# Non-equilibrium and periodically driven quantum systems

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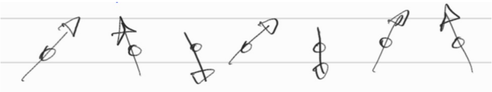
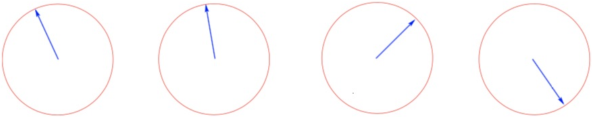
(Lecture 3)



$H = -\sum_j (g\sigma_j^x + \sigma_j^z \sigma_{j+1}^z)$ (interacting wrt "physical spins")

- The same analysis goes through for a time-dependent $g(t)$ (useful for Floquet problems with $g(t+T) = g(t)$)
  $H_k = 2(g(t) - \cos(k))\tau_k^z + 2\sin(k)\tau_k^x$

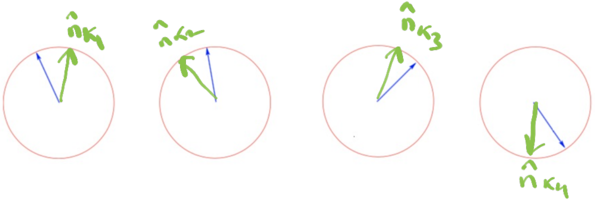


- $i\frac{d}{dt}|\psi_k\rangle = H_k(t)|\psi_k\rangle$
  $|\psi_k\rangle = u_k(t)|\uparrow\rangle_k + v_k(t)|\downarrow\rangle_k$
  $|\psi(t)\rangle = \otimes_{k>0}|\psi_k(t)\rangle$

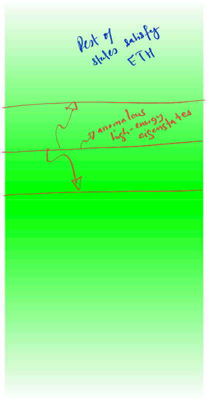
# Floquet version of $S = 1/2$ TFIM

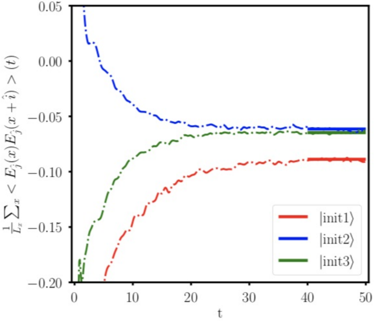
- Let $g(t) = g(t+T)$. Energy is no longer conserved
- However, again there are an extensive number of new conserved quantities if we look at stroboscopic times (e.g., $t = 0, T, 2T, \cdots, nT, \cdots$)
- $|\psi(nT)\rangle = U(T)^n|\psi(0)\rangle$ gives a steady state for local operators when $n \to \infty$
- Dynamics of pseudospins through a time-dependent magnetic field $(\sin(k), 0, g(t) - \cos(k))$ now


- $U_k(T) = \exp(-iH_k(g_2)T/2)\exp(-iH_k(g_1)T/2) = \exp(-i(\hat{n}_k \cdot \vec{\tau})T|\epsilon_k^F|)$
- Thus, GGE replaced by periodic-GGE in this case where the static magnetic field at each $k$ for a constant $g$ replaced by an "effective" field $T|\epsilon_k^F|\hat{n}_k$ at each $k$.



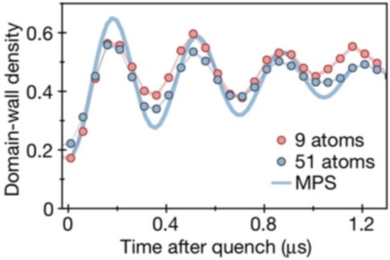
# Many-body quantum scars

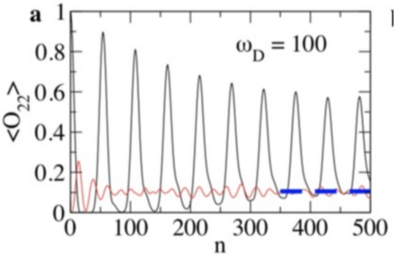



- Anomalous high-energy eigenstates embedded in an otherwise ETH-respecting spectrum
- While most initial states thermalize, some simple initial states may show very anomalous dynamics

