Gravitational waves from the early universe

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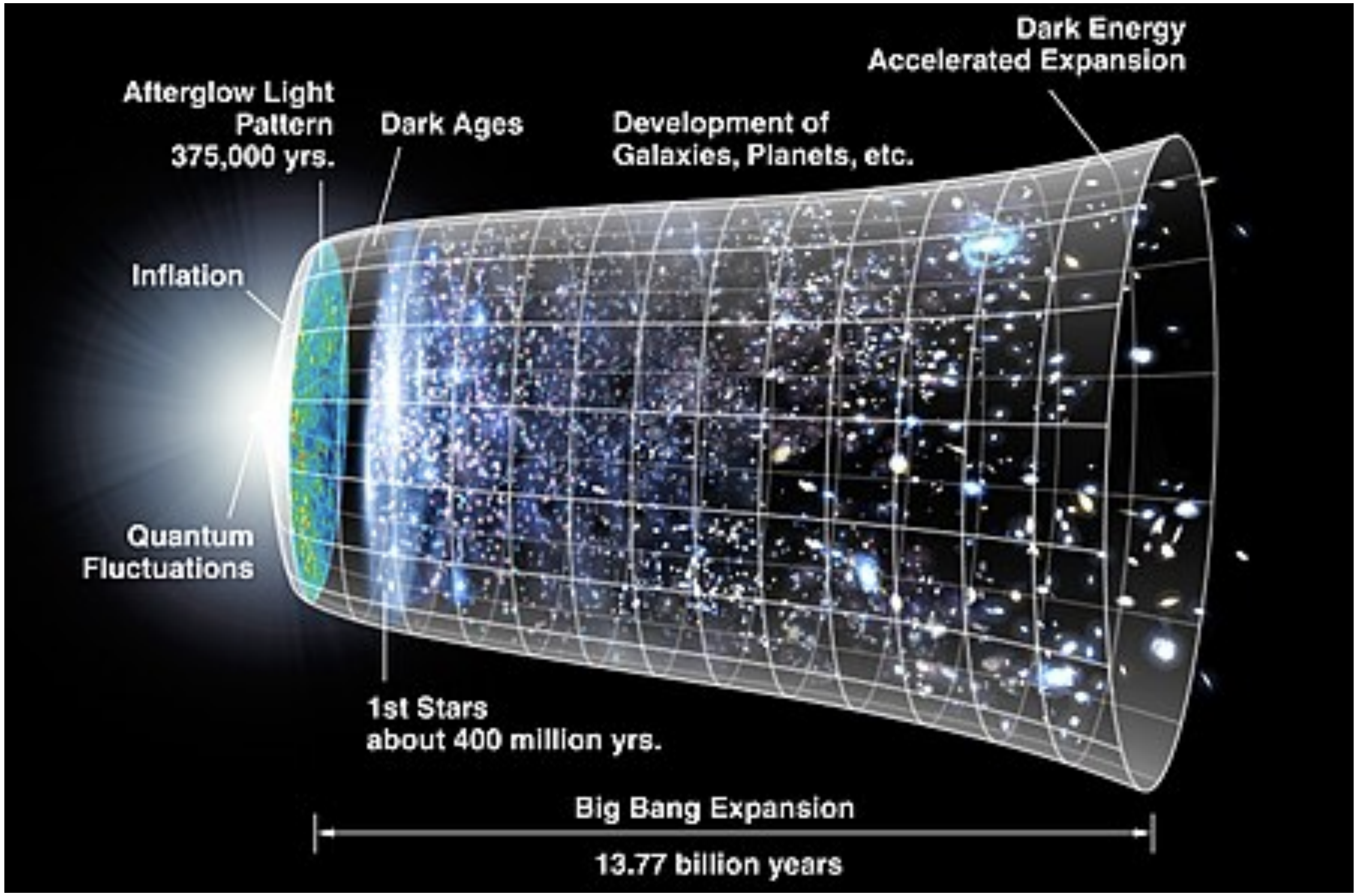
How can GW help to probe the universe?

