Conformal blocks for Galois covers of Algebraic curves

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Let G be a simply-connected, simple complex algebraic group and Σ a reduced projective curve. By a s-pointed curve we mean

$$\left(\Sigma, \vec{p} = (p_1, \cdots, p_s)\right) \qquad s \geq 1$$

where the p_i's are smooth points.

We fix a level (or the central charge) $c > 0$

$$D_c = \{\lambda \in \underline{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geq 0} \ \& \ \lambda(\theta^\vee) \leq c\}$$

$\underline{h}$ = cartan subalgebra of Lie G, $\alpha_i^\vee$ are the simple coroots and $\theta^\vee$ = coroot corresponding to the highest root θ.

Affine Lie algebra: $\hat{\mathfrak{g}}$, $\mathfrak{g} := \text{Lie } G$

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}C$$

$$[x \otimes f, y \otimes g] = [x, y] \otimes fg + \langle x, y \rangle \operatorname{Res}_{t=0}\left((df) \cdot g\right)$$

$\langle x, y \rangle$ is the normalized invariant form on $\mathfrak{g}$ so that $\langle \theta, \theta \rangle = 2$.

Integrable highest wt. $\hat{\mathfrak{g}}$-modules with central charge c (i.e. the action of C)

$$\longleftrightarrow D_c$$

$$\mathcal{H}(\lambda) \longleftarrow\!\!\mid \quad \lambda \in D_c$$

Space of dual conformal blocks:

distinct points

s-pointed curve $(\Sigma, \vec{p})$ $\vec{p} = (p_1, \dots, p_s)$

$\vec{\lambda} = (\lambda_1, \dots, \lambda_s)$ attached to the points

$\vec{p}$, $\lambda_i \in D_c$

$$\mathcal{H}(\vec{\lambda}) = \mathcal{H}(\lambda_1) \otimes \cdots \otimes \mathcal{H}(\lambda_s)$$

Lie algebra $\mathfrak{g} \otimes \mathbb{C}[\Sigma \setminus \vec{p}]$ ↪ affine coordinate ring

dual conformal blocks

$$\boxed{V_\Sigma(\vec{p}, \vec{\lambda})} = \frac{\mathcal{H}(\vec{\lambda})}{\underbrace{\mathfrak{g} \otimes \mathbb{C}[\Sigma - \vec{p}]}_{\text{Lie algebra}} \cdot \mathcal{H}(\vec{\lambda})}$$

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg$$

Action of $\mathfrak{g} \otimes \mathbb{C}[\Sigma \setminus \vec{p}]$ on $\mathcal{H}(\vec{\lambda})$ is given by

$$(X\otimes f)\cdot(v_1\otimes\cdots\otimes v_s) \qquad v_i\in\mathcal{H}(\lambda_i) \ \& \ f\in\mathbb{C}[\Sigma\setminus\vec{p}]$$

$$= \sum_{i=1}^{s} v_1\otimes\cdots\otimes\left(X\otimes(f)_{t_i}\cdot v_i\right)\otimes\cdots\otimes v_s$$

t_i is a local parameter on Σ at p_i

$(f)_{t_i}$ is the Laurent series expansion of f at p_i through the coordinate t_i.

Lemma: $V_\Sigma(\vec{p},\vec{\lambda})$ is a finite dimd. vector space/$\mathbb{C}$. Moreover, it does NOT depend upon the choice of the local parameters t_i (upto a canonical isomorphism) upto a scalar constant

Question: What is the dimension of $V_\Sigma(\vec{p},\vec{\lambda})$.

Conj (E. Verlinde) (1988).

Idea of the proof:

Theorem: ① Propagation of vacua:

$$V_\Sigma(\vec{p},\vec{\lambda}) \simeq V_\Sigma(\vec{p},q_0;\vec{\lambda},0)$$

$q_0\notin\vec{p}$ $\qquad$ q_0 is a smooth point

$\hat{\Sigma} \to$ 

② <u>Factorization theorem</u>:

Let $(\Sigma, \vec{p})$ has a node q

$\hat{\Sigma} \to \Sigma$ normalization at q only

$q', q'' \qquad\quad q$

$$F : V_{\Sigma}(\vec{p}, \vec{\lambda}) \simeq \bigoplus_{\mu \in D_c} V_{\hat{\Sigma}}(\vec{p}, q', q''; \vec{\lambda}, \mu^*, \mu)$$

Arithmetic genus of $\Sigma = g \Rightarrow$ genus of $\hat{\Sigma} = g-1$

③ <u>Sheaf of conformal blocks</u>:

$\pi : \Sigma_T \to T$ a family of s-pointed curves

proper & flat family

sections $\vec{p} = (p_1, \cdots, p_s)$

$p_i \in$ sections of π

mutually disjoint

$\pi^{-1}(t)$ is a s-pointed curve, $t \in T$ geometric point

<u>Sheaf of conformal blocks</u>:

$V_{\Sigma_T}(\vec{p}, \vec{\lambda}) \qquad \vec{\lambda} = (\lambda_1, \cdots, \lambda_s)$

<u>Theorem</u>: Assume that T is smooth & π is a smooth morphism, then

$V_{\Sigma_T}(\vec{b},\vec{\lambda})$ admits a projectively flat connection.

In particular $V_{\Sigma_T}(\vec{b},\vec{\lambda})$ is a vector bundle.