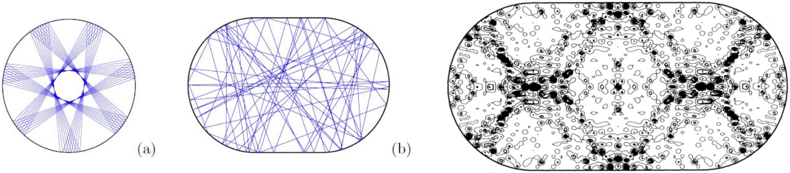
# Experimental realization of weak ergodicity breaking




- Bernien et al., Nature 551, 579 (2017) realized programmable quantum spin model with tunable interactions $\sim$ 51 qubits.
- Certain initial conditions took much longer to relax

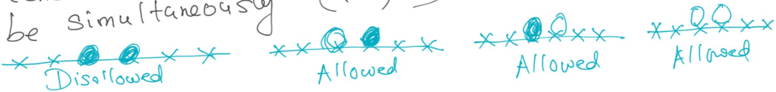
Persistant oscillations in $|1010\cdots\rangle \rightarrow |0101\cdots\rangle \rightarrow |1010\cdots\rangle$ while $|0000\cdots\rangle$ rapidly thermalized




- Classical circular billiard $\rightarrow$ integrable
- Classical stadium billiard $\rightarrow$ chaotic
- However, unstable periodic orbits still exist (classical scars)
- Quantum version also has certain eigenfunctions with anomalous enhancement along unstable orbits Heller (1984)
- Many-body versions??

$\boxed{\text{Minimal } S=\tfrac{1}{2} \text{ model to capture the experiment}}$

* $\sigma_i^z = +1$ denotes a Rydberg excitation
  $= -1$ denotes a Rydberg ground state
* $\sigma_i^x$ converts a Rydberg excitation to ground state and vice-versa
* Strong Rydberg blockade modeled by a hard-constraint that no two neighboring spins can be simultaneously $(+1, +1)$



With this constraint, let us consider the $\{S^z\}$ basis with pbc. The no. of allowed states is clearly much less than $2^L$ (with $L$ sites).

* A transfer matrix approach gives the answer

Let $A = \begin{array}{cc} ++ & +- \\ \left(\begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix}\right) & \\ -+ & -- \end{array}$ ← entry for $++=0$ to reflect hard constraint

$\therefore$ With PBC, the # of states equal

$$\mathrm{Tr}\,(A^L) = F_{L-1} + F_{L+1}$$

where $F_n + F_{n+1} = F_{n+2}$, $F_1 = F_2 = 1$ Fibonacci numbers

When $L \gg 1$, # of states $= \varphi^L$ where $\varphi = \frac{1+\sqrt{5}}{2} \simeq 1.618033\ldots$

$H = \sum_i \sigma_i^x$ does not satisfy this hard-constraint

$$\ldots 00\,1\,\overset{\sigma}{0}\,1\,00\ldots \Rightarrow \ldots 0011100\ldots$$

violates constraints

* What is the most local quantum dynamics?

$$\boxed{H_{PXP} = -w \sum_i P_{i-1}^{\downarrow}\, \sigma_i^x\, P_{i+1}^{\downarrow}}$$

(let's assume pbc)

where $P_j^{\downarrow} = \left(\frac{1-\sigma_j^z}{2}\right) \Rightarrow \left.\begin{array}{l} P_j^{\downarrow}|\uparrow_j\rangle = 0 \\ P_j^{\downarrow}|\downarrow_j\rangle = |\downarrow_j\rangle \end{array}\right\}$ projector

$$\ldots 000 \ldots \Rightarrow \ldots 010 \ldots$$

Now, the hard-constraint is satisfied.