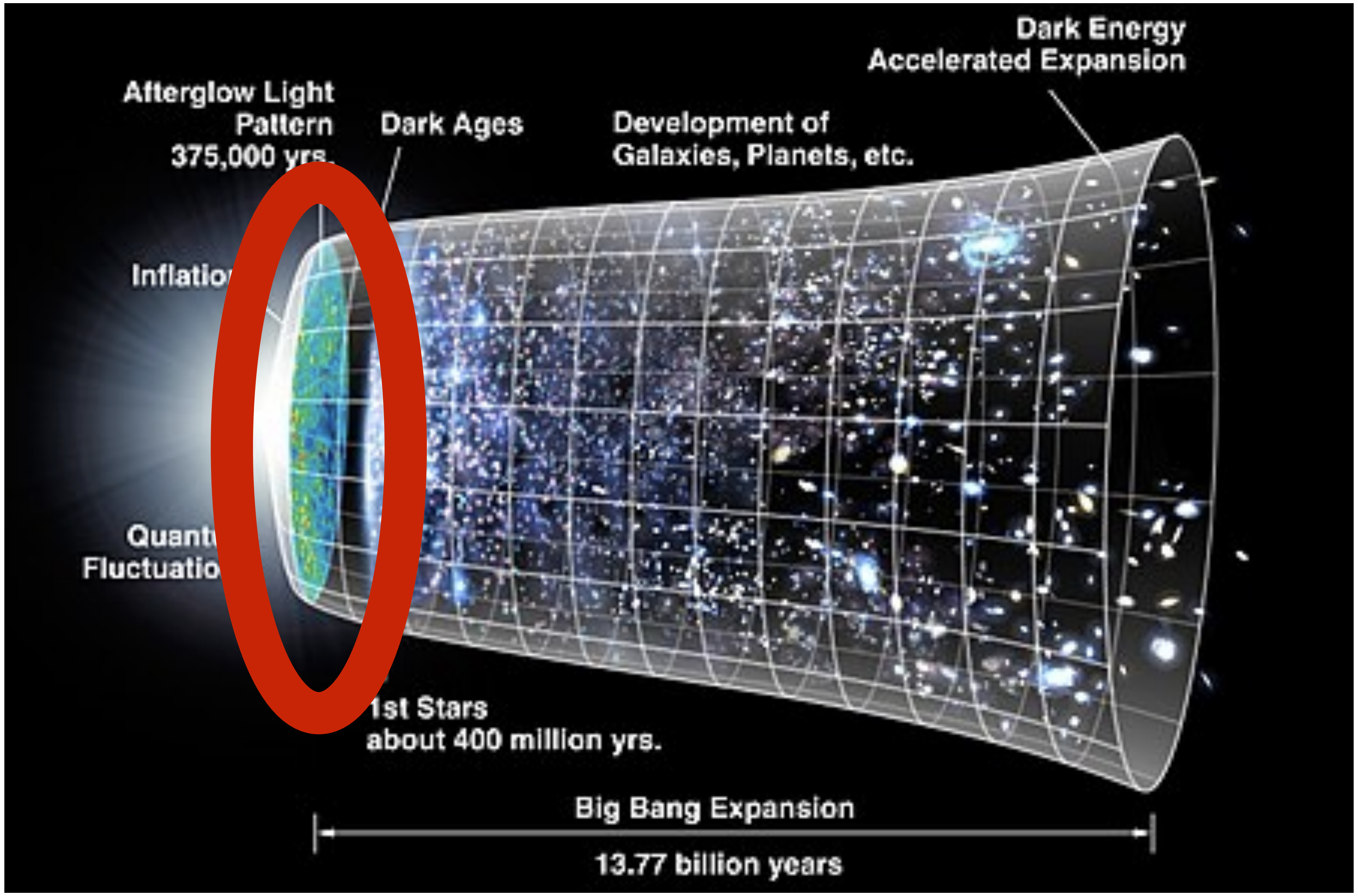


because of the weakness of the gravitational interaction the universe is "transparent" to GWs

$$\frac{\Gamma(T)}{H(T)} \sim \frac{G^2 T^5}{T^2/M_{Pl}} \sim \left(\frac{T}{M_{Pl}}\right)^3 < 1$$

How can GW help to probe the universe?

