Tensor network methods in four dimensional field theory

Daisuke Kadoh
NCTS, National Tsing-Hua Univ.

Numstrings 2021 (online)
Jan 20, 2021

MC method and the sign problem

- Monte Carlo method

expectation value $\langle \mathcal{O} \rangle = \frac{1}{Z} \int D\phi \, \mathcal{O}(\phi) e^{-S}$ → Generate $\{\phi\}_i$ with a probability $e^{-S[\phi_i]}$ → $\langle \mathcal{O} \rangle = \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}(\phi_i)$

$e^{-S[\phi_i]}$: positive real number

- (example) SYM QM

$$S = \frac{N}{2\lambda} \int dt \, \mathrm{tr} \left\{ (D_0 X_i)^2 - \frac{1}{2}[X_i, X_j]^2 + \psi D_0 \psi + \psi \gamma_i [X_i, \psi] \right\}$$

$$Z = \int DADX \, e^{-S_B} \mathrm{pf}(D)$$

$e^{-S_B} \mathrm{pf}(D)$ can be negative

negative

sign problem

Cases with the sign problem

finite density QCD

early universe, neutron star,..

chiral gauge theory

The SM, GUT,..

supersymmetry

AdS/CFT, string theory,…

θ -vacuum

strong CP problem

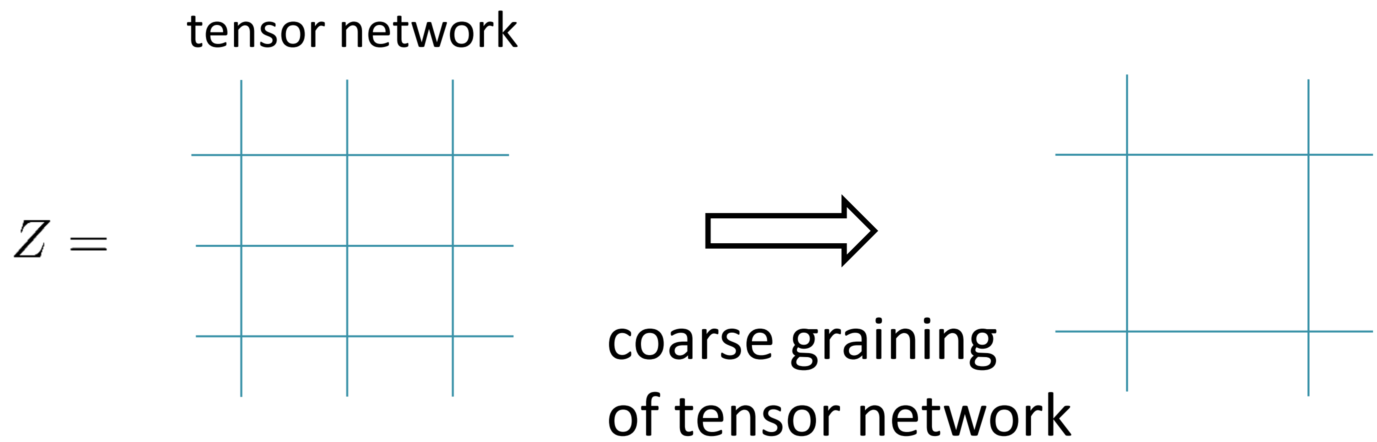
real time simulation

Schwinger-Keldysh formalism

It is important to solve the sign problem.

Tensor renormalization group (TRG)

Levin-Nave, 2007



no stochastic process & no sign problem

Previous studies in field theory:

Y. Kuramashi, S.Takeda, Y. Meurice, K.Jansen, M.C.Banuls, D.C-J.Lin, S. Catterall, J. Unmuth-Yockey, A.Bazavov, R.Sakai, Y. Yoshimura, R.G.Jha, H. Oba, S. Akiyama, …

c.f. improved MC method:
reweighting method, complex Langevin, Lefschetz thimble, …

Talk plan

1. Motivation
2. TRG in 2d Ising model and scalar field theory
3. The triad TRG method
4. Summary and future outlook

2. TRG in 2d Ising model and scalar field theory

2d Ising model

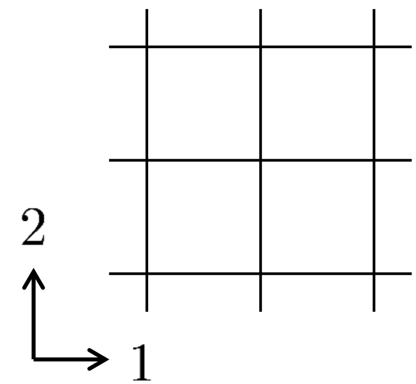
- Hamiltonian

$$H = -J \sum_n \{\sigma_n \sigma_{n-\hat{1}} + \sigma_n \sigma_{n-\hat{2}}\} + h \sum_n \sigma_n$$

$$\sigma_n = \pm 1$$

$$n = (n_1, n_2), \quad n_i \in \mathbf{Z}$$

$\hat{\mu}$: unit vector of μ-direction



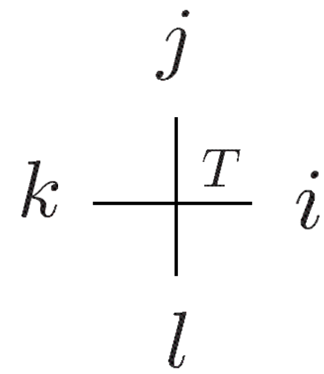
- Partition function

$$Z = \mathrm{Tr}(e^{-\beta H})$$

$$\equiv \prod_n \sum_{\sigma_n = \pm 1} e^{\beta J(\sigma_n \sigma_{n-\hat{1}} + \sigma_n \sigma_{n-\hat{2}}) + \beta h \sigma_n}$$

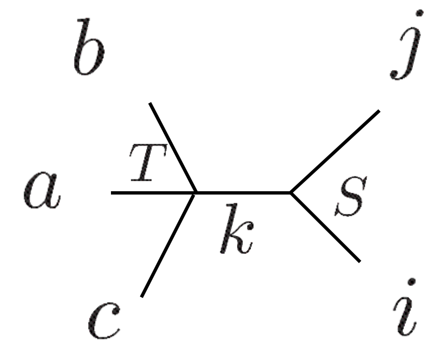
Graphical representation of tensors

$$T_{ijkl} =$$



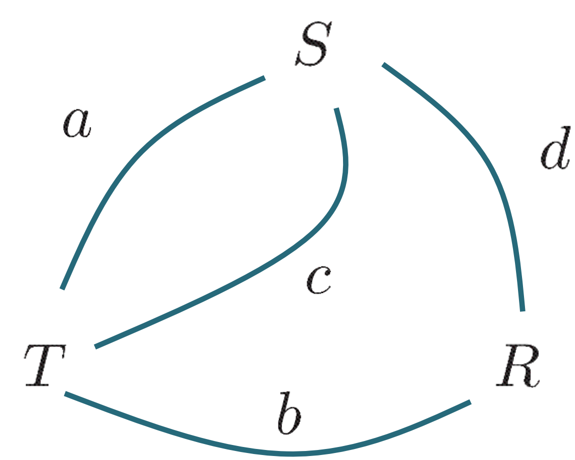
"contraction of tensors"

$$\sum_{k=1}^{N} T_{kabc} S_{ijk} =$$



"tensor network"

$$Z = \sum_{a,b,c,d} T_{bca} S_{acd} R_{bd} =$$

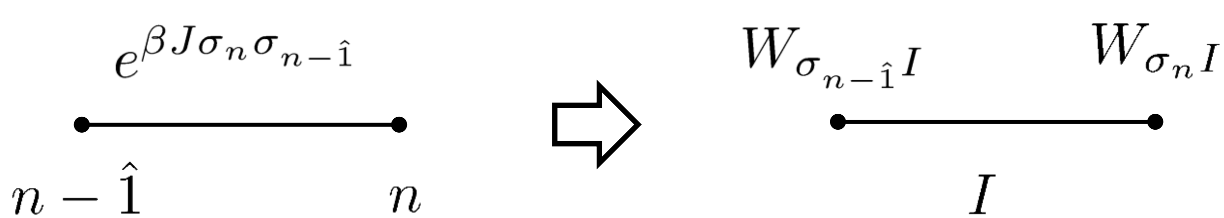


The TN representation of Z

(1) decompose the hopping term

$$
\begin{aligned}
e^{\beta J \sigma_n \sigma_{n-\hat{1}}} &= \cosh(\beta J) + \sigma_n \sigma_{n-\hat{1}} \sinh(\beta J) \\
&= \sum_{I=0,1} \cosh(\beta J)(\sigma_n \sigma_{n-\hat{1}} \tanh(\beta J))^I \\
&= \sum_{I=0,1} W_{\sigma_n I} W_{\sigma_{n-\hat{1}} I}
\end{aligned}
$$

$$
W_{\sigma I} = \sqrt{\cosh(\beta J)} \sqrt{\tanh(\beta J)}^I \sigma^I
$$



The TN representation of Z (cont'd)

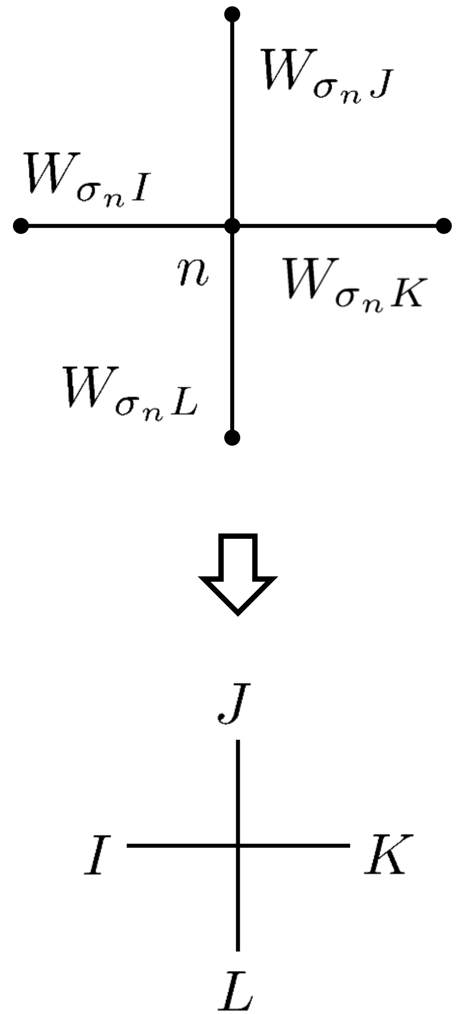
(2) make tensor from $W_{\sigma_n I}$

$$Z = \sum_{I=0,1} \sum_{J=0,1} \sum_{K=0,1} \sum_{L=0,1} \cdots$$

$$\times \left\{ \sum_{\sigma_n=\pm 1} W_{\sigma_n I} W_{\sigma_n J} W_{\sigma_n K} W_{\sigma_n L} \right\} \cdots$$

$$= \sum_{I=0,1} \sum_{J=0,1} \sum_{K=0,1} \sum_{L=0,1} \cdots T_{IJKL} \cdots$$

$$T_{IJKL} = \sum_{\sigma=-1,+1} W_{\sigma I} W_{\sigma J} W_{\sigma K} W_{\sigma L}$$