$\Longrightarrow$ $\dim V_\Sigma(\vec{b},\vec{\lambda})$ only depends upon the genus of Σ and the weights $\vec{\lambda}$.

$$\boxed{m_g(\vec{\lambda}) := \dim V_\Sigma(\vec{b},\vec{\lambda})}$$

Corollary of the Factorization theorem:

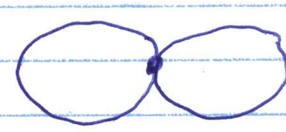
$$m_g(\vec{\lambda}) = \sum_{\mu} m_{g-1}(\vec{\lambda}, \mu^*, \mu)$$

$$= \vdots$$

$$= \sum_{\mu_1,\dots,\mu_g \in D_c} m_0(\vec{\lambda}, \mu_1^*, \mu_1, \mu_2^*, \mu_2, \dots, \mu_g^*, \mu_g)$$

Let us remark that the last theorem remains valid when T is the moduli space of S-pointed stable curves $\Rightarrow$ $V_{\Sigma_T}(\vec{b},\vec{\lambda})$ is a vector bundle.

Further by the Factorization theorem applied to

$\hat{\Sigma} \to \Sigma =$ 

$$m_0(\lambda_1,\lambda_2,\dots,\lambda_s) = \sum_{\mu\in D_c} m_0(\lambda_1,\lambda_2,\mu)\, m_0(\mu^*,\lambda_3,\dots,\lambda_s)$$

This tells us that we are reduced to calculating $m_0(\lambda_1,\lambda_2,\lambda_3)$.
This is achieved by the fusion product on level c representations of $\mathfrak{g}$.

$\longleftrightarrow$

History

conj Verlinde 1988
Tsuchiya-Ueno-Yamada 1989
Beauville-Laszlo (94)
Faltings (94)
Kumar-Narasimhan-Ramanathan (94)
Pauly (96)
Laszlo-Sorger (97)
direct proof for the hol. sections. $\leftarrow$ Bismut-Labourie (93)
Teleman (97)
$G=SL_2$ { Bertram (93), Szenes (91), Thaddeus (94) }

Parallel picture for the space of conformal blocks in terms of hol. sections of line bundles on the moduli space of semistable parabolic G-bundles on Σ.

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Setting: We take a reduced projective curve Σ with the action of a finite group Γ.

Let $\bar{\Sigma} := \Sigma/\Gamma$

We assume that any $\gamma \in \Gamma$, $\gamma \neq 1$ does not fix pointwise any irreducible component of Σ.

$\mathfrak{g}$ as before a simple Lie algebra $/\mathbb{C}$

Γ acts on $\mathfrak{g}$ as Lie algebra automorphisms

Γ fixes a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$.

Definition. A s-pointed Γ-curve is by defn. $\left(\Sigma, \vec{p} = (p_1, \dots, p_s)\right)$ as before, i.e., Σ is a Γ-curve & p_i are distinct smooth points in Σ. Moreover we require that

$\Gamma \cdot p_i \neq \Gamma \cdot p_j$ for all $i \neq j$.

$p_i \rightsquigarrow$ twisted affine Lie algebra

Γ_{p_i} = isotropy of Γ at p_i.

$\hat{\mathfrak{g}}$ as before

$$\hat{\mathfrak{g}}_{p_i} = \left(\hat{\mathfrak{g}}\right)^{\Gamma_{p_i}} \qquad \hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((t_i)) \oplus \mathbb{C}C$$

twisted affine Lie algebra.

Integrable highest wt. modules of $\hat{\mathfrak{g}}_{p_i} \longleftrightarrow$

$D(c, p_i)$

Take $\vec{\lambda} = (\lambda_1, \dots, \lambda_s)$, $\lambda_i \in D(c, p_i)$

$\mathcal{H}(\vec{\lambda}) := \mathcal{H}(\lambda_1) \otimes \dots \otimes \mathcal{H}(\lambda_s)$

$\mathcal{H}(\lambda_i)$ is a module for $\hat{\mathfrak{g}}_{p_i}$

$\mathcal{H}(\vec{\lambda})$ is a module for $\hat{\mathfrak{g}}_{p_1} \oplus \dots \oplus \hat{\mathfrak{g}}_{p_s}$.

<u>Defn.</u> <u>Twisted dual conformal blocks:</u>

$(\Sigma, \vec{p}, \vec{\lambda}) \rightsquigarrow V_{\Sigma, \Gamma}(\vec{p}, \vec{\lambda})$

space of twisted conformal blocks

$$V_{\Sigma,\Gamma}(\vec{p}, \vec{\lambda}) := \frac{\mathcal{H}(\vec{\lambda})}{\mathfrak{g}[\Sigma \setminus \Gamma \cdot \vec{p}]^{\Gamma} \cdot \mathcal{H}(\vec{\lambda})}$$

$\Sigma \setminus \Gamma \cdot \vec{p} := \Sigma \setminus \bigcup_{i=1}^{s} \Gamma p_i$

the action of $\mathfrak{g}[\Sigma \setminus \Gamma \cdot \vec{p}]$ on $\mathcal{H}(\vec{\lambda})$ is as before.

Lemma. $V_{\Sigma,\Gamma}(\vec{p},\vec{\lambda})$ is finite diml. and it does not depend upon the choice of the local parameters t_i at p_i.

Theorems:

- Propagation of vacua
- Factorization theorem
- Existence of projectively flat connection on a smooth family
- Extension of the connection to the Hurwitz stack with logarithmic singularities.
- Identification of twisted conformal blocks with the global sections on some moduli stack (cf Pappas–Rapoport)

Precise dimension formula is given by Deshpande — Mukhopadhyay

Frenkel-Szczesny
Kuroki-Takebe
Damiolini

← earlier works