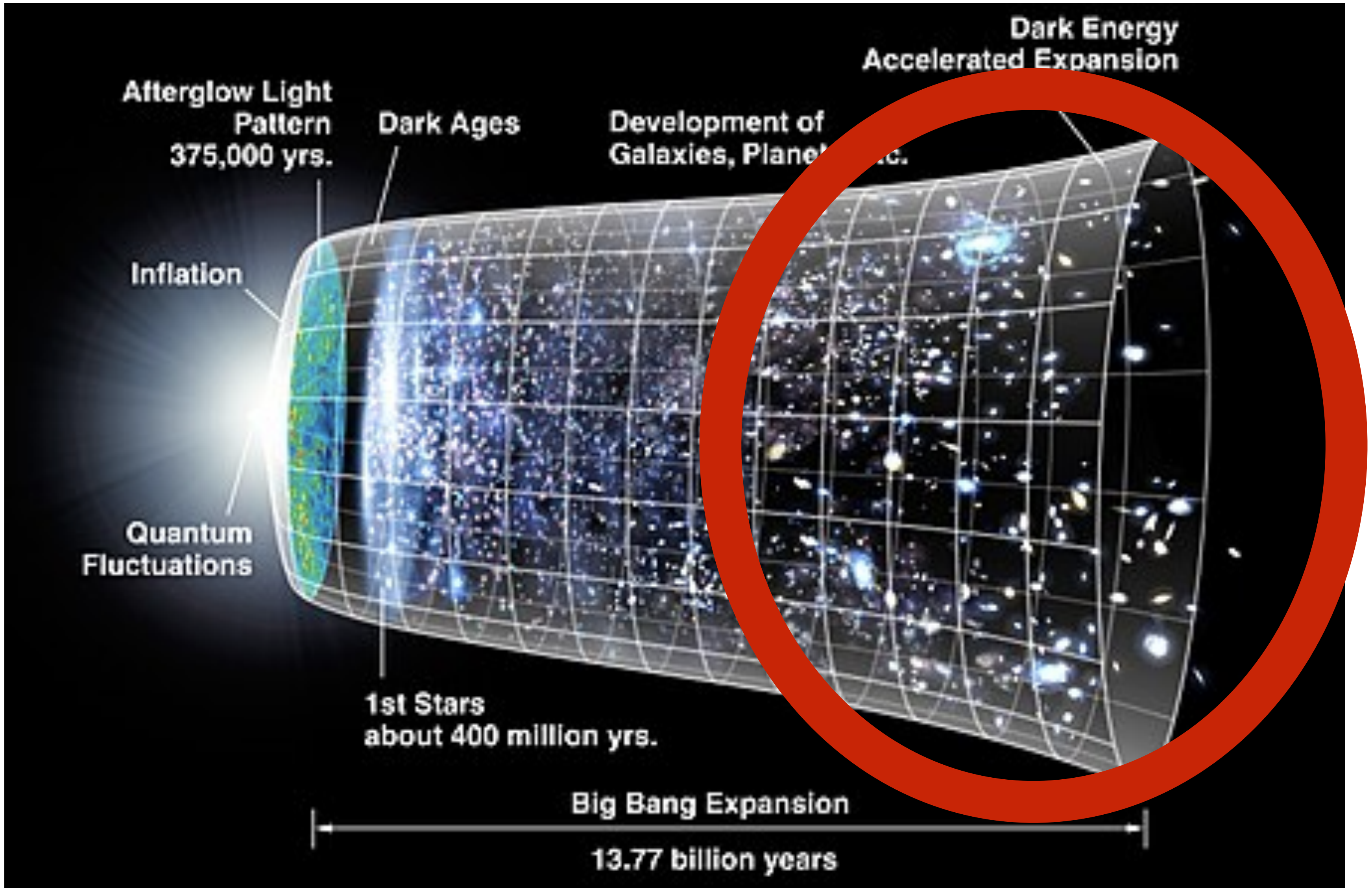
early
universe



GW can bring direct information from the early universe:
phenomena occurring in the early universe can produce stochastic GW backgrounds (SGWB) a fossil radiation like the CMB

tests of high energy phenomena

How can GW help to probe the universe?