A generalized version of this model first appeared in Sachdev, Sengupta, Girvin, arXiv:cond-mat/0205169

*Mott insulators in strong electric fields, Sachdev, Sengupta, Girvin (2002)*

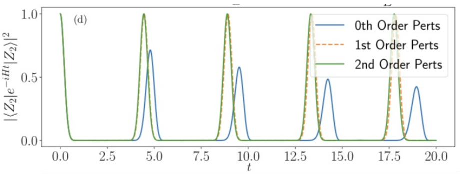
- $L$ spin-$1/2$ on a 1D lattice with a constrained Hilbert space, not all $2^L$ configurations allowed

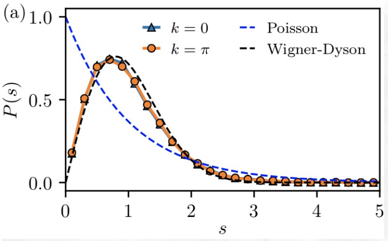
  $\boxed{\cdots\downarrow\downarrow\cdots, \cdots\uparrow\downarrow\cdots, \cdots\downarrow\uparrow\cdots, \cdots\uparrow\uparrow\cdots}$

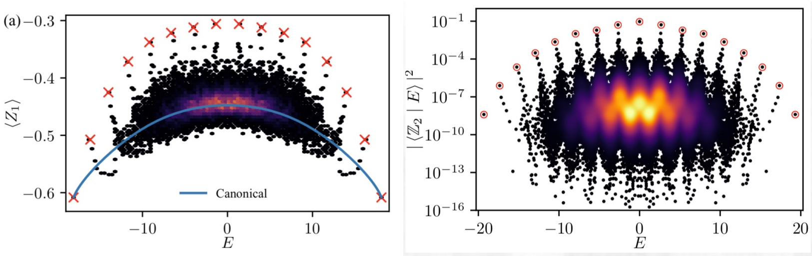
- HSD for $L$ sites equals $F_{L-1} + F_{L+1}$ where $F_1 = F_2 = 1$ and $F_n + F_{n+1} = F_{n+2}$; $HSD = \varphi^L$ for $L \gg 1$ where $\varphi = \frac{\sqrt{5}+1}{2}$

- $H_{\mathrm{PXP}} = -w\sum_i P^{\downarrow}_{i-1}\sigma^x_i P^{\downarrow}_{i+1}$ $\boxed{\cdots\downarrow\uparrow\downarrow\cdots \Leftrightarrow \cdots\downarrow\downarrow\downarrow\cdots}$

- $P(s) = \frac{\pi}{2} s e^{-\frac{\pi}{4}s^2}$ (Wigner-Dyson), $P(s) = e^{-s}$ (Poisson)



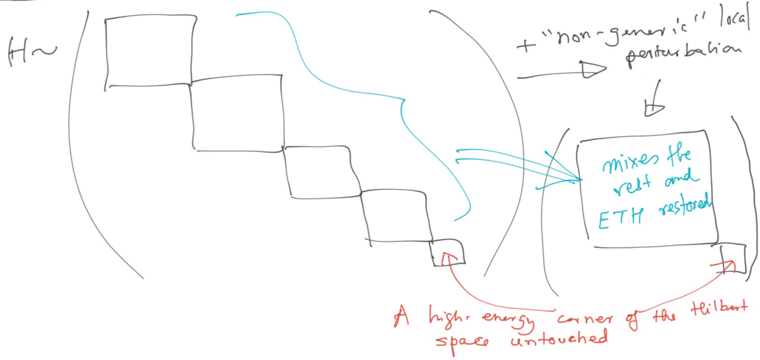



Turner *et al.*, Nat. Phys. 14, 745 (2018), PRB 98, 155134 (2018)

- Subspace spanned by $O(L)$ special eigenstates. High overlap with Néel state of alternating $\uparrow$ and $\downarrow$
- Approximately equally spaced eigenvalues
- These eigenstates are highly atypical

  (half-chain entanglement entropy $S \sim \ln L$ instead of $S \sim L$)
- Reminds one of a “giant” $SU(2)$ spin “protected” in the Hilbert space

## "Non-generic" breaking of symmetry



## An example

* Consider $L$ $s=\frac{1}{2}$ spins on a ring (pbc)
* Define $\mathcal{D}$ as common null space of $L$ projectors $P^{singlet}_{i,i+1}$ → n.n.

