Non-equilibrium and periodically driven quantum systems

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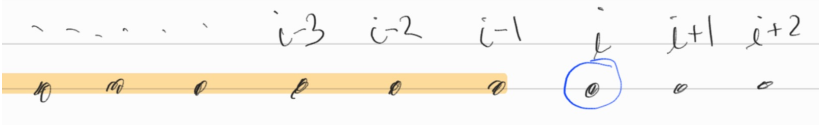
(Lecture 2)

$S = 1/2$ transverse field Ising model in 1D

- $H = -\sum_j (g\sigma_j^x + \sigma_j^z \sigma_{j+1}^z)$ (assume pbc s.t. $\sigma_{L+1}^\alpha = \sigma_1^\alpha$ with $\alpha = x, y, z$)
- $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- For $L \to \infty$, the ground state ferromagnetic when $-1 < g < 1$ and paramagnetic otherwise, with continuous quantum critical points at $g = \pm 1$
- This model can be solved exactly for any finite L using a well-known mapping of the spins to spinless fermions (**Jordan-Wigner transformation**) [e.g., see S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, 2011)]



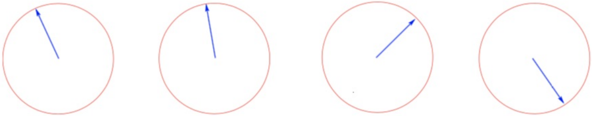
- The JW transformation can also be used for calculating unitary dynamics
- $\sigma_i^x = 1 - 2c_i^\dagger c_i$, $\sigma_i^z = -\left[\prod_{j<i}(1-2c_j^\dagger c_j)\right](c_i^\dagger + c_i)$
  Schematically, spin = fermion $\times$ string where string = $+1$ (-1) if total no. of fermions on string left of site i is even (odd)
- Vacuum of c-fermions $|0\rangle$ is the state with $|\cdots \rightarrow\rightarrow\rightarrow\rightarrow \cdots\rangle$ (i.e., $\sigma_i^x = +1$ for all sites i) [ground state when $g \rightarrow \infty$].

- $H = -\sum_j (g\sigma_j^x + \sigma_j^z \sigma_{j+1}^z)$ (assume pbc)
- $\sigma_i^x = 1 - 2c_i^\dagger c_i$, $\sigma_j^z = -\left[\prod_{j<i}(1 - 2c_j^\dagger c_j)\right](c_i^\dagger + c_i)$
- $H = 2g\sum_{j=1}^{L} c_j^\dagger c_j - \sum_{j=1}^{L-1}(c_j^\dagger c_{j+1} + c_j^\dagger c_{j+1}^\dagger + \mathrm{h.c.}) + (-1)^{N_F}[c_L^\dagger c_1 + c_L^\dagger c_1^\dagger + \mathrm{h.c.}]$
- We restrict to even N_F since ground state at any g has even number of fermions. This implies $c_{L+1} = -c_1$
- $c_k = \frac{\exp(i\pi/4)}{\sqrt{L}} \sum_j \exp(-ikj) c_j$ with $k = \frac{2\pi m}{L}$ with $m = -\frac{L-1}{2}, \cdots, -\frac{1}{2}, \frac{1}{2}, \cdots, \frac{L-1}{2}$

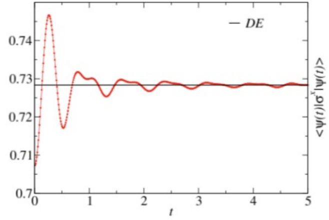
- $H = \sum_{k>0} H_k$ where
  $H_k = 2(g - \cos(k))[c_k^\dagger c_k - c_{-k} c_{-k}^\dagger] + 2\sin(k)[c_{-k} c_k + c_k^\dagger c_{-k}^\dagger]$
- Since N_F is even, need to only consider processes where fermions are created/destroyed in pairs
- Introduce **pseudospins** $|\uparrow\rangle_k = |k, -k\rangle = c_k^\dagger c_{-k}^\dagger |0\rangle$ and $|\downarrow\rangle_k = |0\rangle$.
- $H_k = 2(g - \cos(k))\tau_k^z + 2\sin(k)\tau_k^x$

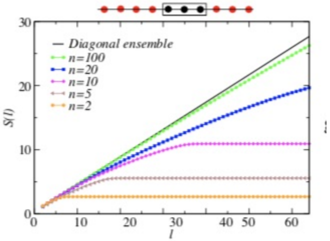
$H = -\sum_j (g\sigma_j^x + \sigma_j^z \sigma_{j+1}^z)$ (interacting wrt “physical spins”)

- The same analysis goes through for a time-dependent $g(t)$ (useful for Floquet problems with $g(t+T) = g(t)$)
  $H_k = 2(g(t) - \cos(k))\tau_k^z + 2\sin(k)\tau_k^x$



