar{X}$ with values in $\mathcal{B}(\mathcal{H}_\omega)$

Twisted Patterson-Sullivan density: $a = (a_x)$.

Twisted Patterson-Sullivan measures

Main properties

- (Support) a_x is supported on ∂X (cheated a bit)
- (Normalization) $\int \mathbb{1} da_o \in \mathcal{B}(\mathcal{H}_\omega)$ has norm 1.
- (Twisted equivariance) $g_* a_x = \rho_\omega(g)^{-1} a_{gx}$ for every $g \in G$ and $x \in X$.
- (Conformality)
$$\frac{da_x}{da_y}(\xi) = e^{-h_\rho \beta_\xi(x,y)} \mathrm{Id}$$
for every $x, y \in X$ and $\xi \in \partial X$.

Twisted Patterson-Sullivan measures

Shadow Lemma

Let $\nu = (\nu_x)$ be an h_G-conformal, G-invariant density on ∂X.

There exists $C > 0$ and r, such that for every $g \in G$,

$$\frac{1}{C} e^{-h_G d(o,go)} \leqslant \nu_o\left(\mathcal{O}_o(go, r)\right) \leqslant C e^{-h_G d(o,go)}$$

Twisted Patterson-Sullivan measures

Shadow Lemma

Let $\nu = (\nu_x)$ be an h_G-conformal, G-invariant density on ∂X.

There exists $C > 0$ and r, such that for every $g \in G$,

$$\frac{1}{C} e^{-h_G d(o,go)} \leqslant \nu_o\left(\mathcal{O}_o(go, r)\right) \leqslant C e^{-h_G d(o,go)}$$

Half shadow Lemma

For every $r > 0$, there exists $C > 0$, such that for every $g \in G$,

$$\left\| a_o\left(\mathcal{O}_o(go, r)\right)\right\| \leqslant C e^{-h_\rho d(o,go)}$$

Heuristic of the proof

Assumption. $h_\rho = h_G$.

Consequence. For every $g \in G$,

$$\|a_o\left(\mathcal{O}_o(go, r)\right)\| \prec e^{-h_\rho d(o,go)} \asymp e^{-h_G d(o,go)} \asymp \nu_o\left(\mathcal{O}_o(go, r)\right),$$

where ν_o (standard) Patterson-Sullivan measure of G.

In other words $a_o \ll \nu_o$.

Heuristic of the proof

Introduce

$$D = \frac{da_o}{d\nu_o} \colon \partial X \to \mathcal{B}(\mathcal{H}_\omega)$$

- Equivariance/Conformality of ν_o/a_o implies

$$D(g\xi) = \rho_\omega(g)D(\xi), \quad \forall g \in G,\ \forall \xi \in \partial X,\ \nu_o\text{-as.}$$

- Ergodicity of G on $(\partial X \times \partial X, \nu_o \otimes \nu_o)$ implies

$$D \text{ is constant } \nu_o\text{-as.}$$

Conclusion. The image of D consists of ρ_ω-invariant vectors.

From shadows to absolute continuity

Radial limit set

Given $K \subset X$ compact, define Λ^K_{rad}.

$$\Lambda_{\mathrm{rad}} = \bigcup_{\substack{K \subset X \\ \text{compact}}} \Lambda^K_{\mathrm{rad}}$$

Absolute continuity

Proposition

If the action of G on X is strongly positively recurrent, then there exists $K \subset X$ compact such that $\Lambda_{\mathrm{rad}}^{K}$ has full measure for both ν_o and a_o.

Consequence (using the Shadow Lemmas)

Absolute continuity

Corollary

There exists a unique linear continuous map

$$D\colon \mathcal{H}_\omega \to L^\infty(\partial X, \mathcal{H}_\omega)$$

such that for every $f \in C(\bar{X})$, for every $\phi \in \mathcal{H}_\omega$,

$$\left(\int f da_o\right)\phi = \int f D(\phi) d\nu_o,$$

Roughly speaking,

$$D = \frac{da_o}{d\nu_o}$$

Exploiting equivariance/conformality

Proposition

Choose $\phi \in \mathcal{H}_\omega$ and let $\Psi = D(\phi)$

For every $g \in G$,

$$\Psi \circ g = \rho_\omega(g)\Psi, \quad \nu_o\text{-a.s.}$$

Exploiting equivariance/conformality

Hopf-Tsuji-Sullivan theorem

Theorem

Let G be a group acting properly by isometries on a CAT(-1) space X. The following statements are equivalent

1. The action of G is divergent
2. The Patterson-Sullivan measure ν_o only charges the radial limit set.
3. The geodesic flow on SX/G is ergodic (for the Bowen-Margulis measure).
4. The diagonal action of Γ on $(\partial X \times \partial X, \nu_o \otimes \nu_o)$ is ergodic.

Reminder: $G \curvearrowright X$ strongly positively recurrent $\Longrightarrow \nu_o$ only charges the radial limit set.

Hopf-Tsuji-Sullivan theorem

Exploiting ergodicity

Proposition

Choose $\phi \in \mathcal{H}_\omega$. The map $\Psi = D(\phi)$ is essentially constant.

Exploiting ergodicity

Summary.

Assumption: $h_\rho = h_G$.

1. Build a twisted Patterson-Sullivan density $a = (a_x)$.
2. a_o is absolutely continuous with respect to ν_o (uses strong positive recurrence).
3. $D\colon \mathcal{H}_\omega \to L^\infty(\partial X, \mathcal{H}_\omega)$ "Radon-Nikodym" derivative.
4. D is non-zero : the norm of $\displaystyle\int \mathbb{1} da_0$ is 1.
5. Choose $\phi \in \mathcal{H}_\omega$, such that $\Psi = D(\phi)$ is non-zero.
6. Ψ is essentially constant and ρ_ω-equivariant.

Conclusion: the essential value ψ of Ψ is a non-zero invariant vector for ρ_ω.

…hence the initial representation ρ almost has invariant vectors.

That’s all folks!

Radial limit set

Proposition

If the action of G on X is strongly positively recurrent, then there exists $K \subset X$ compact such that Λ^K_{rad} has full measure for both ν_o and a_o.

Radial limit set

Radial limit set

Radial limit set