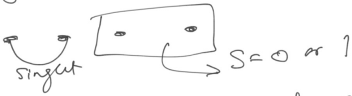
$$P^{singlet}_{i,i+1} |singlet\rangle = |singlet\rangle$$
$$P^{singlet}_{i,i+1} |triplet\rangle = 0$$

$$P^{singlet}_{i,i+1} \propto 1 - \vec{\sigma}_i \cdot \vec{\sigma}_j$$

* What is $\mathcal{D}$ ?
* Go to largest possible spin $S = \frac{L}{2}$ → Total $L+1$ states

All these $(L+1)$ states annihilated by all $P^{singlet}_{i,i+1}$ no matter where $i$ is.

E.g. take four spin $S = \frac{1}{2}$




$\Rightarrow$ $\therefore$ total $S = 0$ or $1$

The $S = 2$ states do not have any

$$H_{toy} = \frac{\Omega}{2} \sum_i \sigma_i^x + \sum_i \underbrace{V_{i-1,\, i+2}}_{\text{these need only be diff. from } i,\, i+1} P^{\text{singlet}}_{i,\, i+1}$$

$$V_{i,j} = \sum_{\mu,\nu} J_{ij}^{\mu\nu} \sigma_i^\mu \sigma_j^\nu$$

This term still annihilates the $(L+1)$ states in 2D

$H_{toy}$ does not commute with either $P^{\text{singlet}}_{i,\, i+1}$ or $S^x = \sum_i \sigma_i^x$

Thus, $\left| S = \frac{L}{2},\ S^x_{tot} = m_x \right\rangle \in 2)$ still

eigenstates with $E = \frac{\Omega}{2} m_x$

Suppose, I start with an initial state

$\left| S = \frac{L}{2},\ S^z_{tot} = \frac{L}{2} \right\rangle \longrightarrow$ undergoes perfect oscillations $\boxed{\omega_{osc} \sim \Omega}$

The other initial states do not do this.

- An exact Krylov subspace in the spectrum of an interacting model

$$|\mathbb{Z}_2\rangle = \underset{1}{\bullet}\ \underset{2}{\circ}\ \underset{3}{\bullet}\ \underset{4}{\circ}\ \underset{5}{\bullet}\ \circ \cdots$$

$$|\mathbb{Z}_2'\rangle = \circ\ \bullet\ \circ\ \bullet\ \circ\ \bullet \cdots$$

$$H_+ = \sum_{j\in e} P^{\downarrow}_{j-1}\sigma_j^+ P^{\downarrow}_{j+1} + \sum_{j\in o} P^{\downarrow}_{j-1}\sigma_j^- P^{\downarrow}_{j+1}$$

$$H_- = \sum_{j\in e} P^{\downarrow}_{j-1}\sigma_j^- P^{\downarrow}_{j+1} + \sum_{j\in o} P^{\downarrow}_{j-1}\sigma_j^+ P^{\downarrow}_{j+1}$$

$$\boxed{\begin{array}{l} H_{PXP} = H_+ + H_- \text{ s.t.} \\ H_+ = (H_-)^\dagger \\ \text{chosen in such a way} \\ \text{s.t. } H_- \text{ annihilates } |\mathbb{Z}_2\rangle \\ \text{and } H_+ \ \ {}'' \ \ |\mathbb{Z}_2'\rangle \end{array}}$$

$$\left(|\mathbb{Z}_2\rangle, H_+|\mathbb{Z}_2\rangle, (H_+)^2|\mathbb{Z}_2\rangle, \ldots, \underbrace{(H_+)^L|\mathbb{Z}_2\rangle}_{=|\mathbb{Z}_2'\rangle}\right) \to \underline{\underline{L+1 \text{ states}}}$$

form an orthogonal $(L+1)$-dim Krylov-like space

Krylov space $\rightarrow$ $\underbrace{H^0|\psi\rangle, H^1|\psi\rangle, H^2|\psi\rangle, \ldots, H^K|\psi\rangle}$ used heavily in Lanczos algorithm in Exact diagonalization

$H_+$ and $H_-$ act as raising and lowering operator of a fictitious $S = \frac{L}{2}$ spin. Revivals are like precession of this large spin

Write $H = H_+ + H_-$ in this orthogonal $(L+1)$ basis:

$H_{FSA}$ =

$(L+1) \times (L+1)$ → Tridiagonal matrix.

polynomial complexity!