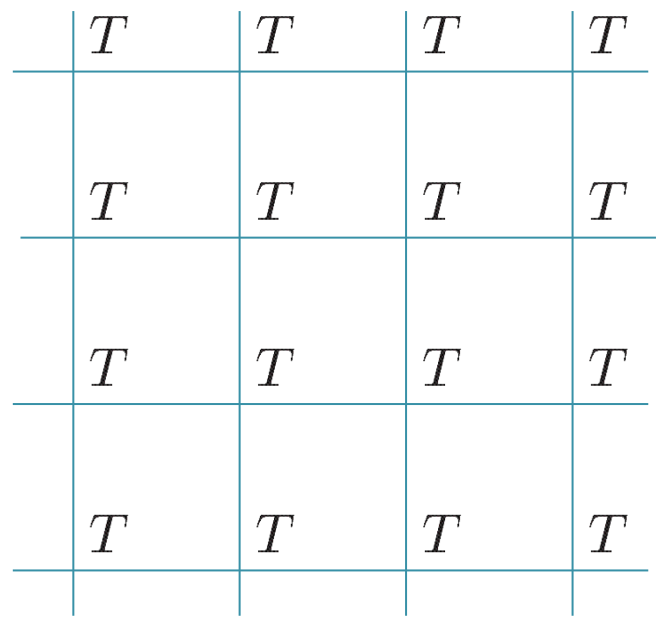


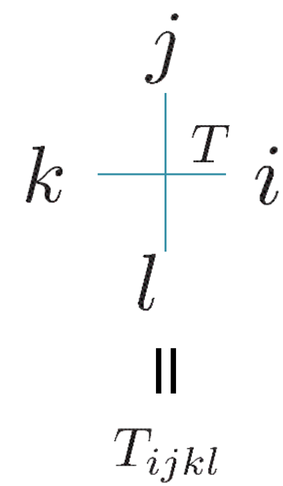
The TN representation of Z (cont'd)

$$Z = \mathrm{Tr} e^{-\beta H}$$

$$= \sum_{l,o,\ldots} T_{ijkl} T_{lmno} T_{opqr} \cdots$$

$=$





Lattice field theory (translational invariance)
→ local and homogeneous tensor networks

Singular value decomposition (SVD)

- SVD of N x N matrix T_{IJ}

$$T_{IJ} = \sum_{m=1}^{N} U_{Im} \sigma_m V_{mJ}$$

singluar values

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N \geq 0$$

$$= \sum_{m=1}^{N} S_{Im} S'_{mJ}$$

$$S_{Im} = \sqrt{\sigma_m} U_{Im} \qquad S'_{mJ} = \sqrt{\sigma_m} V_{mJ}$$

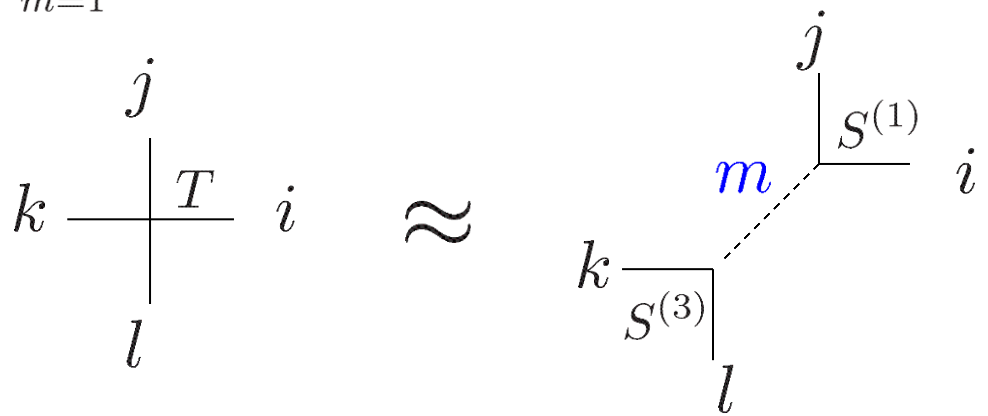
- low rank approximation

$$T_{IJ} \approx \sum_{m=1}^{D_{\text{cut}}} S_{Im} S'_{mJ}$$

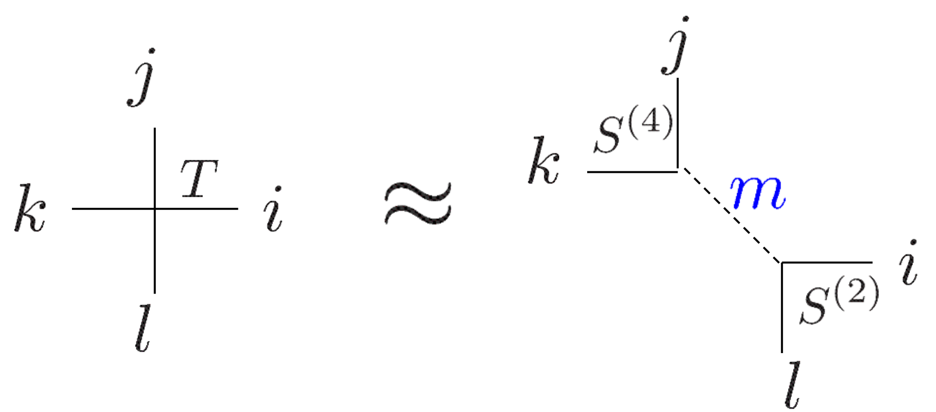
$$D_{cut} < N$$

SVD for tensors

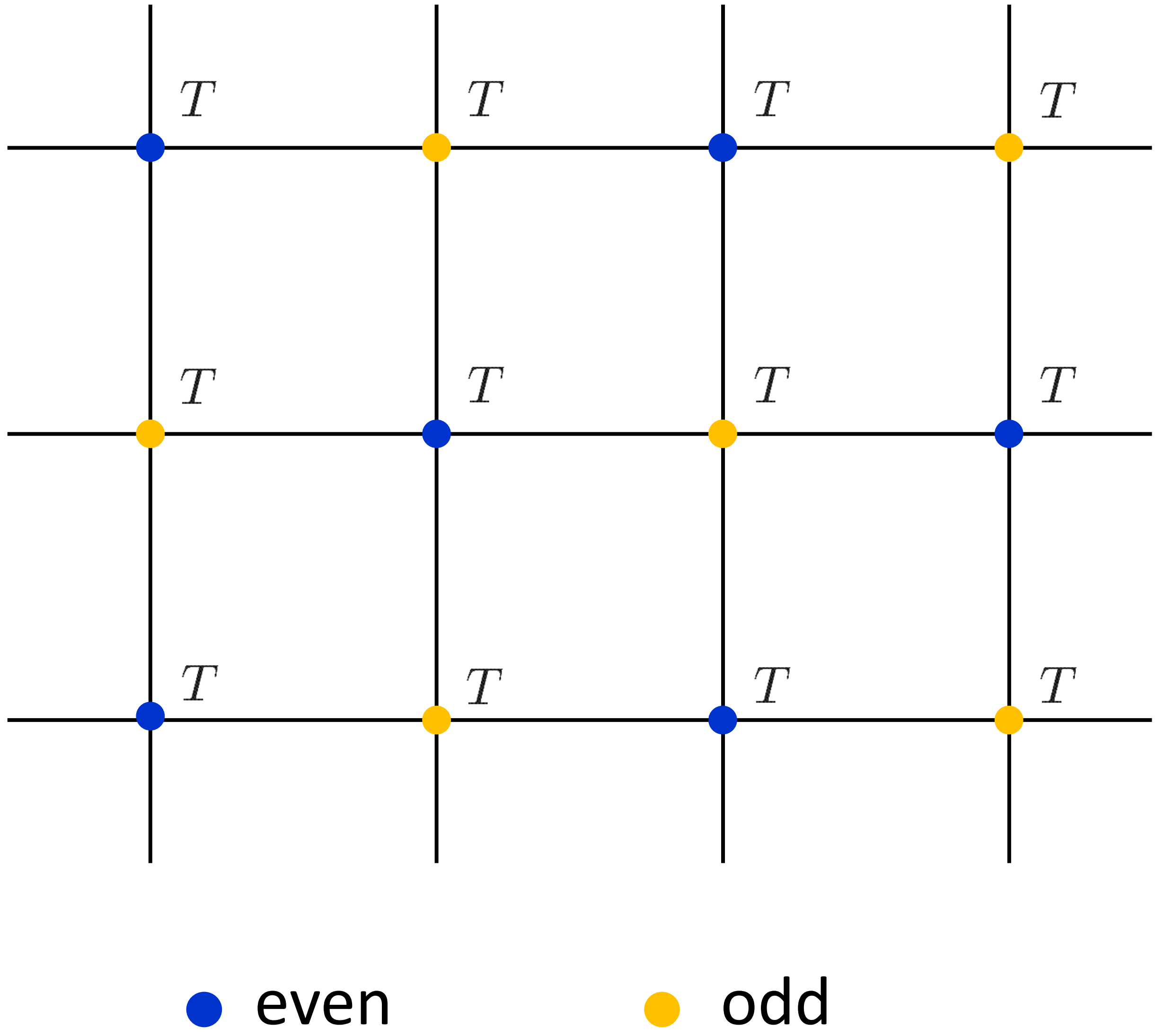
- $T_{ijkl} \approx \sum_{m=1}^{D_{cut}} S^{(1)}_{ijm} S^{(3)}_{klm}$ on even sites ($n_1 + n_2 = \text{even}$)



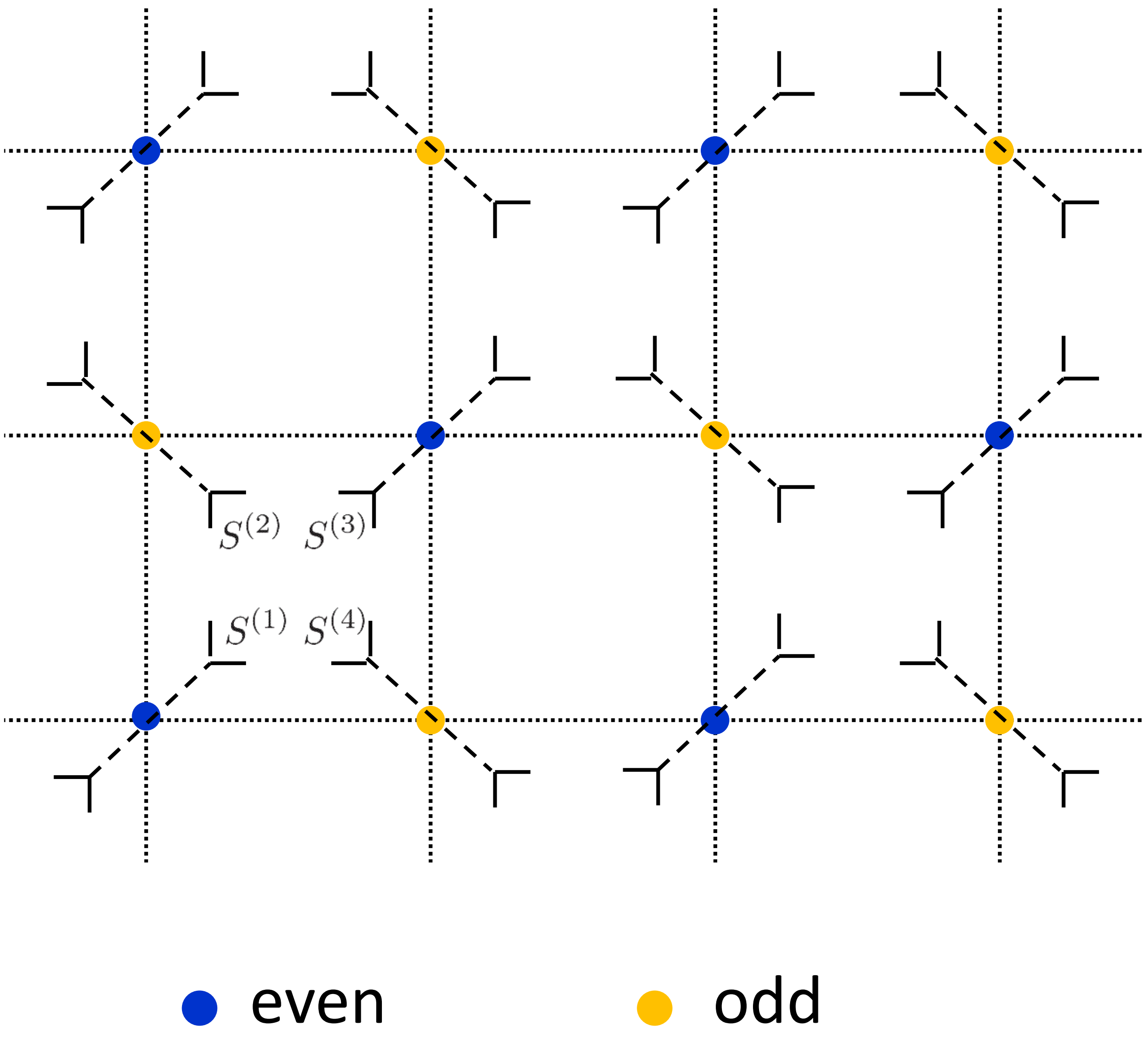
- $T_{ijkl} \approx \sum_{m=1}^{D_{cut}} S^{(2)}_{lim} S^{(4)}_{jkm}$ on odd sites ($n_1 + n_2 = \text{odd}$)