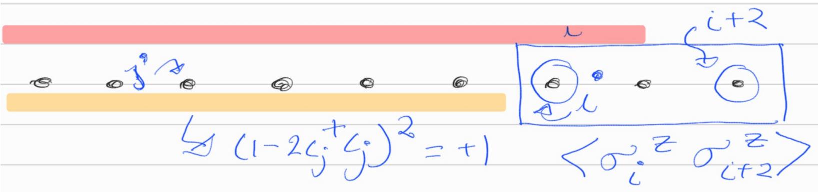
- $i\frac{d}{dt}|\psi_k\rangle = H_k(t)|\psi_k\rangle$
  $|\psi_k\rangle = u_k(t)|\uparrow\rangle_k + v_k(t)|\downarrow\rangle_k$
  $|\psi(t)\rangle = \otimes_{k>0}|\psi_k(t)\rangle$



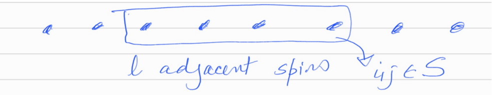


Several local quantities and even the entanglement entropy of l adjacent spins can be calculated for a quench from a simple initial state, e.g. $|\psi(0)\rangle = \otimes_{k>0}|\downarrow\rangle_k$ which is nothing but the c-fermion vacuum or the state $|\cdots \rightarrow\rightarrow\rightarrow \cdots\rangle$ in terms of the physical spins

Correlations involving σ_i^z harder to calculate due to JW string



- Calculating entanglement entropy where subsystem has l adjacent “physical” spins in terms of the fermions
- All correlation functions with an even number of σ_i^z (for example) defined in the subsystem can be expressed in terms of fermions within the subsystem (JW strings outside the subsystem cancel in pairs)
- Since the fermions are non-interacting, all higher point fermionic correlations can be expressed in terms of two-point fermionic correlations



- The two-point fermionic correlations can be expressed in terms of two $l \times l$ matrices. Denote by **C** and **F** where $C_{ij} = \langle\psi(t)|c_i^\dagger c_j|\psi(t)\rangle$ and $F_{ij} = \langle\psi(t)|c_i^\dagger c_j^\dagger|\psi(t)\rangle$
- On general grounds, $\rho_S = \frac{1}{Z_s}\exp(-H_S)$ where $H_S = \sum_{l=1}^{l} \epsilon_\alpha \eta_\alpha^\dagger \eta_\alpha$. The factor Z_s ensures that $\mathrm{Trace}(\rho_s) = 1$.
- Bogoluibov transformation that gives diagonal representation $\eta, \eta^\dagger$ from $c, c^\dagger$ for the subsystem

A simple problem

$$H = \epsilon(c_1^\dagger c_1 + c_2^\dagger c_2) + \lambda(c_1^\dagger c_2^\dagger + c_2 c_1)$$

Assume $\lambda \in \mathbb{R}$ for simplicity

Let
$$\left.\begin{aligned} c_1^\dagger &= u d_1^\dagger + v d_2 \\ c_2^\dagger &= u d_2^\dagger - v d_1 \end{aligned}\right\}$$
where u and v can be chosen to be real since $\lambda \in \mathbb{R}$

Assume that $d_1, d_1^\dagger, d_2, d_2^\dagger$ also satisfy fermionic algebra. Then easy to see that $\{c_1^\dagger, c_2^\dagger\} = \{c_1, c_2\} = 0$ already

$$\{c_1, c_1^\dagger\} = u^2\{d_1, d_1^\dagger\} + v^2\{d_2, d_2^\dagger\} = u^2 + v^2 = 1$$

Parametrize $u = \cos\theta$ and $v = \sin\theta$

$$\mathcal{H} = \frac{1}{2} \begin{pmatrix} c_1^\dagger & c_2 & c_2^\dagger & c_1 \end{pmatrix} \left(\begin{array}{cc|cc} \varepsilon & \lambda & 0 & 0 \\ \lambda & -\varepsilon & 0 & 0 \\ \hline 0 & 0 & \varepsilon & -\lambda \\ 0 & 0 & -\lambda & -\varepsilon \end{array}\right) \begin{pmatrix} c_1 \\ c_2^\dagger \\ c_2 \\ c_1^\dagger \end{pmatrix}$$

+ constant

Focus on the upper block $\rightarrow$ $\begin{pmatrix} c_1^\dagger & c_2 \end{pmatrix} \begin{pmatrix} \varepsilon & \lambda \\ \lambda & -\varepsilon \end{pmatrix} \begin{pmatrix} c_1 \\ c_2^\dagger \end{pmatrix}$

