Lifting. (w/ simulators)

def composition, state lifting thm, give other dir, give intuition, sketch alg.
def ρ-structured, give invariants, see how it proves efficiency.
explain partitioning and state query compat and full range lemma.
see partitioning alg and go deep into proof.

simulation algorithm:

start w/ R = all.

on each comm bit, go to larger half of R (not only prob).

then apply partition algorithm, and choose a nice row block.

query associated indices, and move into appropriate column block.

def (ρ-structured):

(i) X, Y fixed on dom ρ, and $\mathrm{Ind}(X,Y)=\rho$ on dom ρ.

(ii) $X[\text{free}]$ has bw-min-entropy $\geq 0.95 \log m$

(iii) $Y[\text{free}]$ has deficiency $\leq n \log m$

lemma (query compatibility)

sup X has bw-min-entropy $\geq 0.95 \log m - O(1)$

Y has deficiency $\leq n \log m - 1$

then $\exists j$ st $\forall \beta \in \{0,1\}^{I_j}$,
↳ result of partition

$Y[I_j = \eta] \geq |Y| / 2^{|I_j| \cdot (m + 2\log m)}$
↳ result of partition

most surprising, I think?

proof uses full-range lemma.

simulation invariants

(i) X, Y fixed on dom ρ, and $\mathrm{Ind}(X,Y)=\rho$ on dom ρ → Claim 4.2

(ii) $X[\text{free}]$ has bw-min-entropy $\geq 0.95 \log m$ → Claim 4.3

(iii) $Y[\text{free}]$ has deficiency $\leq 2|\rho|\log m + t$ → QC lemma.

simulation efficiency

invariant (iv): $D(X[\text{free}]) \leq 2t - 0.05|\rho|\log m$ → Claim 4.4

then: $|\rho| \leq (2t - D)/0.05 \log m \leq 2t/0.05\log m = O(d)$ ok.

simulation correctness: full-range lemma.

partition algorithm:

X side
- find maximal I st $\exists\alpha$ $\log \Pr[X[I]=\alpha] > -0.95|I|\log m$.
- remove $X[I]=\alpha$ from X
- repeat until removed more than 1/2 fraction of X.

Y side
- for each (I,α) found and $\beta \in \{0,1\}^I$: "conditioned on"
- pick η that maximises $\{y[I]=\eta \mid \mathrm{Ind}(x[I], y[I]) = \beta\}$

Lifting. Prelims.

def min-entropy $H_\infty(X) = \min_x - \log \Pr[X=x]$ (cf $H(x) = \mathbb{E}_x - \log \Pr[x]$)

def bw-min-entropy $= \min_I \frac{1}{|I|} H_\infty(X[I])$. $\hookrightarrow \{x[I] : x \in X\}$, weighted appropriately.

def deficiency $D(X) = \log |X| - H_\infty(X)$

th $cc(f \circ \mathrm{Ind}) = \mathrm{depth}(f) \cdot \Theta(\log m)$ where $m = n^{1.1}$.

Warmup. Then F proof of width $\log s$

Sp $F \circ \oplus$ proof of size s.

proof. fix one of x_1/x_2 at random and to 0/1.

$(F \circ \oplus)\restriction_\rho = F$, and $\pi \restriction \rho$ proof of F.

$\Pr[C \text{ survives}] \leq (3/4)^{w(C)}$.

union bound $\to \Pr[\text{some survives}] \leq s \cdot (3/4)^{\log s} < 1$.

Lifting Query Compatibility Lemma.

lem. X has bw-min-entropy $0.95 \log m - O(1)$.
$|Y| > 2^{mn - n\log m + 1}$.
Then exists row block X_j st $\forall \beta \in \{0,1\}^I$ $\quad I = I_j$
$|Y[I = \eta]| \geq |Y| / 2^{m|I| + 2|I| \log m}$ $\quad \eta = \eta_{j,\beta}$

proof.

* show $|Y \mid Ind(\alpha, y[I]) = \beta| \geq |Y| / 2^{2|I| \log m}$.

obs we can write this as $\bigcup_{\eta} \{ Y \mid \dots \wedge y[I] = \eta \}$,
and there are $2^{m|I|}$ such sets, and the largest is the η we picked,
so by averaging we're done.

* proof by contradiction.
for all j exists β st $|Y \mid \dots = \beta| < \dots$.
let $Y_{bad} = \bigcup_j$ previous sets. claim: $|Y_{bad}| < |Y|/2$.
apply full range lemma to $Y_{missing} = Y \setminus Y_{bad}$
we have x^* st $\forall \beta \; \exists y$ st $Ind(x^*, y) = \beta$. but this breaks
the definition of $Y_{missing}$.

* proof of claim.

$\binom{n}{k} < n^k < m^{0.9k}/2$

wait
group (I, α)s by size: $\mathcal{F}_k$ has / group size k.
obs $|\mathcal{F}_k| \leq \binom{n}{k} \cdot m^k < 2^{(0.9k \log m - 1) + k \log m}$.
also we don't have an empty group, otherwise we'd pick $X_\emptyset$ as our output.

then $|Y_{bad}| \leq \sum_j |Y| / 2^{|I| \log m} =$

$$= \sum_{k=1}^{n} \sum_{\substack{j \\ |I| = k}} |Y| / 2^{k \log m} \leq |Y| \cdot \underbrace{\sum_{k=1}^{n} \frac{2^{1.9 k \log m - 1}}{2^{k \log m}}}_{\leq 1/2 \text{ geom. series.}}$$

not 0

Lifting. Full Range Lemma.

X has low-min-entropy $0.95 \log m - O(1)$.
$|Y| > 2^{mN - n\log m}$.
exists $x^* \in X$ st $\forall \beta \in \{0,1\}^N$ exists $y \in Y$ st $\mathrm{Ind}(x, y) = \beta$.

proof.

by sunflower lemma

$$\Pr_{y\ \mathrm{unif}} [\forall x,\ \mathrm{Ind}(x,y) \neq 1^n] \leq \varepsilon.$$

proof by contradiction. sp $\forall x\ \exists \beta\ \forall y\ \mathrm{Ind}(x,y) \neq \beta$.

$$\begin{aligned} |Y| &\leq |y : \forall x,\ \mathrm{Ind}(x,y) \neq \beta| \\ &\leq |y : \forall x,\ \mathrm{Ind}(x,y) \neq 1^n| \\ &= 2^{mn} \cdot \Pr[\forall x,\ \mathrm{Ind}(x,y) \neq 1^n] \\ &\leq 2^{mn} \cdot \varepsilon \end{aligned}$$

because different constraints more restrictive than agreeing constraints

Sunflower Lemma.

If X has low-min-entropy $\log r$ then

$$\Pr_{y\ \mathrm{unif}} [\forall x,\ \mathrm{Ind}(x,y) \neq 1^n] \leq \varepsilon$$

for $r \geq k \cdot \log(n/\varepsilon)$

Using FRL to prove correctness.

DT thinks that fix (ρ) is a solution. we need to prove any extension of ρ to all assignment works.
by FRL, every extension can be found in our rectangle. done.