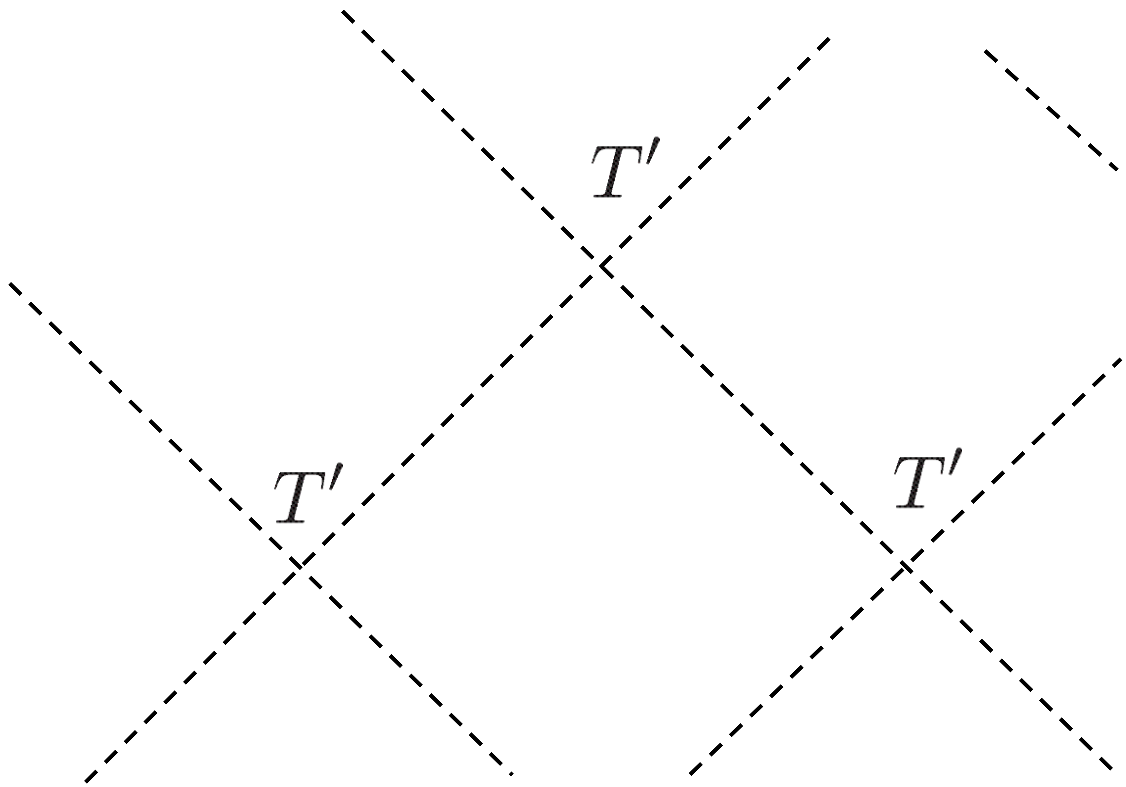
Renormalization step



Renormalization step



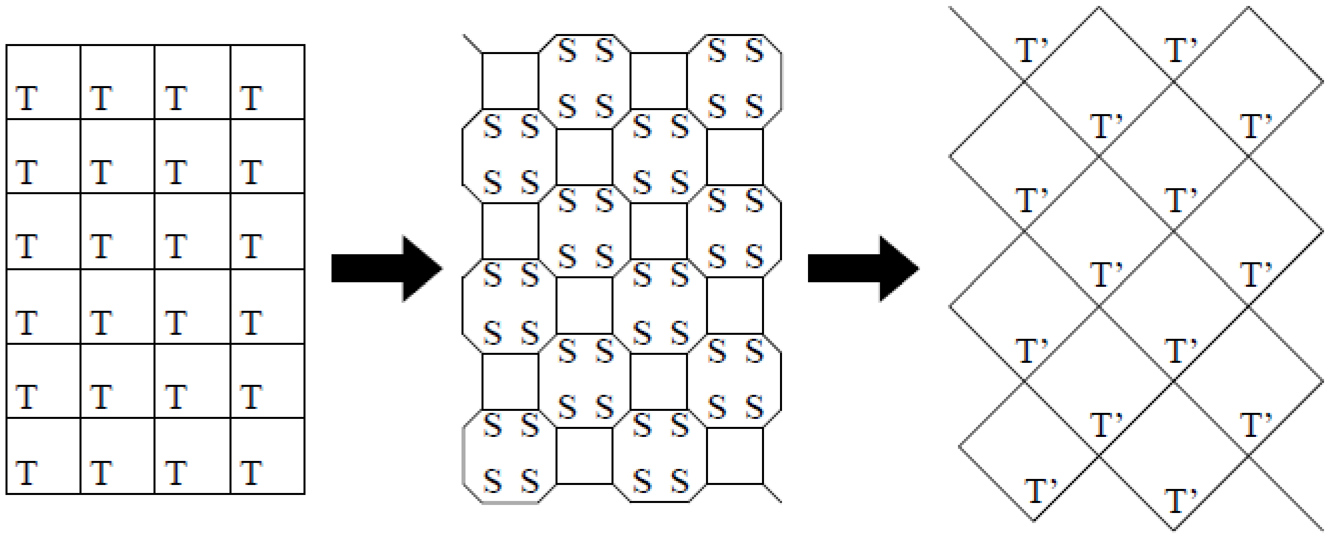
Renormalization step



$$T'_{ijkl} = \sum_{abcd} S^{(1)}_{abi} S^{(2)}_{bcj} S^{(3)}_{cdk} S^{(4)}_{dal}$$

Tensor renormalization group

Levin-Nave, 2007



$$Z = \sum_{i,j,\ldots} T_{ijkl} T_{lmno} \cdots \approx \sum_{i,j,\cdots} S^{(1)} S^{(2)} S^{(3)} S^{(4)} \ldots = \sum_{m,n,\ldots} T'_{mnpq} T'_{mkrs} \cdots$$

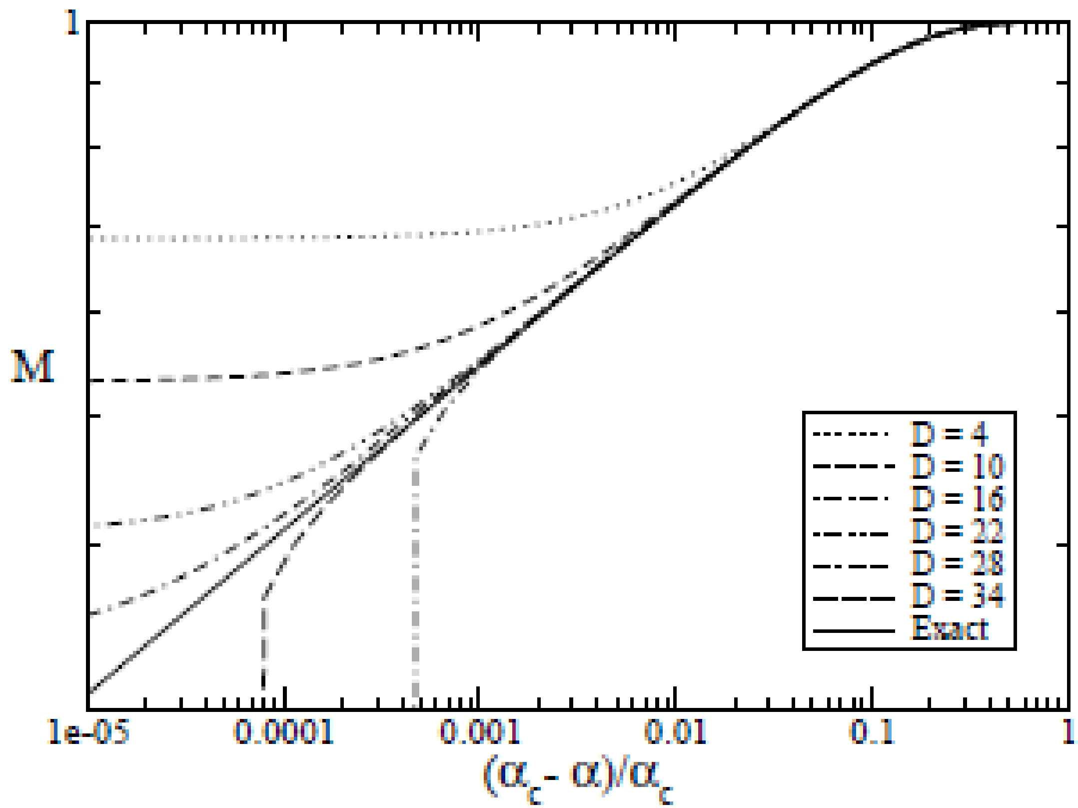
$$Z \approx \sum_{i,j=1}^{D_{\rm cut}} \tilde{T}_{ijij}$$

$$T'_{ijkl} = \sum_{abcd} S^{(1)}_{abi} S^{(2)}_{bcj} S^{(3)}_{cdk} S^{(4)}_{dal}$$

$$\#\ \text{of}\ T' = \frac{\#\ \text{of}\ T}{2}$$

Magnetization

Levin-Nave, 2007



$$\alpha = e^{-2\beta J}$$

Properties of TRG

(1) no sign problem, no statistical errors

(2) systematic error from finite Dcut

(3) partition function is directly calculable

(4) computational cost is $\log(V) \times D_{cut}^{p}$

$p = 5 \quad \text{for } d = 2$ (TRG,2007)

$p = 4d - 1$ (HOTRG,2012)

$p = 2d + 1$ (ATRG,2019)

2d complex ϕ^4-theory at finite density

- continuum action

$$S = \int d^2x \; \{|\nabla_\rho \phi|^2 + (m^2 - \mu^2)|\phi|^2 + \mu(\partial_0\phi^*\phi - \phi^*\partial_0\phi) + \lambda\phi^4\}$$

$m^2, \lambda > 0$	mass and coupling constant
$\mu \geq 0$	chemical potential

- sign problem at finite chemical potential

$$S[\mu]^* = S[-\mu]$$

- bulk observables are independent of the chemical potential for $\mu < \mu_c$ (Silver Blaze phenomenon)

Discretization of scalar fields

- Gaussian quadrature

$$\int_{-\infty}^{\infty} d\phi F(\phi) \approx \sum_{\phi \in S_K} g_K(\phi) F(\phi)$$

e.g. Gauss-Hermite quadrature

S_K = a set of roots of $H_K(\phi)$

$$g_K(\phi) = \frac{2^{K-1} K! \sqrt{\pi}}{K^2 H_{K-1}^2(\phi)} e^{\phi^2}$$

- discretization of the path integral measure

$$\int d\phi_1 d\phi_2 \cdots \phi_N \approx \sum_{\phi_1 \in S_K} \sum_{\phi_2 \in S_K} \cdots \sum_{\phi_N \in S_K} g_K(\phi_1) g_K(\phi_2) \cdots g_K(\phi_N)$$

TN representation of scalar theory

- decompose the hopping term

$$h(\phi_n, \phi_{n-\hat{1}}) = \exp\left\{-\frac{1}{2}(\phi_n - \phi_{n-\hat{1}})^2 - \frac{1}{8}\mu_0^2(\phi_n^2 + \phi_{n-\hat{1}}^2) - \frac{\lambda_0}{16}(\phi_n^4 + \phi_{n-\hat{1}}^4)\right\}$$

$$= \sum_{I=1}^{K^2} U_{\phi_n I} \sigma_I V^\dagger_{I \phi_{n-\hat{1}}}$$ (SVD of KxK matrix)

$$= \sum_{I=1}^{K^2} W^{(1)}_{\phi_n I} W^{(2)}_{\phi_{n-\hat{1}} I}$$

$$W^{(1)}_{\phi I} = U_{\phi I}\sqrt{\sigma_I} \qquad W^{(2)}_{\phi I} = \sqrt{\sigma_I} V^\dagger_{I\phi}$$