The transformation is

$$\begin{pmatrix} c_1 \\ c_2^\dagger \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} d_1 \\ d_2^\dagger \end{pmatrix}$$

Then, $\begin{pmatrix} c_1^\dagger & c_2 \end{pmatrix} \begin{pmatrix} \varepsilon & \lambda \\ \lambda & -\varepsilon \end{pmatrix} \begin{pmatrix} c_1 \\ c_2^\dagger \end{pmatrix}$.

$$\begin{pmatrix} d_1^\dagger & d_2 \end{pmatrix} \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \varepsilon & \lambda \\ \lambda & -\varepsilon \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}} \begin{pmatrix} d_1 \\ d_2^\dagger \end{pmatrix}$$

$$\begin{pmatrix} \tilde{\varepsilon} & 0 \\ 0 & -\tilde{\varepsilon} \end{pmatrix} \quad \text{where} \quad \tilde{\varepsilon} = \sqrt{\varepsilon^2 + \lambda^2}$$

$$\eta_k = \sum_i \left(g_{ki}\, c_i + h_{ki}\, c_i^\dagger \right) \quad \text{and} \quad \eta_k^\dagger = \sum_i \left(g_{ki}^*\, c_i^\dagger + h_{ki}^*\, c_i \right)$$

$$\Rightarrow \begin{pmatrix} \eta \\ \eta^\dagger \end{pmatrix} = T \begin{pmatrix} c \\ c^\dagger \end{pmatrix}$$

where T is a $2l \times 2l$ unitary matrix

(expected because this is a basis transformation)

$$T = \begin{pmatrix} g & h \\ h^* & g^* \end{pmatrix}$$

Shorthand: $|\phi\rangle = \begin{pmatrix} c \\ c^\dagger \end{pmatrix}$ and $|\psi\rangle = \begin{pmatrix} \eta \\ \eta^\dagger \end{pmatrix}$;

$$|\psi\rangle = T|\phi\rangle$$

$$H_S = \phi^\dagger M \phi = \phi^\dagger T^\dagger \, T M T^\dagger \, T\phi$$

$$= \eta^\dagger \, T M T^\dagger \, \eta$$

$TMT^\dagger \to$ diagonal matrix $\begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix}$

$$\Rightarrow \quad M = T^\dagger \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix} T$$

Now, consider the following

$$G = \left(|\phi\rangle\langle\phi| \right) = T^\dagger T |\phi\rangle\langle\phi| T^\dagger T$$

$$= T^\dagger \left(|\psi\rangle\langle\psi| \right) T$$

$(|\psi\rangle\langle\psi|)$ easy to calculate

Just use $\langle \eta^\dagger_{k_1} \eta_{k_2} \rangle = \dfrac{\delta_{k_1,k_2}}{\exp(\varepsilon_k)+1}$

FD distribution

$$C_r' = T^\dagger \begin{pmatrix} \frac{1}{\exp(\varepsilon)+1} & 0 \\ 0 & \frac{1}{\exp(-\varepsilon)+1} \end{pmatrix} T$$

p ; $1-p$

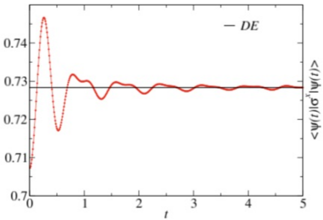
$S_{ent} = -\sum_{i=1}^{2l} p_i \ln p_i$ [eigenvalues come in pairs of $p, 1-p$]

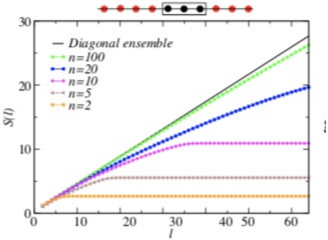
What is G in terms of C_{ij} and F_{ij} ?

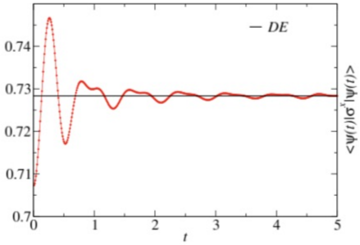
$$G = |\phi\rangle\langle\phi| = \begin{pmatrix} c \\ c^\dagger \end{pmatrix} \begin{pmatrix} c^\dagger & c \end{pmatrix}$$

$$= \left(\begin{array}{c|c} c\,c^\dagger & c\,c \\ \hline c^\dagger c^\dagger & c^\dagger c \end{array}\right) = \left(\begin{array}{c|c} \mathbb{I} - C & F^* \\ \hline F & C \end{array}\right)$$