late
universe

Binaries of compact objects (black holes, neutron stars…) orbiting around each other and possibly merging emit GWs

Provide information on binary formation and evolution, cosmological structure formation, black hole growth and environment, tests of General Relativity in strong and weak regime, tests of the cosmic expansion…

Summary of the course

- FIRST PART: GW definition, GW energy momentum tensor, GW in FLRW space-time, GW equation of motion, relevant solutions

- SECOND PART: Stochastic GW background from the early universe (with a digression on PTA measurement), and a few examples of SGWB sources

C.C. and D.G. Figueroa, “Cosmological backgrounds of GWs”, arXiv:1801.04268
M. Maggiore, “Gravitational waves”, volume 1 and 2, Oxford University Press

What are gravitational waves?

- GWs emerge naturally in General Relativity:

  Newtonian theory + special relativity = a causal theory of gravitation

  There must be some form of radiation propagating information causally: GWs!

- “waves” in physics are propagating perturbations over a background. In General Relativity:

  1. take a background space-time metric (the gravitational field)
  2. define a small perturbation over this background metric
  3. insert it into the equations that describe the space-time dynamics (Einstein equations)
  4. (if everything goes well) one finds a dynamical solution for the perturbation which is propagating as a wave -> GWs!

Which background metric to choose?
Simplest choice: flat space-time

GWs in linearised theory over Minkowski

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,, \qquad |h_{\mu\nu}(x)| \ll 1$$

Linearise in $h_{\mu\nu}$, raise and lower indices with $\eta_{\mu\nu}$

Affine connection $$\Gamma^{\alpha}{}_{\mu\nu} \simeq \frac{1}{2}(\partial_\nu h^{\alpha}{}_{\mu} + \partial_\mu h^{\alpha}{}_{\nu} - \partial^\alpha h_{\mu\nu})$$

Riemann tensor $$R^{\alpha}{}_{\mu\nu\beta} \simeq \frac{1}{2}(\partial_\mu\partial_\nu h^{\alpha}{}_{\beta} + \partial_\beta\partial^\alpha h_{\mu\nu} - \partial_\nu\partial^\alpha h_{\mu\beta} - \partial_\beta\partial_\mu h^{\alpha}{}_{\nu})$$

Einstein tensor $$G_{\mu\nu} \simeq \frac{1}{2}(\partial_\alpha\partial_\nu \bar{h}^{\alpha}{}_{\mu} + \partial^\alpha\partial_\mu \bar{h}_{\nu\alpha} - \Box\bar{h}_{\mu\nu} - \eta_\mu\nu\partial_\alpha\partial^\beta\bar{h}^{\alpha}{}_{\beta})$$

$$\Box \equiv \partial_\alpha\partial^\alpha$$

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\,h$$

trace-reversed metric perturbation (OK - still small)

"Gravitational Waves", M. Maggiore, Oxford University Press 2008

GWs in linearised theory over Minkowski

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Linearise in $h_{\mu\nu}$, raise and lower indices with $\eta_{\mu\nu}$

Affine connection $$\Gamma^{\alpha}{}_{\mu\nu} \simeq \frac{1}{2}(\partial_\nu h^{\alpha}{}_{\mu} + \partial_\mu h^{\alpha}{}_{\nu} - \partial^{\alpha} h_{\mu\nu})$$

Riemann tensor $$R^{\alpha}{}_{\mu\nu\beta} \simeq \frac{1}{2}(\partial_\mu\partial_\nu h^{\alpha}{}_{\beta} + \partial_\beta\partial^\alpha h_{\mu\nu} - \partial_\nu\partial^\alpha h_{\mu\beta} - \partial_\beta\partial_\mu h^{\alpha}{}_{\nu})$$