- make tensor from W:

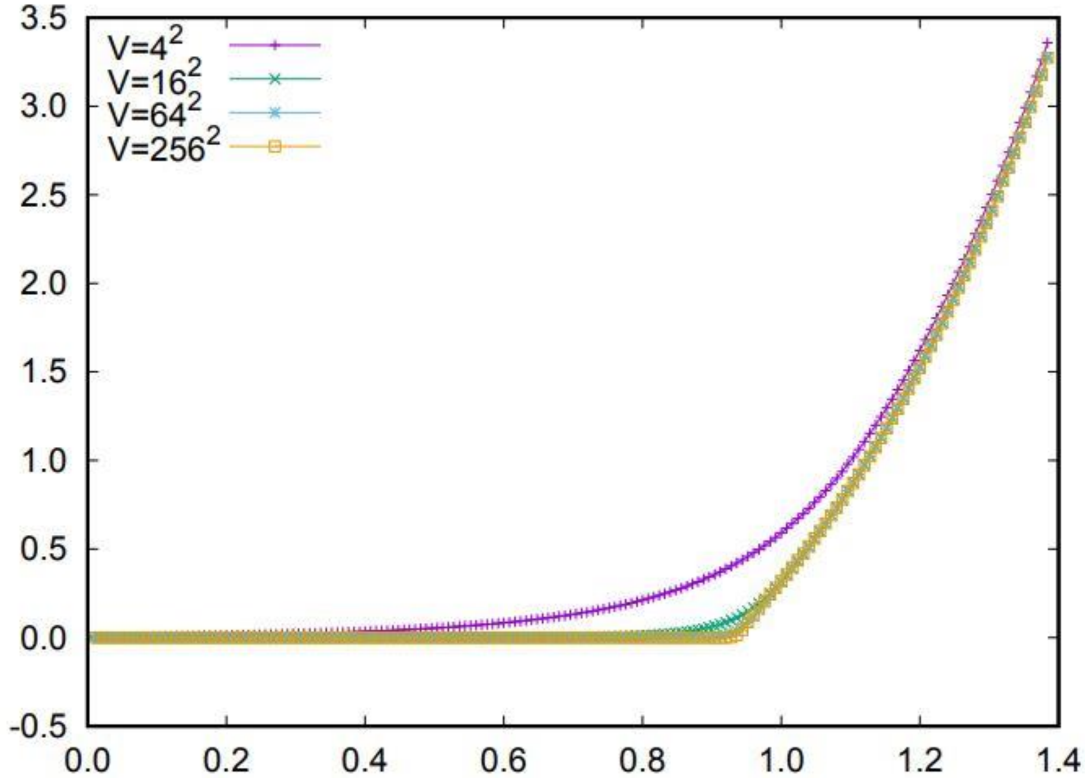
$$T_{IJKL} = \sum_{\phi \in S_K} g_K(\phi) W^{(1)}_{\phi I} W^{(1)}_{\phi J} W^{(2)}_{\phi K} W^{(2)}_{\phi L}$$

$$I, J, K, L = 1, 2, \ldots, D$$

The particle number density

[D.K., Kuramashi, Nakamura, Sakai, Takeda, Yoshimura (2019)]

$\langle n \rangle$



μ

$\lambda = 1$

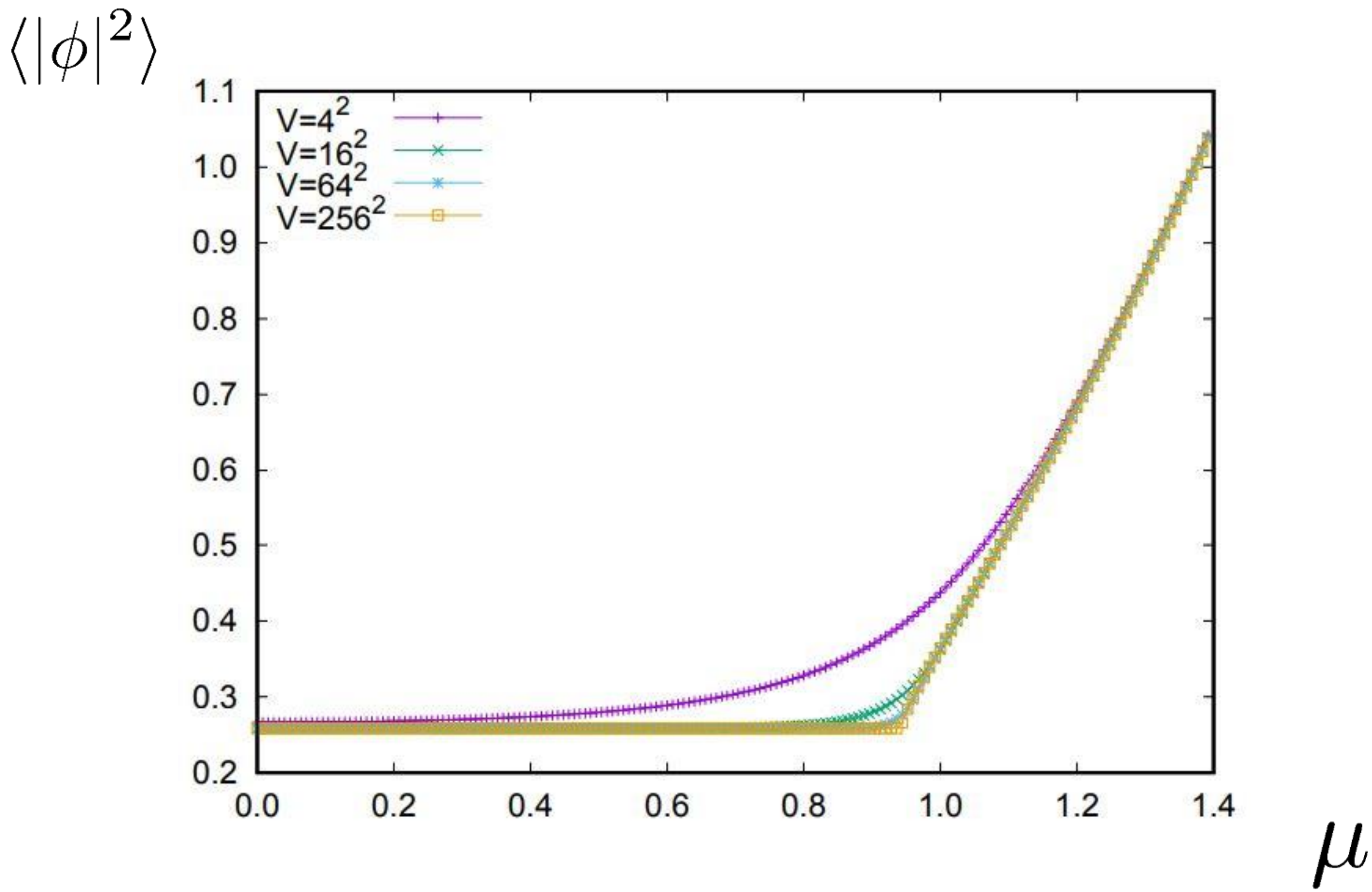
$m^2 = 0.01$

$K = 64$

$D_{cut} = 64$

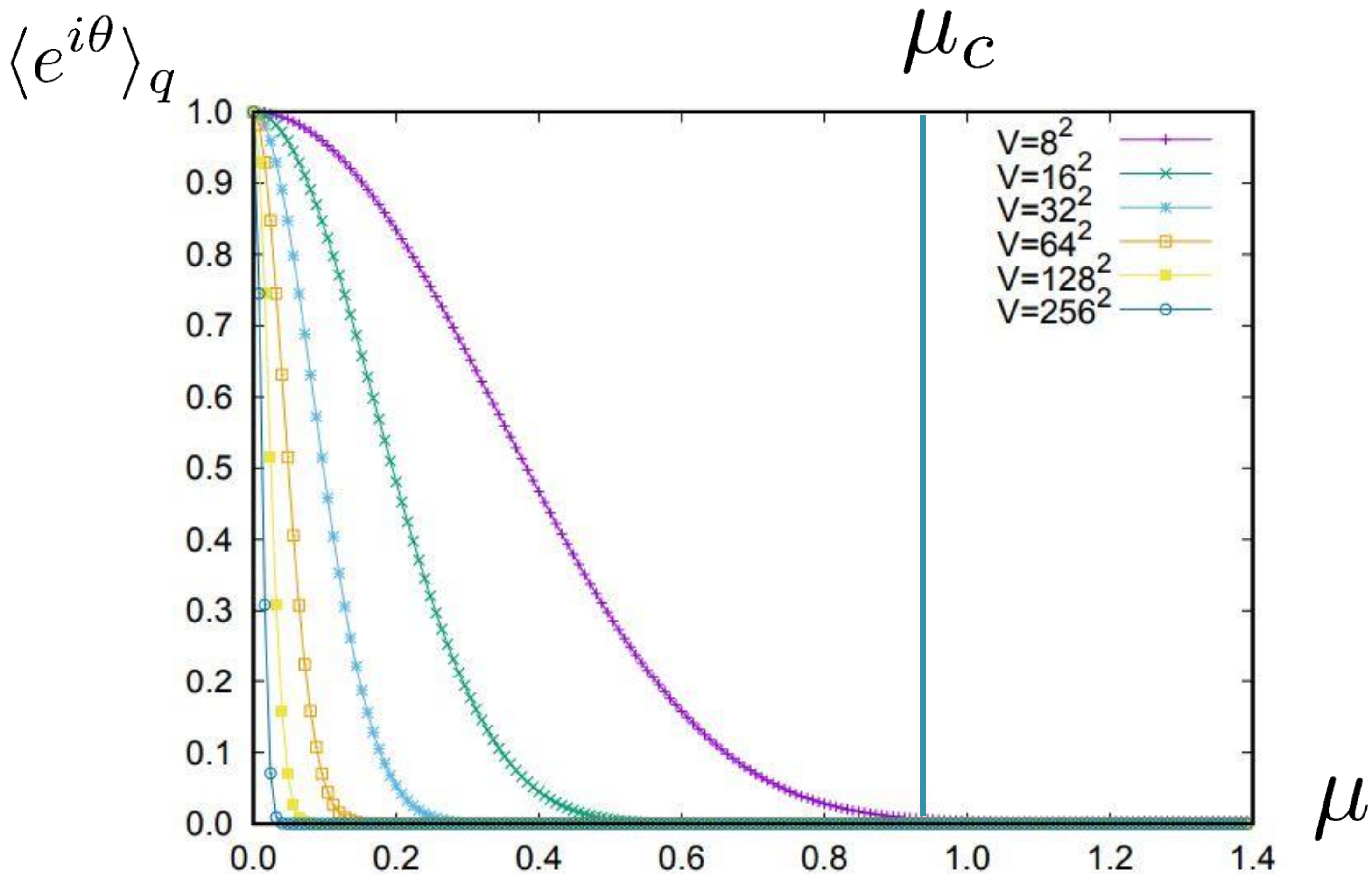
The Silver Blaze is clearly observed for large volume lattices.