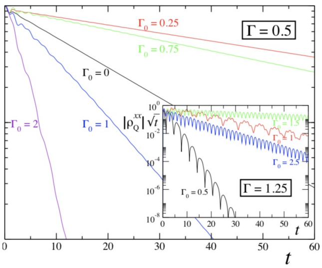
- $C_{ij} = \frac{2}{L} \sum_{k>0} |u_k(t)|^2 \cos[k(i-j)]$ and
  $F_{ij} = \frac{2}{L} \sum_{k>0} u_k^*(t) v_k(t) \sin[k(i-j)]$ (exercise)







- $\delta C_{ij} \sim \int dk f_1(k)\cos(2t\epsilon_k)$ where $\epsilon_k = \sqrt{(g-\cos(k))^2+\sin(k)^2}$ and $f_1(k) = -(1-\hat{n}_{k3})\cos(k(i-j))$
- At late time, integral dominated by stationary points of ϵ_k (these are only at $k = 0, \pm\pi$).
- $\int dk f_1(k)\cos(2t\epsilon_k) \sim \frac{1}{\sqrt{n}}(f_1(k_0) + \frac{i f_1''(k_0)}{2\epsilon_k''(k_0)t} + \cdots)$
- $\langle\sigma^x\rangle$ reaches its final value as $t^{-3/2}$



- See Rossini, Silva, Mussardo, Santoro, arXiv:0810.5508
- The autocorrelation function $\rho^{zz}(t) = \langle\psi(0)|\exp(iHt)\sigma_j^z\exp(-iHt)\sigma_j^z|\psi(0)\rangle$ decays as $\exp(-t/\tau)$
- The operator $\sigma_j^z(t)\sigma_j^z(0)$ connects states with different c-parity. Instead use $C^x(t;L) = \langle\sigma_1^z(t)\sigma_1^z(0)\sigma_{\frac{L}{2}+1}^z(t)\sigma_{\frac{L}{2}+1}^z(0)\rangle$
- $[\rho^{zz}(t)]^2 = \lim_{L\to\infty} C^x(t;L)$

- Let us consider eigenstates. Each pseudospin sees an effective magnetic field of $[2\sin(k), 0, 2(g - \cos(k)]$.
- $|\psi\rangle = \otimes_{k>0}|\psi_k\rangle$ where $|\psi_k\rangle = U_{kn}(g)|\uparrow\rangle_k + V_{kn}(g)|\downarrow\rangle_k$
- $(U_{k0}(g), V_{k0}(g)) = (-\sin(\theta_k^g/2), \cos(\theta_k^g/2))$ and $(U_{k1}(g), V_{k1}(g)) = (-\cos(\theta_k^g/2), -\sin(\theta_k^g/2))$
- Here $\sin(\theta_k^g/2) = \frac{\sin(k)}{\sqrt{(g-\cos(k)^2+\sin(k)^2}}$
- Any string of 0 and 1 at each allowed $k > 0$ denotes an eigenstate with $E = \sum_{k>0} \epsilon_k(g)(2n_k - 1)$ and $\epsilon_k(g) = 2\sqrt{(g - \cos(k))^2 + \sin(k)^2}$
- Since fermions are (un) occupied together for $(k, -k)$, these eigenstates represent $2^{L/2}$ of all eigenstates

Generalized Gibbs ensemble

- The free fermion picture clearly suggests that there are an extensive number of conservation laws
- $H = \sum_{k>0} H_k$ where
  $H_k = 2(g - \cos(k))[c_k^\dagger c_k - c_{-k} c_{-k}^\dagger] + 2\sin(k)[c_{-k} c_k + c_k^\dagger c_{-k}^\dagger]$
- $H = \sum_{k>0} \epsilon_k(g)(A_k^\dagger A_k + A_{-k}^\dagger A_{-k} - 1)$ with
  $A_k = V_{k0}(g) c_k - U_{-k0}(g) c_{-k}^\dagger$ (Bogoluibov transformation)
- Clearly $\langle n_k \rangle = \langle A_k^\dagger A_k \rangle = \langle n_{-k} \rangle$ conserved quantities in dynamics
- $\rho_{GGE} = \frac{1}{Z_{GGE}} \exp(-\sum_k \lambda_k n_k)$
  Question: Do we really need all these conservation laws to describe a local S? Also, n_k are not local in space

- See Fagotti and Essler, arXiv: 1302.6944
- Can define local in space conservation laws $I_n = \sum_k \cos(nk)\epsilon_k(g)n_k$. These involve $n+2$ neighboring physical spins
- $\mathrm{Trace}[\rho_{GGE} I_n] = \langle\psi|I_n|\psi\rangle$
- Truncated GGE $\rho_{GGE}^{(y)} = \frac{1}{Z_y}\exp(\sum_{n=0}^{y-1}\lambda_{n,y}I_n)$