Einstein tensor $$G_{\mu\nu} \simeq \frac{1}{2}(\partial_\alpha\partial_\nu \bar{h}^{\alpha}{}_{\mu} + \partial^\alpha\partial_\mu \bar{h}_{\nu\alpha} - \Box\bar{h}_{\mu\nu} - \eta_{\mu}\nu\partial_\alpha\partial^\beta\bar{h}^{\alpha}{}_{\beta})$$

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu} \longrightarrow$$ we would like to set $\partial^\mu \bar{h}_{\mu\nu}(x) = 0$

GWs in linearised theory over Minkowski

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,, \qquad |h_{\mu\nu}(x)| \ll 1$$

GR is invariant under general coordinate transformation

the linearised theory is invariant under
infinitesimal (slowly varying) coordinate transformation

$$x^\mu \longrightarrow x^\mu + \xi^\mu \qquad h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

$$|\partial_\alpha \xi_\beta| \lesssim |h_{\alpha\beta}| \quad \longrightarrow \quad |h'_{\mu\nu}(x')| \ll 1$$

$$\partial^\mu \bar{h}_{\mu\nu}(x) \longrightarrow \partial'^\mu \bar{h}'_{\mu\nu}(x') = \partial^\mu \bar{h}_{\mu\nu}(x) - \Box \xi_\nu$$

By a suitable coordinate transformation, it is always possible to go to the LORENTZ GAUGE

$$\partial'^\mu \bar{h}'_{\mu\nu}(x') = 0$$

GWs in linearised theory over Minkowski

IN LORENTZ GAUGE EINSTEIN EQUATIONS TAKE THE FORM OF A WAVE EQUATION

$$\Box \bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu}$$

$T_{\mu\nu}$ source energy momentum tensor

From the Lorentz gauge condition $\partial^\mu \bar{h}_{\mu\nu}(x) = 0$ one gets

$$\partial^\mu T_{\mu\nu} = 0$$

The energy-momentum tensor of the source is conserved

the source does not loose energy and momentum by the GW emission

in linearised theory, the background space-time is flat, i.e. the source is described by Newtonian gravity

linearised theory does not describe how GW emission influences the source, but it can be use to describe the behaviour of test masses

GWs in linearised theory over Minkowski

IN LORENTZ GAUGE EINSTEIN EQUATIONS TAKE THE FORM OF A WAVE EQUATION

$$\Box \bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu}$$

$T_{\mu\nu}$ source energy momentum tensor

$\bar{h}_{\mu\nu} = \bar{h}_{\nu\mu}$ (16-6) $\quad \partial^\mu \bar{h}_{\mu\nu}(x) = 0$ (-4) $\longrightarrow$ 6 radiative components

ARE THESE ALL PHYSICAL?

$x^\mu \longrightarrow x^\mu + \xi^\mu$ satisfying $\Box \xi_\mu = 0$ to remain in the Lorentz gauge

$\bar{h}_{\mu\nu} \longrightarrow \bar{h}_{\mu\nu} + \xi_{\mu\nu}$ with $\xi_{\nu\mu} \equiv \eta_{\nu\mu}\partial^\alpha \xi_\alpha - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$

IF IN VACUUM: $T_{\mu\nu} = 0$ $\quad \Box' \bar{h}'_{\mu\nu} \simeq \Box(\bar{h}_{\mu\nu} + \xi_{\mu\nu}) = 0$

GWs in linearised theory over Minkowski, in vacuum

Restricting to vacuum space-time, the residual coordinate freedom can be used to fix 4 constraints

TRANSVERSE TRACELESS GAUGE

$$\bar{h}'^{\mu}_{\ \mu} = 0 \qquad h'_{0i} = 0 \qquad \begin{array}{r} \partial^i h'_{ij} = 0 \\ h'_{00} = 0 \end{array}$$

come for free

Is this OK? $\Box \bar{h}_{\mu\nu} = 0 \rightarrow \Box \bar{h} = 0$

$$\bar{h}' = \bar{h} + 4\partial^\alpha \xi_\alpha - \partial^\mu \xi_\mu - \partial^\mu \xi_\mu = \bar{h} - 2\partial^\mu \xi_\mu$$

$\bar{h}'^{\mu}_{\ \mu} = 0$ implies $\bar{h} = 2\partial^\mu \xi_\mu$ and therefore $\Box \bar{h} = 0$

If we were not in vacuum: $\Box \bar{h} = -16\pi G T$ contradiction!