→ $H_{FSA} = \sum_{n=0}^{L} \beta_n \left( |n\rangle\langle n+1| + h.c. \right)$

$\beta_n = \langle n+1 | H_+ | n \rangle = \langle n | H_- | n+1 \rangle$

$$\text{error}(n) = \left\| \frac{\langle n | H_+ H_- | n \rangle}{\beta_{n-1}^2} - 1 \right\|$$

* for $L=32$, maximum error(n) $\approx 0.2\%$

→ Hilbert space dim = 4870847

* for $L=32$, eigenvalue error from FSA → $\frac{\Delta E}{E} \approx 1\%$

For SU(2) algebra,

$S_z = \frac{1}{2}[S_+, S_-]$

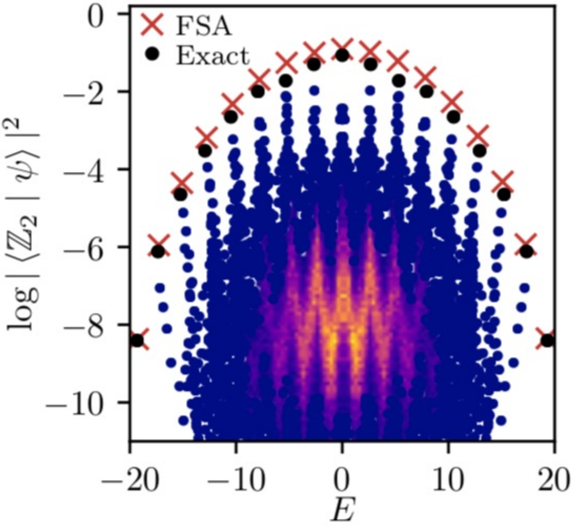
$\pm S_\pm = [S_z, S_\pm]$

Here define $H_z = \frac{1}{2}[H_+, H_-]$

But $[H_z, H_+] = H_+ + \delta$

error

# Forward scattering approximation in PXP



$H_z = \frac{1}{2}[H_+, H_-]$

$[H_z, H_+] = H_+ + \delta \longrightarrow \delta = \sum_n c_n V_n$

Replace $H_+ \to H_+ + \sum \lambda_n V_n$ and $H_- = (H_+)^\dagger$

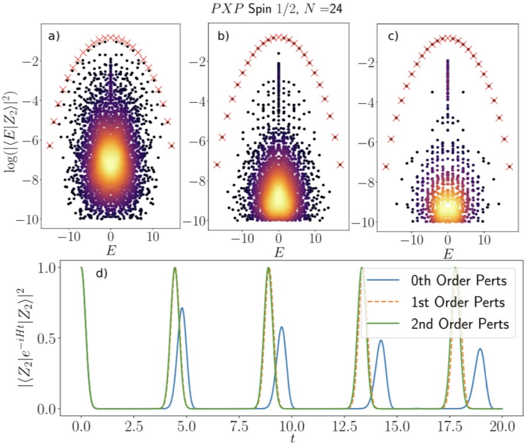
This becomes $H = H_+ + \hat{H}_- = H_{PXP} +$ additional terms

* Optimize $\lambda_n$ to enhance first revival in fidelity $|\langle\psi(0)| e^{-iHt} |\psi(0)\rangle|^2$

* $H_{deformed} = H_{PXP} + \lambda \left( P^{\downarrow}_{n-1} \sigma^x_n P^{\downarrow}_{n+1} P^{\downarrow}_{n+2} + P^{\downarrow}_{n-2} P^{\downarrow}_{n-1} \sigma^x_n P^{\downarrow}_{n+1} \right)$

where $\lambda \cong 0.1$

Fig from Bull, Desaules and Papić, arXiv:2001.08232




Similar approach of introducing small perturbations greatly enhance revivals from $|\mathbb{Z}_3\rangle$ and $|\mathbb{Z}_4\rangle$ as well