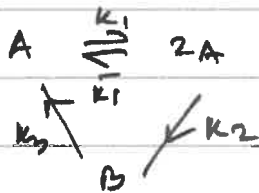


(20)

In this example, that we considered yesterday, let's write down the rate eqns.



$$Y = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} -k_1 & \bar{k}_1 & k_3 \\ k_1 & -\bar{k}_1 - k_2 & 0 \\ 0 & k_2 & -k_3 \end{pmatrix}$$

$$\Rightarrow YA = \begin{pmatrix} k_1 & -\bar{k}_1 - 2k_2 & k_3 \\ 0 & k_2 & -k_3 \end{pmatrix}$$

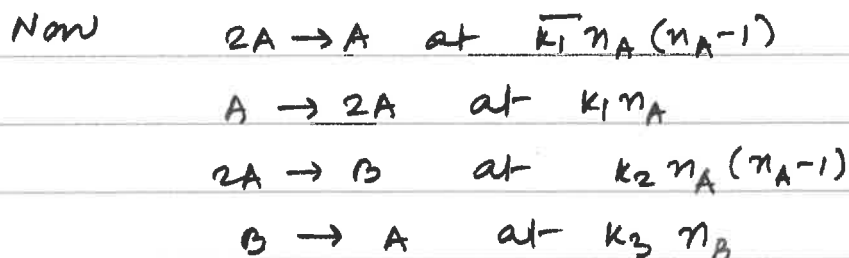
$$\Rightarrow \frac{d}{dt} \begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} k_1 & -\bar{k}_1 - 2k_2 & k_3 \\ 0 & k_2 & -k_3 \end{pmatrix} \begin{pmatrix} x_A \\ x_A^2 \\ x_B \end{pmatrix}$$

So we get

$$\begin{cases} \frac{d}{dt} x_A = k_1 x_A - (\bar{k}_1 + 2k_2) x_A^2 + k_3 x_B \\ \frac{d}{dt} x_B = k_2 x_A^2 - k_3 x_B \end{cases}$$

compare with "complex-balance" conditions.

(21)

Stochastic ModelingMaster eqn

$$\begin{aligned}
 \frac{d}{dt} P(n_A, n_B) = & P(n_A + 1, n_B) \bar{k}_1 n_A (n_A + 1) - \bar{k}_1 n_A (n_A - 1) P(n_A, n_B) \\
 & + P(n_A - 1, n_B) k_1 (n_A - 1) - k_1 n_A P(n_A, n_B) \\
 & + P(n_A + 2, n_B - 1) k_2 (n_A + 2) (n_A + 1) \\
 & \quad - P(n_A, n_B) k_2 n_A (n_A - 1) \\
 & + P(n_A - 1, n_B + 1) k_3 (n_B + 1) - P(n_A, n_B) k_3 n_B
 \end{aligned}$$

$$\text{std st} \Rightarrow \text{RHS} = 0$$

Detailed-balance $\Rightarrow$ pair-wise cancellations in RHS.
 Similarly, we will see that "complex-balance"
 gives us some rules for setting $\text{RHS} = 0$ &
 solving for the prob. distb.

Before solving for the steady-state, let's get
 the equation for the first moments $\langle n_A \rangle$ & $\langle n_B \rangle$

(22)

$$\begin{aligned}
\frac{d}{dt} \langle n_A \rangle = & \sum_{n_A, n_B} \bar{k}_1 \left[n_A^2 (n_A + 1) P(n_A + 1, n_B) - n_A^2 (n_A - 1) P(n_A, n_B) \right] \quad (1) \\
& + \sum_{n_A, n_B} k_1 \left[n_A (n_A - 1) P(n_A - 1, n_B) - n_A^2 P(n_A, n_B) \right] \quad (2) \\
& + \sum_{n_A, n_B} k_2 \left[n_A (n_A + 1) (n_A + 2) P(n_A + 2, n_B - 1) \right. \\
& \quad \left. - n_A^2 (n_A - 1) P(n_A, n_B) \right] \quad (3) \\
& + \sum_{n_A, n_B} k_3 \left[(n_A + n_B + 1) P(n_A - 1, n_B + 1) \right. \\
& \quad \left. - n_A n_B P(n_A, n_B) \right] \quad (4)
\end{aligned}$$

$$\text{In } (1) \sum_{n_A} \bar{k}_1 n_A^2 (n_A + 1) P(n_A + 1, n_B) \rightarrow \sum_{n_A} \bar{k}_1 (n_A - 1)^2 n_A P(n_A, n_B)$$

$$\Rightarrow (1) \rightarrow -\bar{k}_1 \langle n_A (n_A - 1) \rangle$$

Similarly

$$(2) \rightarrow k_1 \langle n_A \rangle$$

$$(3) \rightarrow -2k_2 \langle n_A (n_A - 1) \rangle$$

$$(4) \rightarrow k_3 \langle n_B \rangle$$

$$\Rightarrow \frac{d}{dt} \langle n_A \rangle = k_1 \langle n_A \rangle - (\bar{k}_1 + 2k_2) \langle n_A (n_A - 1) \rangle + k_3 \langle n_B \rangle$$

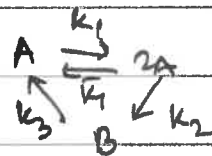
$$\frac{d}{dt} \langle n_B \rangle = k_2 \langle n_A (n_A - 1) \rangle - k_3 \langle n_B \rangle$$

This looks pretty close to the equations we get in the deterministic description!