The expectation value of $|\phi|^2$



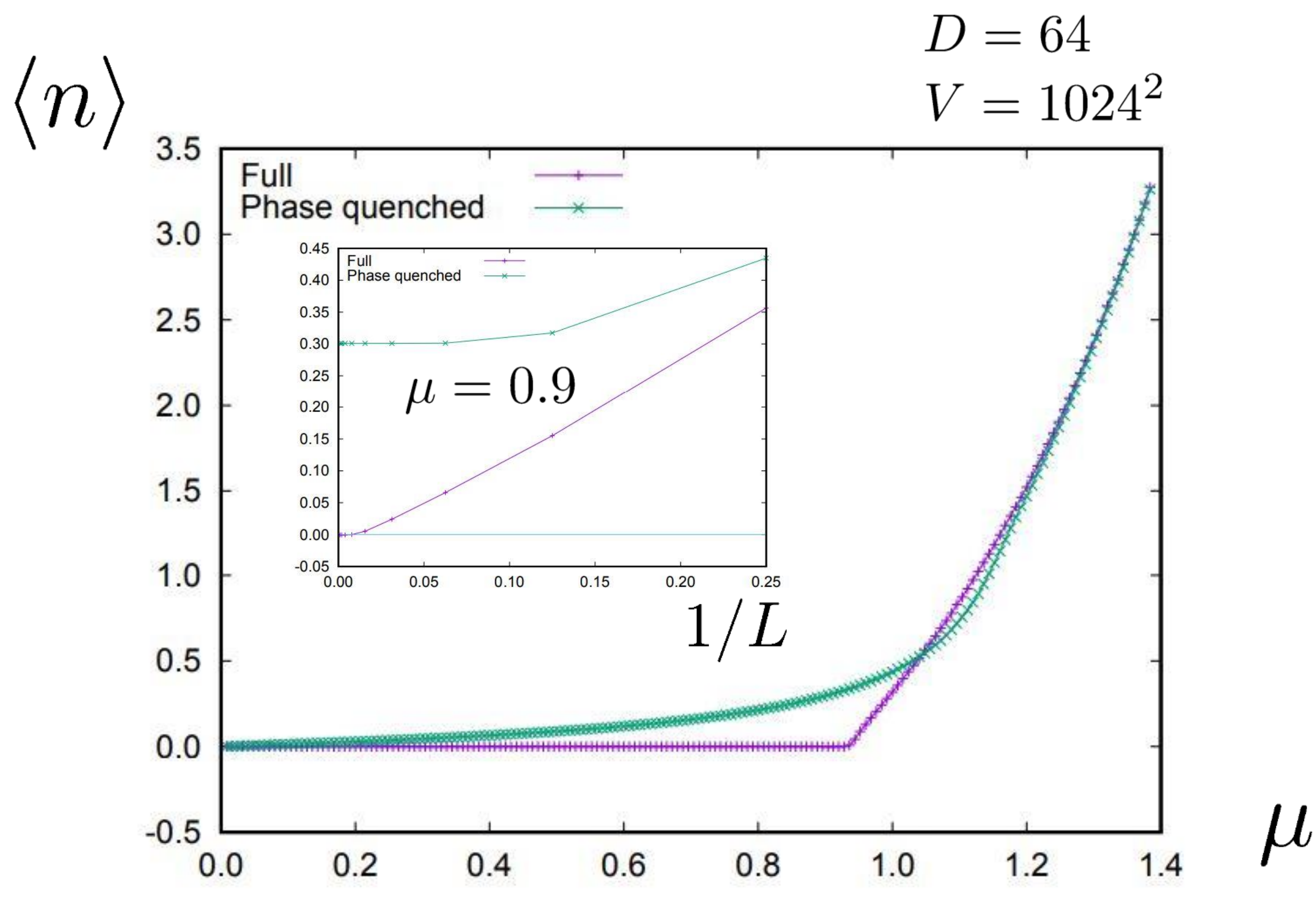
Results similar to number density can be seen.

Average phase factor



The sign problem becomes severe for large volume lattices.

Silver Blaze vs. phase



The Silver braze is not obtained without the phase, and TRG properly works for LFT with severe sign problem.

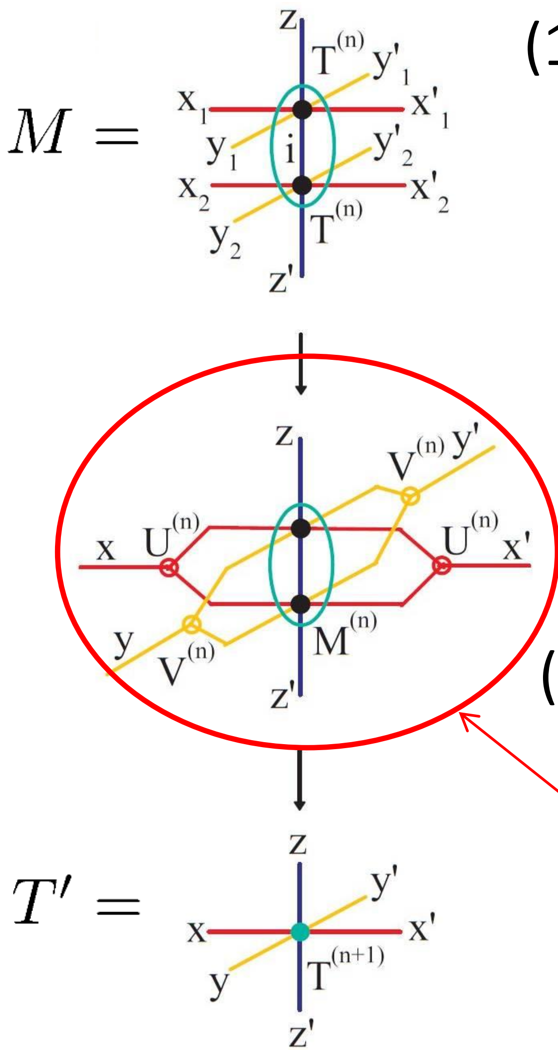
3. The triad TRG method

Kadoh and Nakayama, arXiv:1912.02414

Higher-order TRG (HOTRG)

Xie et al., 2012

3d case



(1) Make projectors from two Ts

$$M = T \cdot T$$

$\mathcal{O}(D^{2d+2})$

diagonalization:

$$(UMM^{\dagger}U^{\dagger})_{XX'} = \sigma_X \delta_{XX'}$$

U_{X,x_1x_2} : projector for x-direction

(2) Take contractions with projectors

→ a renormalized tensor T'

$\mathcal{O}(D^{4d-1})$

Can we wait?

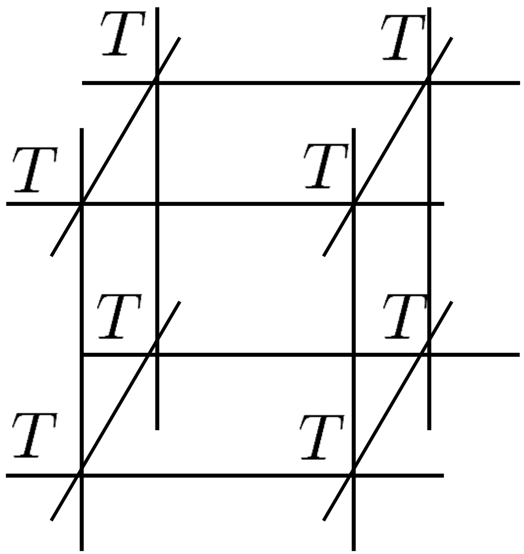
computational time is a few hours for D=32 in 2d.
however, in 4d, it becomes…

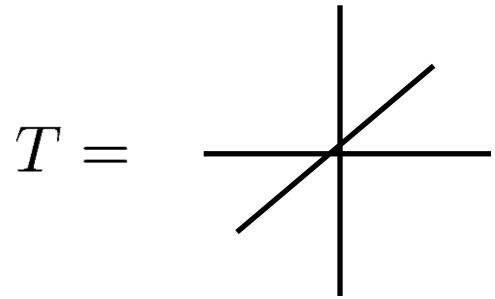
$$1.5[h] \times 32^8 \fallingdotseq \text{200 million years!!!}$$

$$D^7 \longleftrightarrow D^{15}$$

→ need to create a low-cost scheme applicable to higher dimensions

Why HOTRG's cost is high?





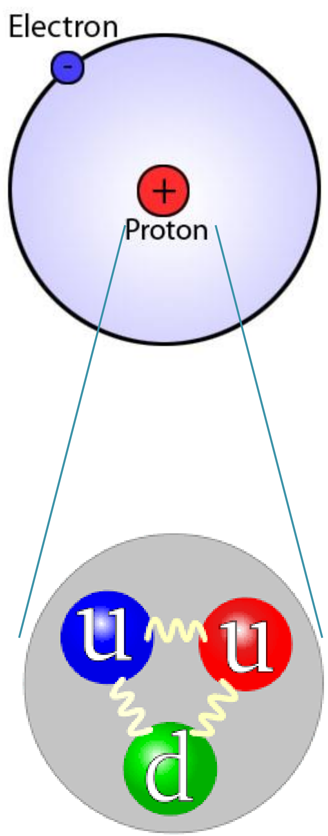
2d-rank tensor

tensor networks on
hyper cubic lattice

The cost of contracting
two 2d-rank tensors
is high for large d.

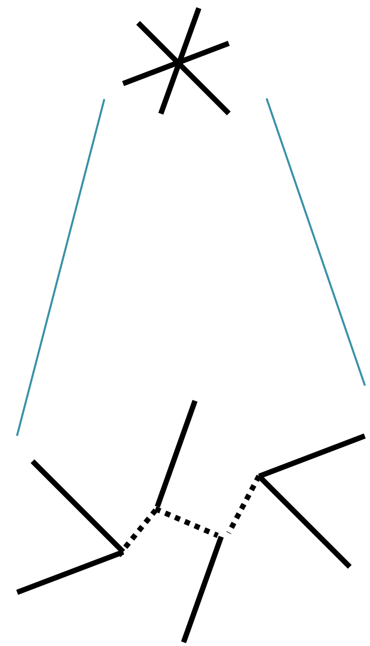
We should reconsider a theory of tensor networks at a fundamental level.

Fundamental building blocks



quarks

rank-6 tensor



rank-3 tensors

“triads”

Hidden structure

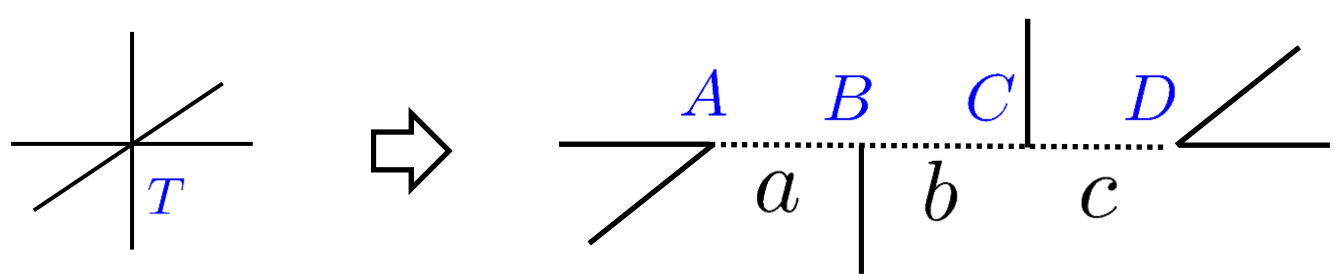
tensor in the polyadic decomposition

$$T_{ijklmn} = \sum_{a=1}^{r} W^{(1)}_{ia} W^{(2)}_{ja} W^{(3)}_{ka} W^{(4)}_{la} W^{(5)}_{ma} W^{(6)}_{na}$$