GWs in linearised theory over Minkowski, in vacuum

Restricting to vacuum space-time, the residual coordinate freedom can be used to fix 4 constraints

TRANSVERSE TRACELESS GAUGE

$$\bar{h}'^{\mu}_{\ \mu} = 0 \qquad h'_{0i} = 0 \qquad \partial^i h'_{ij} = 0 \quad h'_{00} = 0$$

come for free

How can one choose the 4 functions ξ_μ ?

"General relativity", N. Straumann, Springer 2004

GWs in linearised theory over Minkowski, in vacuum

Restricting to vacuum space-time, the residual coordinate freedom can be used to fix 4 constraints

TRANSVERSE TRACELESS GAUGE

$$\bar{h}'^{\mu}_{\ \mu} = 0 \qquad h'_{0i} = 0 \qquad \partial^i h'_{ij} = 0$$

$$h'_{00} = 0$$

come for free

There are only 2 remaining *physical* degrees of freedom in the metric

$$\Box h_{ij}(\mathbf{x}, t) = 0$$

$$h_{ij}(\mathbf{x}, t) = \sum_{r=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \, h_r(\mathbf{k}) \, e^r_{ij}(\hat{\mathbf{k}}) e^{-ik(t - \hat{k}\cdot\mathbf{x})}$$

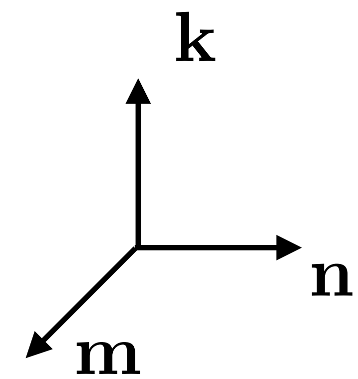
Plane waves, transverse, moving at the speed of light $k = \omega$ with two independent polarisation components +, ×

GWs in linearised theory over Minkowski, in vacuum

- Polarisation tensors

$$e^{+}_{ij}(\hat{\mathbf{k}}) = \hat{m}_i \, \hat{m}_j - \hat{n}_i \, \hat{n}_j$$

$$e^{\times}_{ij}(\hat{\mathbf{k}}) = \hat{m}_i \, \hat{n}_j + \hat{n}_i \, \hat{m}_j$$



- Free wave traveling in the z direction

$$h_{ij}(z,t) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos\left[\omega(t-z)\right]$$

$$\mathbf{k} = \omega \hat{\mathbf{z}}$$

- Metric line element

$$ds^2 = -dt^2 + dz^2 + (1 + h_+ \cos[\omega(t-z)])dx^2 + $$
$$+ (1 - h_+ \cos[\omega(t-z)])dy^2 + 2h_\times \cos[\omega(t-z)]dxdy$$

- Polarisation states are related to the spin of the massless particle expected upon quantisation