12 13

In addition, we can check

$$\frac{d}{dt} \begin{bmatrix} \langle n_A \rangle \\ \langle n_B \rangle \end{bmatrix} = Y_A \begin{bmatrix} \langle n_A \rangle \\ \langle n_A(n_A-1) \rangle \\ \langle n_B \rangle \end{bmatrix}$$



$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -k_1 - k_1 & k_3 \\ k_1 - k_1 - k_2 & 0 \\ 0 & k_2 - k_3 \end{bmatrix} \begin{bmatrix} \langle n_A \rangle \\ \langle n_A(n_A-1) \rangle \\ \langle n_B \rangle \end{bmatrix}$$

So the rate eqns we got before look like the eqns. for the 1st moment.

ACK showed that the steady-state prob. distn can be obtained by "complex-balancing" the probabilities

$$(1) \quad P(n_A+1, n_B) \bar{k}_1 n_A(n_A+1) - \bar{k}_1 n_A P(n_A, n_B) + k_3 (n_B+1) P(n_A-1, n_B+1) = 0$$

$$(2) \quad k_1 (n_A-1) P(n_A-1, n_B) - k_2 n_A (n_A-1) P(n_A, n_B) = 0 - \bar{k}_1 n_A (n_A-1) P(n_A, n_B)$$

$$(3) \quad k_2 (n_A+1)(n_A+2) P(n_A+2, n_B-1) - k_3 n_B P(n_A, n_B) = 0$$

Product of

2 and a Poisson solves these eqns [2 by

Perron-Frobenius, this is guaranteed to be the soln.]

$$P(n_A, n_B) = \frac{(x_A)^{n_A}}{n_A!} e^{-x_A} \cdot \frac{(x_B)^{n_B}}{n_B!} e^{-x_B} \quad \begin{matrix} x_A = \langle n_A \rangle \\ x_B = \langle n_B \rangle \end{matrix}$$

$$(1) \Rightarrow \bar{k}_1 x_A^2 - k_1 x_A + k_3 x_B = 0$$

$$(2) \Rightarrow k_1 - (k_2 + \bar{k}_1) x_A = 0$$

$$(3) \Rightarrow k_2 x_A^2 - k_3 x_B = 0$$

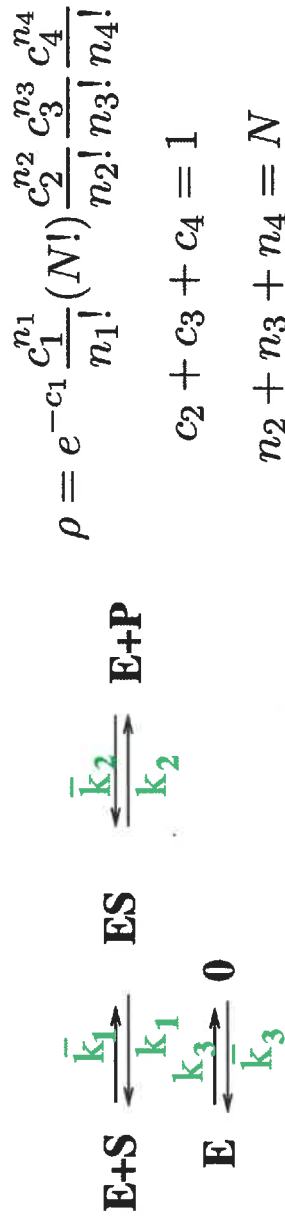
exactly what we got before for the deterministic system!

Anderson-Craciun-Kurtz (ACK) Theorem

Theorem

If a CRN modeled deterministically is complex balanced with a complex balanced fixed point, then the stochastically modeled (mass action) system has a product-form stationary distribution.

Anderson, Craciun and Kurtz, *Bull. Math. Biol.* 72,1947, 2010

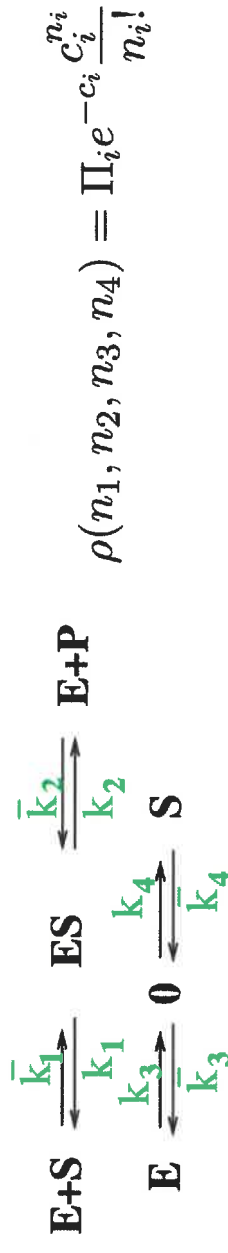


Anderson-Craciun-Kurtz (ACK) Theorem

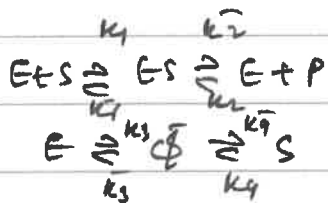
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EX