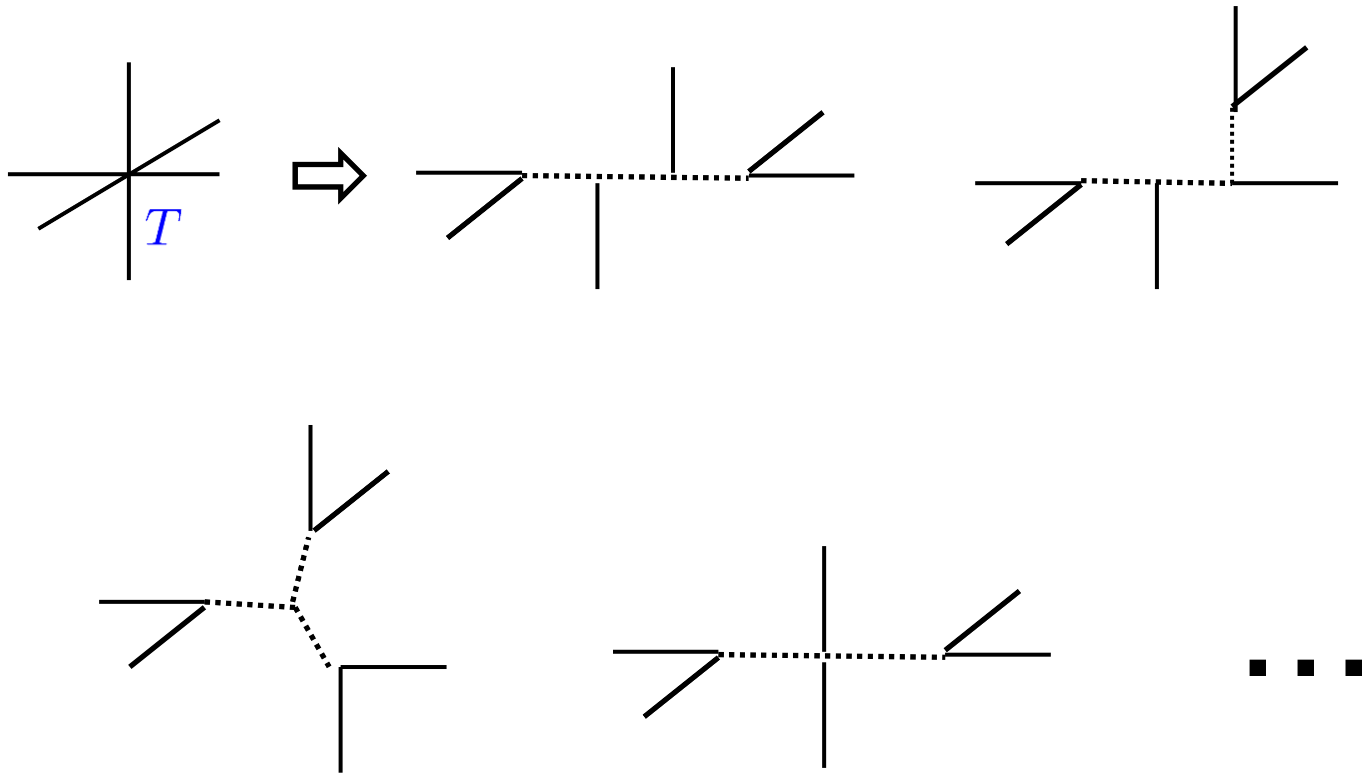
$$= \sum_{a,b,c=1}^{r} A_{ija}\, B_{akb}\, C_{blc}\, D_{cmn}$$

$$A_{ija} = W^{(1)}_{ia} W^{(2)}_{ja}, \quad D_{cmn} = W^{(5)}_{mc} W^{(6)}_{nc}$$

$$B_{akb} = W^{(3)}_{ak} \delta_{ab}, \quad C_{blc} = W^{(4)}_{lb} \delta_{bc}$$

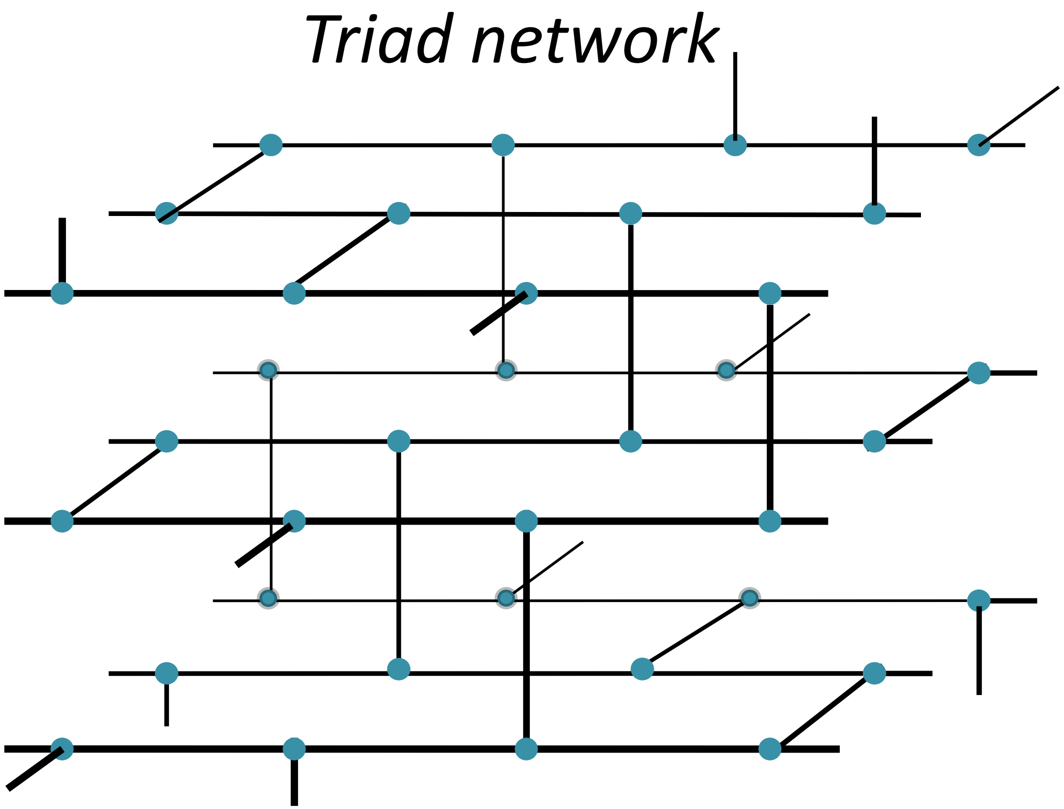


Many kinds of triad representation



tetrad mixture

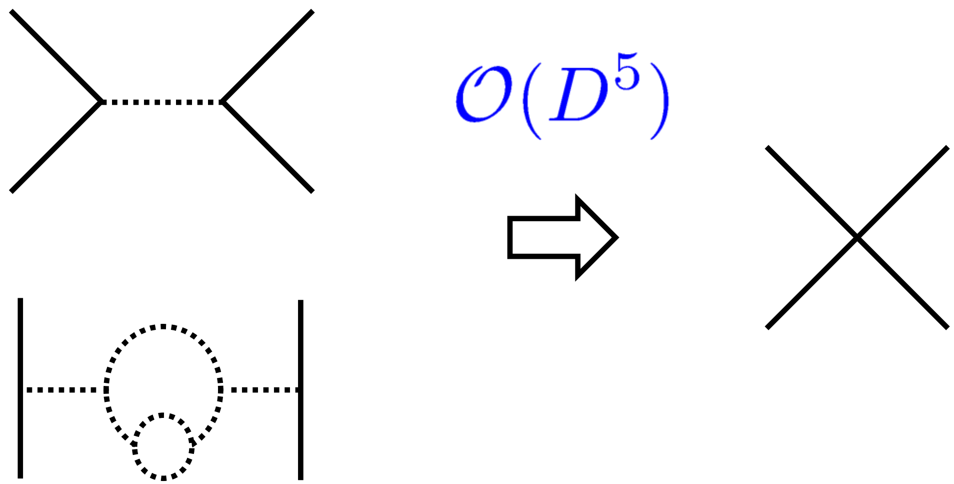
Triad networks and RGs



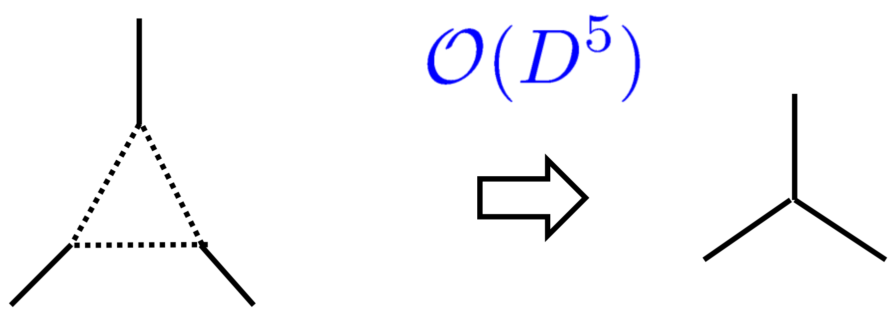
The cost of RGs on a triad network is naturally reduced because ...

The cost for rank-3 tensors

network of rank-3 tensors



rank-4 tensor

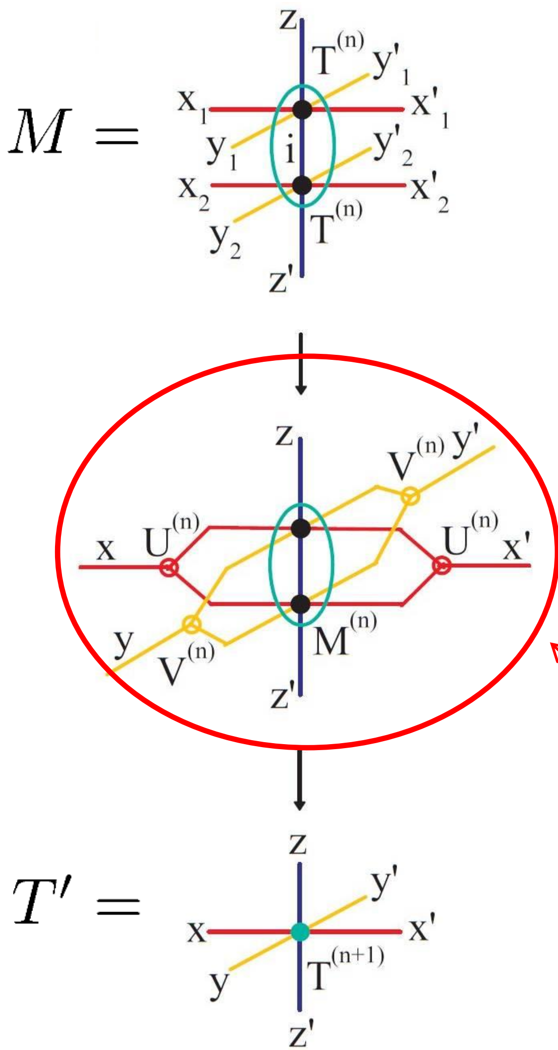


rank-3 tensor

Higher-order TRG (HOTRG)