$$S = \frac{2\pi}{\theta}$$

Misner, Thorne, Wheeler
"Gravitation"
Chapter 35.6

Where θ is the rotation angle under which the polarisation modes are invariant

GWs in linearised theory over Minkowski, in vacuum

invariant under rotation around the z-axis of $\theta = \pi$

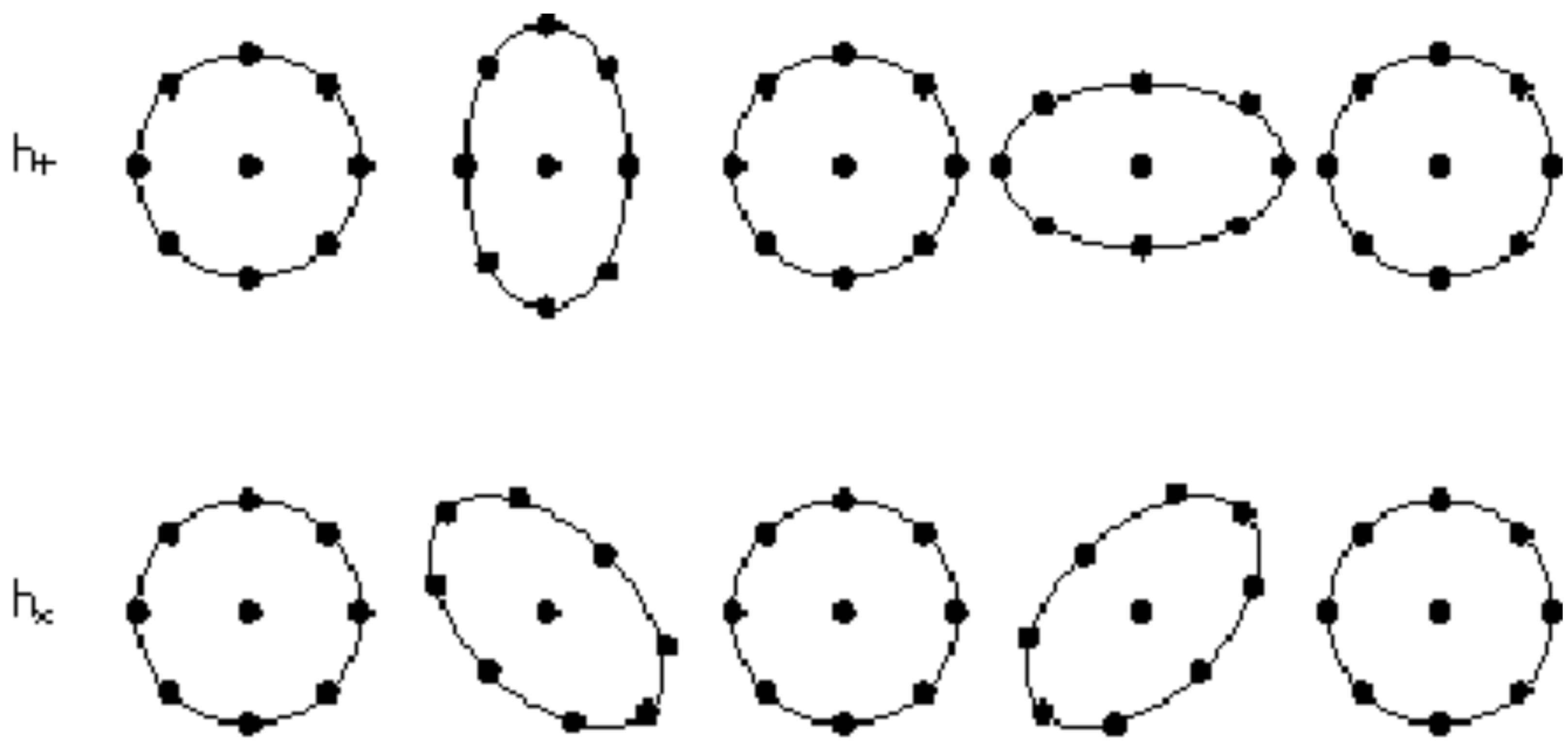
$$h_+^\theta = h_+ \cos 2\theta - h_\times \sin 2\theta$$
$$h_\times^\theta = h_\times \cos 2\theta + h_+ \sin 2\theta$$

Graviton is a spin 2 particle

- Effect of GW on a ring of test masses

Geodesic deviation equation $\ddot{\xi}^i = -R^i{}_{0j0}\xi^j = \frac{1}{2}\ddot{h}_{ij}\xi^j$

$h_+, h_\times$



GWs in linearised theory over Minkowski, with matter

To exhibit the two physical d.o.f. of GWs we had to restrict to vacuum

However, the fact that GWs have only two physical components is a manifestation of the intrinsic nature of the gravitational interaction, mediated by the graviton, a spin-two massless field that has only two independent helicity states

It should be true also in space-time with matter

WHAT WE DO NEXT:

1. Drop the condition of vacuum
2. Exploit the invariance of Minkowski space-time under spatial rotations, and split the metric perturbation into irreducible components under rotations (scalar, vector, tensor)
3. Construct metric perturbation variables that are invariant under infinitesimal coordinate transformations
4. Find the metric perturbation variable that obeys a wave equation -> *we define GWs without restricting to vacuum*