$$P(n_E, n_S, n_{ES}, n_P) = \frac{(n_E)!}{n_E!} e^{-n_E} \frac{(n_S)!}{n_S!} e^{-n_S} \frac{(n_{ES})!}{n_{ES}!} e^{-n_{ES}} \frac{(n_P)!}{n_P!} e^{-n_P}$$

$\{n_E, n_S, n_{ES}, n_P\}$ are obtained from the deterministic description

$$(1) (n_E n_S) k_1 P(n_E, n_S, n_{ES}, n_P) = \bar{k}_1 P(n_E - 1, n_S - 1, n_{ES} + 1, n_P) (n_{ES} + 1)$$

this gives $k_1 n_E n_S = \bar{k}_1 n_{ES}$

$$(2) \bar{k}_1 n_{ES} P(n_E, n_S, n_{ES}, n_P) = k_1 P(n_E + 1, n_S + 1, n_{ES} - 1, n_P) (n_E + 1) (n_S + 1)$$

which gives the same condition

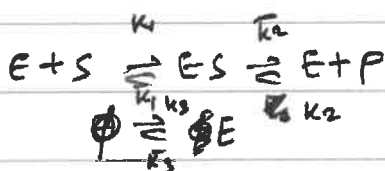
Here complex balance is in fact detail balanced.

So the pop factorised distribution holds

if we can solve for the $\{n_E, n_S, n_{ES}, n_P\}$.

But from the $\delta=0$ theorem, we know we can do that. so the ACK theorem holds.

EX



$$(N_S + N_{ES} + N_P = N \text{ (conserved)})$$

$$A \rightarrow \begin{bmatrix} \Phi & E & E+S & ES & E+P \\ -k_3 & \bar{k}_3 & 0 & 0 & 0 \\ k_3 & -k_3 & 0 & 0 & 0 \\ 0 & 0 & -k_1 & -\bar{k}_1 & 0 \\ 0 & 0 & k_1 & -\bar{k}_2 + \bar{k}_1 & k_2 \\ 0 & 0 & 0 & \bar{k}_2 & -k_2 \end{bmatrix}$$

(25)

"complex balance" gives

$$① \quad \bar{\kappa}_3 P(n_E - 1, n_S, n_{ES}, n_P) = \bar{\kappa}_3 n_E P(n_E, n_S, n_{ES}, n_P)$$

$$② \quad \bar{\kappa}_1 P(n_E - 1, n_S - 1, n_{ES} + 1, n_P) = \kappa_1 n_E n_S P(n_E, n_S, n_{ES}, n_P)$$

$$(n_{ES} + 1)$$

$$= (\bar{\kappa}_1 + \bar{\kappa}_2) n_{ES} P(n_E, n_S, n_{ES}, n_P)$$

$$③ \quad \left\{ \begin{aligned} & \kappa_1 (n_E + 1) (n_S + 1) P(n_E + 1, n_S + 1, n_{ES} - 1, n_P) \\ & + \kappa_2 P(n_E + 1, n_S, n_{ES} - 1, n_P + 1) \end{aligned} \right.$$

$$(n_E + 1) (n_P + 1)$$

if we put our ansatz

$$P(n_E, n_S, n_{ES}, n_P) \sim \frac{(\kappa_E)^{n_E} (\kappa_S)^{n_S} (\kappa_P)^{n_P} (\kappa_{ES})^{n_{ES}}}{n_E! n_S! n_P! n_{ES}!} \text{ Norm}$$

we get

$$① \quad \bar{\kappa}_3 \kappa_E = \kappa_3$$

$$② \quad \kappa_1 \kappa_E \kappa_S = \bar{\kappa}_1 \kappa_{ES}$$

$$③ \quad (\bar{\kappa}_1 + \bar{\kappa}_2) \kappa_{ES} = \kappa_1 \kappa_E \kappa_S + \kappa_2 \kappa_E \kappa_P$$

the same eqns we got before!

$$x_S + x_P + x_{ES} = N$$

$$\text{So } P(n_E, n_S, n_{ES}, n_P) = \frac{e^{-x_E} (x_E)^{n_E}}{n_E!} \left(\frac{N!}{n_S! n_P! n_{ES}!} \right) (\kappa_S)^{n_S} (\kappa_P)^{n_P} (\kappa_{ES})^{n_{ES}}$$

$$c_S + c_P + c_{ES} = 1 \quad \left(= \frac{x_S + x_P + x_{ES}}{N} \right)$$

$$\text{Norm} = \frac{N!}{N^N}$$