Xie et al., 2012

3d case



$$M = T \cdot T$$

$$(UMM^{\dagger}U^{\dagger})_{XX'} = \sigma_X \delta_{XX'}$$

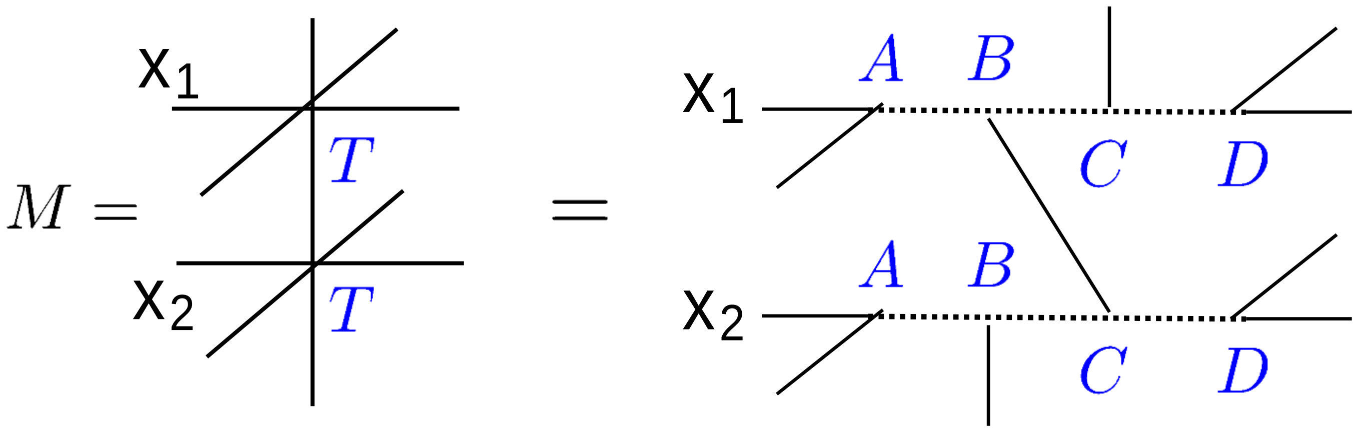
U_{X,x_1x_2} : projector

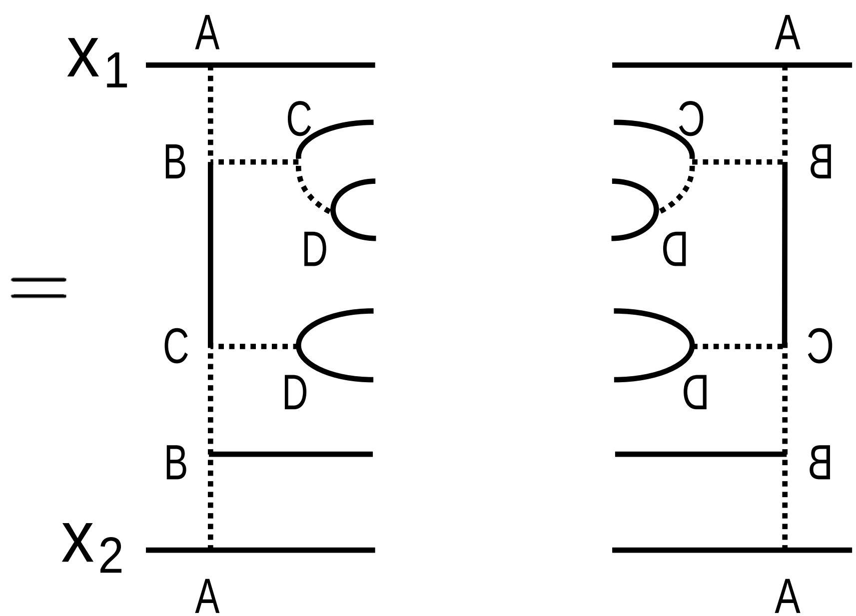
$\mathcal{O}(D^{2d+2})$

Contractions for making a renormalized tensor

$\mathcal{O}(D^{4d-1})$

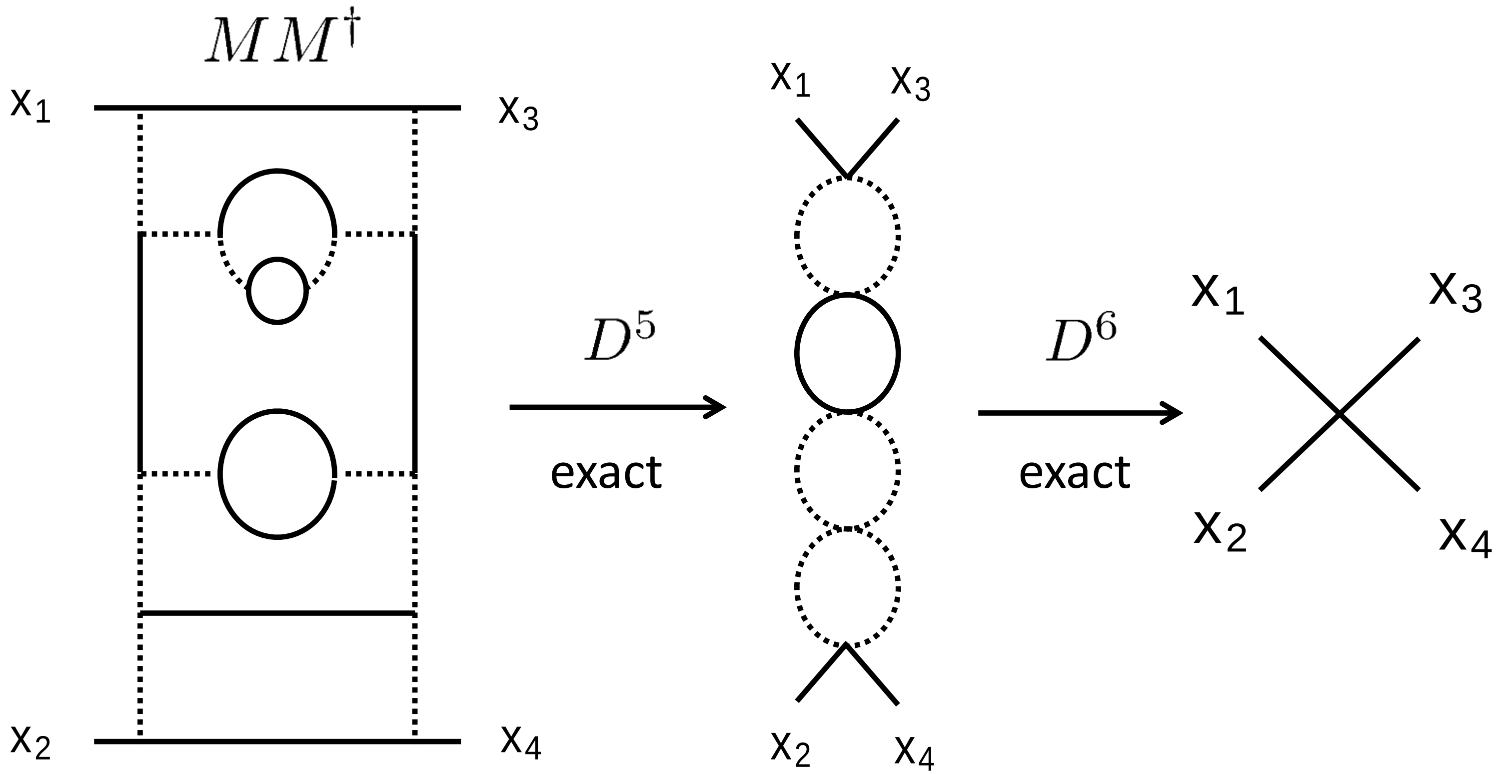
M in the triad representation





$M^\dagger$ is a mirror image of M.

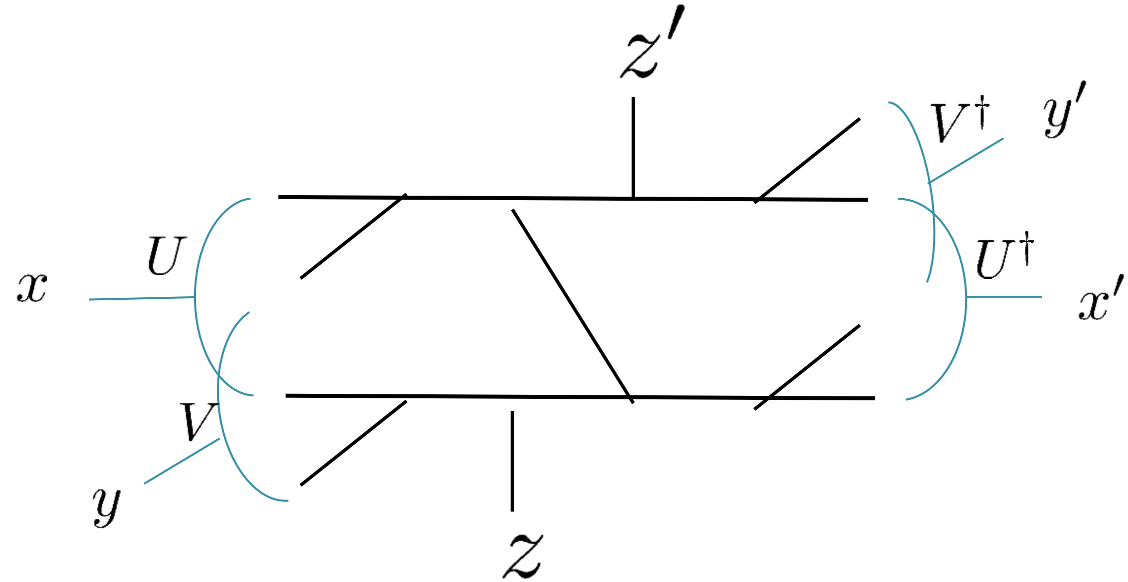
Steps of making projectors



projectors U can be exactly prepared at an $\mathcal{O}(D^6)$ cost in any dimension!