GWs in linearised theory over Minkowski, with matter

2. Exploit the invariance of Minkowski space-time under spatial rotations, and split the metric perturbation into irreducible components under rotations

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,, \qquad |h_{\mu\nu}(x)| \ll 1$$

$$\begin{aligned}
h_{00} &= -2\phi \\
h_{0i} &= \partial_i B + S_i \quad (\partial_i S_i = 0) \\
h_{ij} &= -2\psi\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)E + \partial_i F_j + \partial_j F_i + H_{ij} \\
&\quad (\partial_i F_i = 0,\ \partial_i H_{ij} = 0,\ H_{ii} = 0)
\end{aligned}$$

scalars	vectors	tensor
$\phi,\ B,\ \psi,\ E$	$S_i,\ F_i$	H_{ij}

E.E. Flanagan and S.A. Hughes, “The basics of GW theory”, arXiv:gr-qc/0501041

“Space-time and geometry: an introduction to GWs”, S. Carroll, Pearson Education Limited, 2014

“The Cosmic Microwave Background”, R. Durrer, Cambridge University Press, 2008

GWs in linearised theory over Minkowski, with matter

3. Construct metric perturbation variables that are invariant under infinitesimal coordinate transformations

$$x^\mu \longrightarrow x^\mu + \xi^\mu \qquad h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

$$\xi_\mu = (\xi_0, \xi_i) \equiv (d_0, \partial_i d + d_i) \qquad \text{with} \quad \partial_i d_i = 0$$

Two scalars, one vector and one tensor gauge invariant variables

$$\Phi \equiv \phi + \dot{B} - \ddot{E}/2$$
$$\Theta \equiv -2\psi - \nabla^2 E/3$$
$$\Sigma_i \equiv S_i - \dot{F}_i \qquad \text{with} \quad \partial_i \Sigma_i = 0$$
$$H_{ij} \qquad \text{with} \quad \partial_i H_{ij} = 0 \;\; H^i_i = 0$$

Automatically gauge invariant (no tensor gauge transformation)

16 free functions - 6 contraints - 4 constraints =
6 physical degrees of freedom

GWs in linearised theory over Minkowski, with matter

1. Drop the condition of empty space-time

$$
\begin{aligned}
T_{00} &= \rho \\
T_{0i} &= \partial_i u + u_i \quad (\partial_i u_i = 0) \\
T_{ij} &= p\delta_{ij} + (\partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\sigma + \partial_i v_j + \partial_j v_i + \Pi_{ij} \\
& \quad (\partial_i v_i = 0,\ \partial_i \Pi_{ij} = 0,\ \Pi_{ii} = 0)
\end{aligned}
$$

scalars	vectors	tensor
$\rho,\ u,\ p,\ \sigma$	$u_i,\ v_i$	Π_{ij}

Gauge invariant automatically but four further constraints given by energy-momentum conservation (OK since we are still in linearised theory!)

$$\partial_\mu T^{\mu\nu} = 0$$

16 free functions - 6 contraints - 4 constraints =
6 physical degrees of freedom

GWs in linearised theory over Minkowski, with matter

4. Find the metric perturbation variable that obeys a wave equation -> *we define GWs in non-vacuum space-times*

Write Einstein equations in terms of the 6 gauge invariant variables

$$\nabla^2\Theta = -8\pi G\rho \qquad \nabla^2\Phi = 4\pi G\left(\rho + 3p - 3\dot{u}\right)$$

$$\nabla^2\Sigma_i = -16\pi G S_i \qquad \Box H_{ij} = -16\pi G\,\Pi_{ij}$$

Three Poisson-like equations, one wave equation

Only the TT metric components are radiative

Cosmological case: the same procedure, exploiting the symmetries (homogeneity and isotropy) of FLRW spacetime, but:

- There is a energy-momentum tensor also in the background
- The equation of the tensor perturbations contains Hubble friction (see later)
- One finds a wave equation also for the Bardeen potential (sound waves in the fluid)