→ $\mathcal{O}(D^5)$ by using a randomized SVD

Contraction of two triads



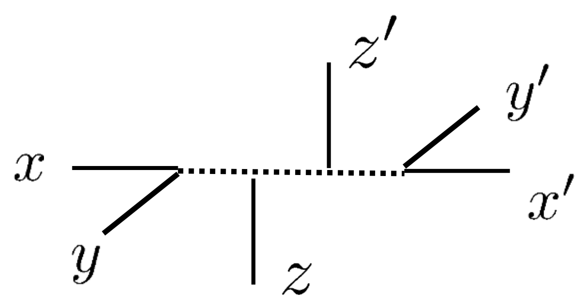
2 projectors

U_{X,x_1x_2}

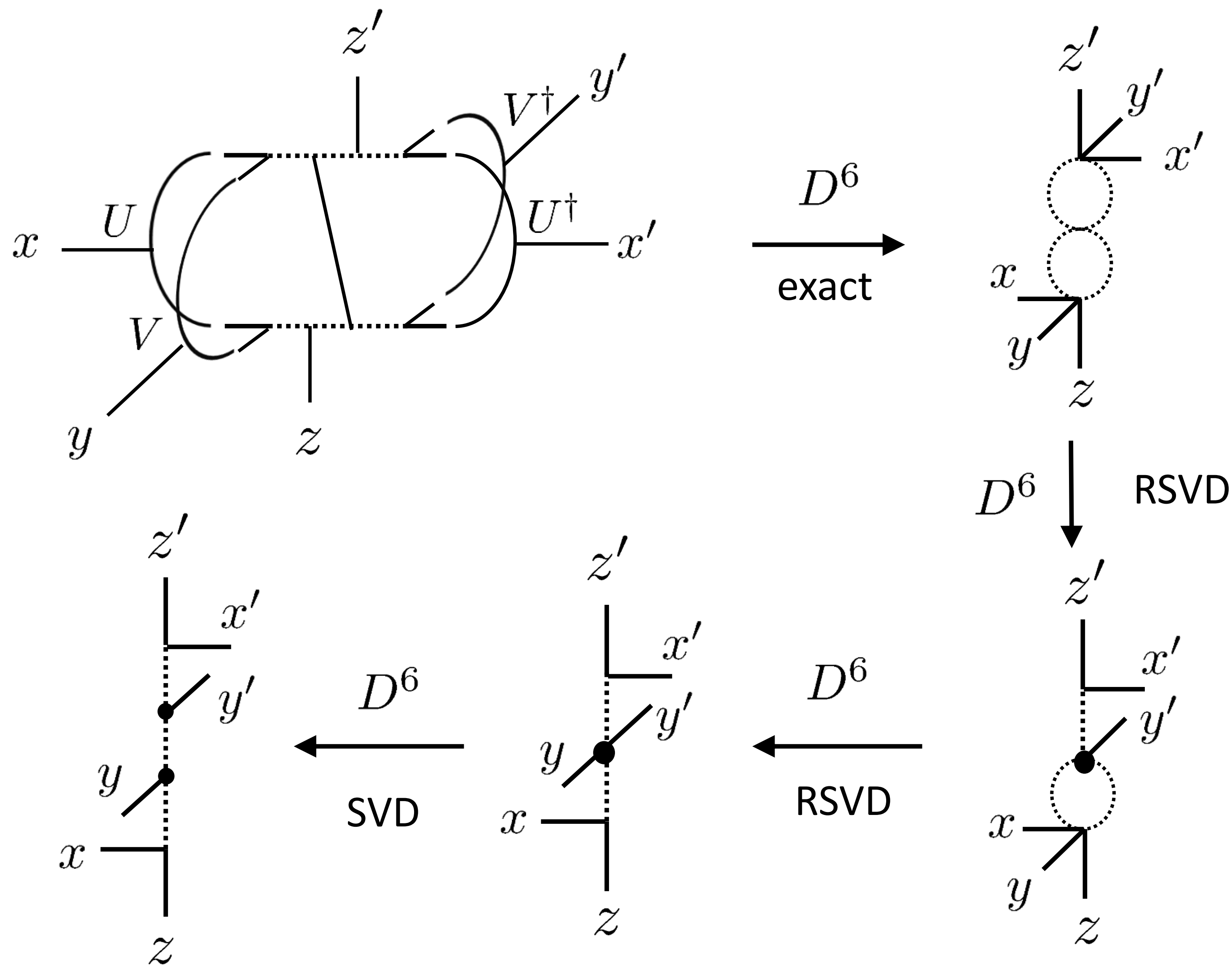
V_{Y,y_1y_2}

contractions & RSVDs whose cost are $\mathcal{O}(D^{d+3})$

renormalized triad



Steps of making a renormalized triad

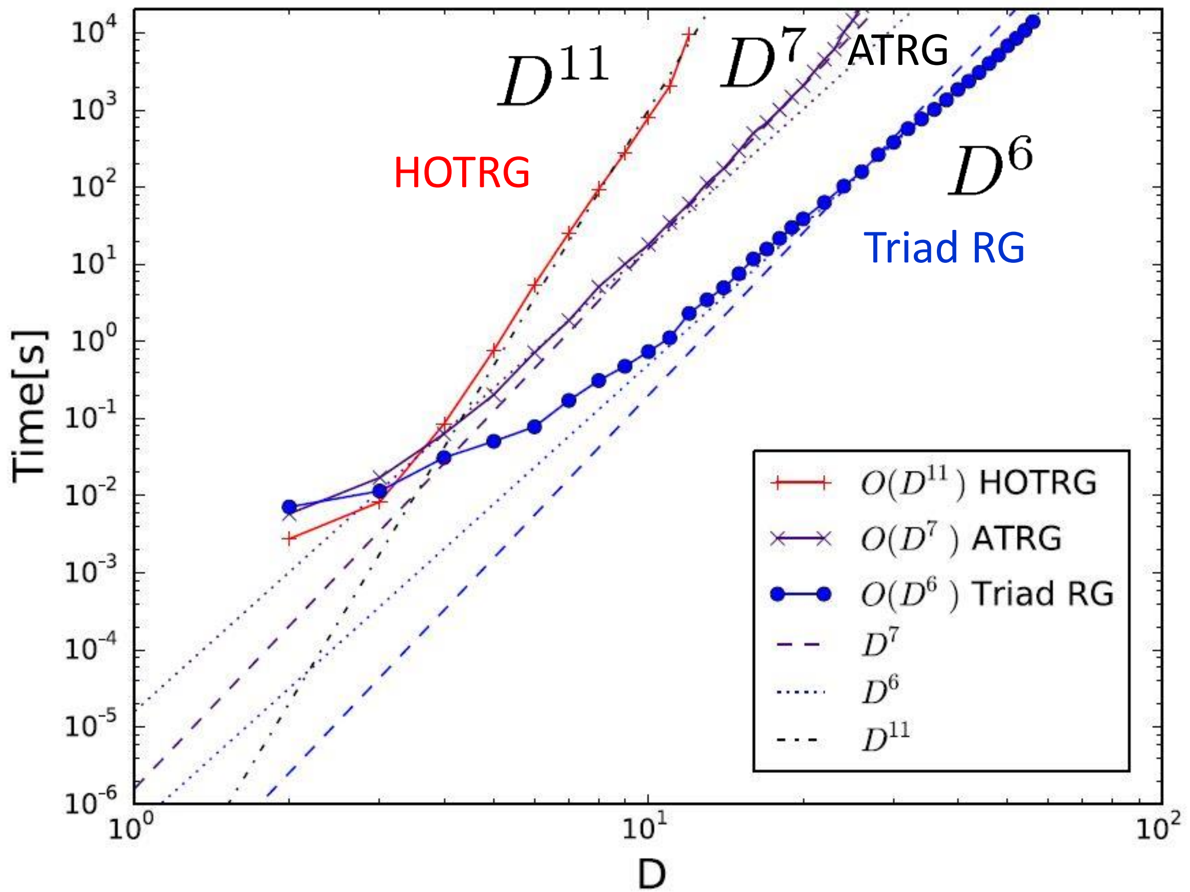


Theoretical cost

	dimensionality			
	2	3	4	d
TRG	D^5	—	—	—
HOTRG	D^7	D^{11}	D^{15}	D^{4d-1}
Anisotropic TRG [Adachi et al.,2019]	D^5	D^7	D^9	D^{2d+1}
Triad TRG (this work)	D^5	D^6	D^7	D^{d+3}

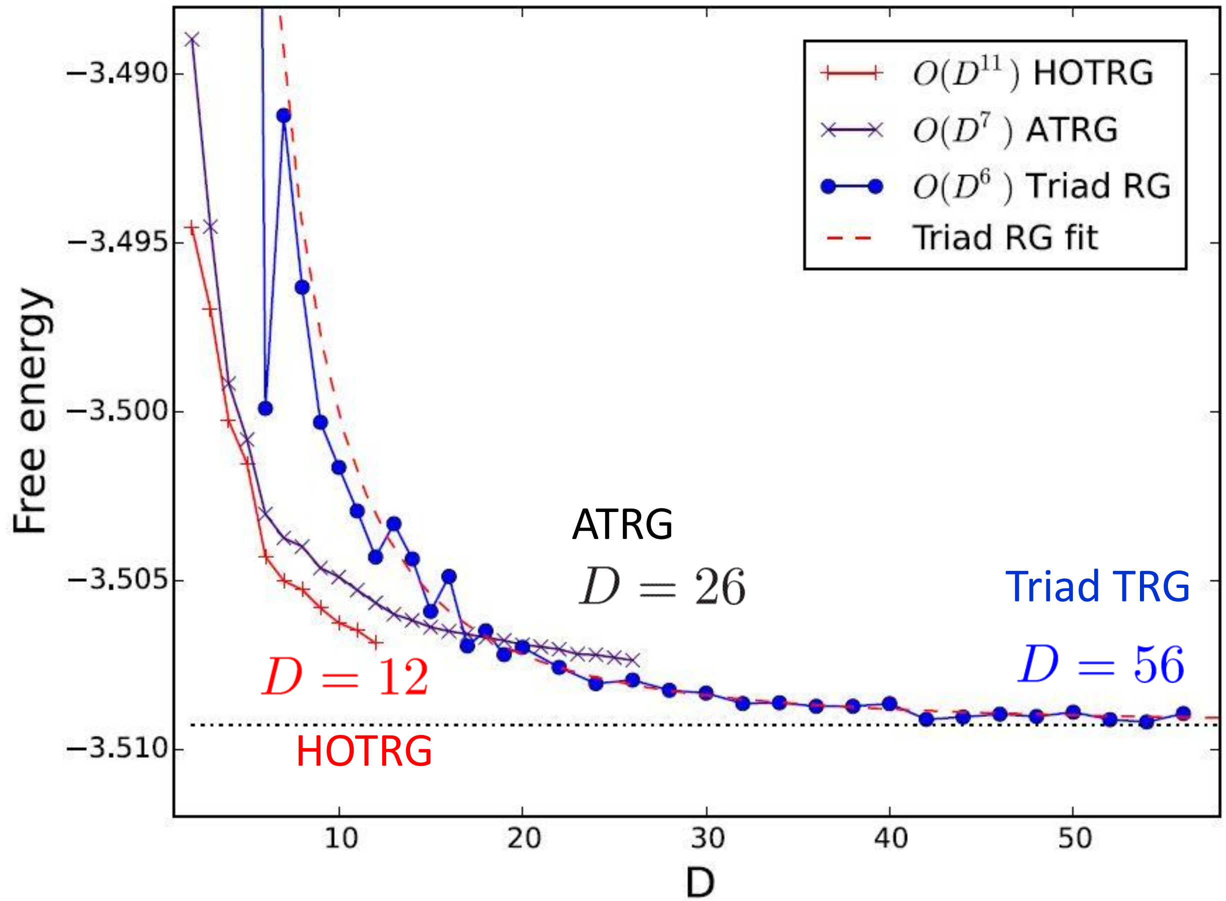
Numerical test in 3d Ising model at Tc

Computational time vs. D



Theoretical D-dependence is properly reproduced in actual computations.

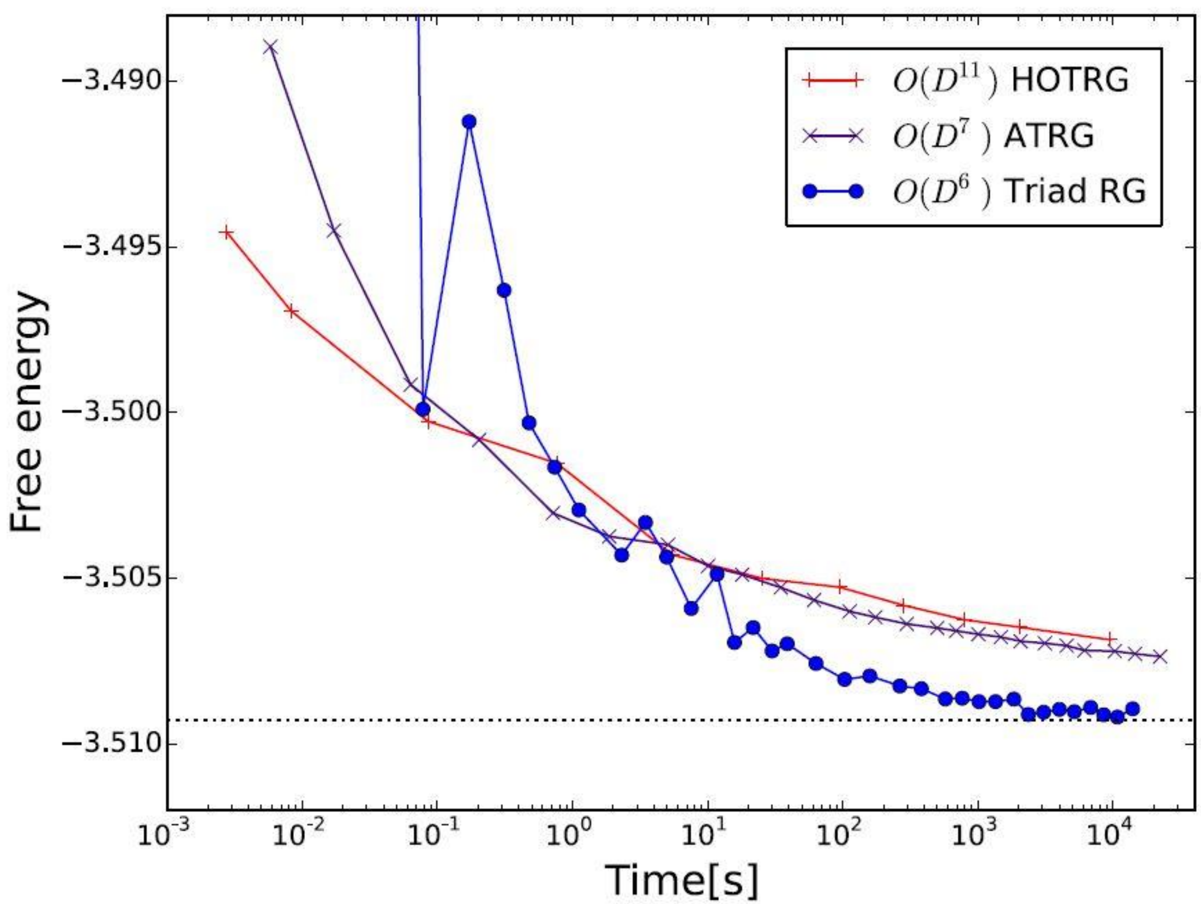
D-dependence of free energy



a few hours using laptop computers

The Triad TRG method shows good convergence as D increases.

Free energy vs. computational time



HOTRG
ATRG
Triad TRG

The other methods need much more time to approach a converged value around -3.509.

4. Summary and future outlook

Conclusions

- The tensor network approach:

  Good: no sign problem & no statistical error
  free energy is directly calculable.
  the large volume limit is easily taken.

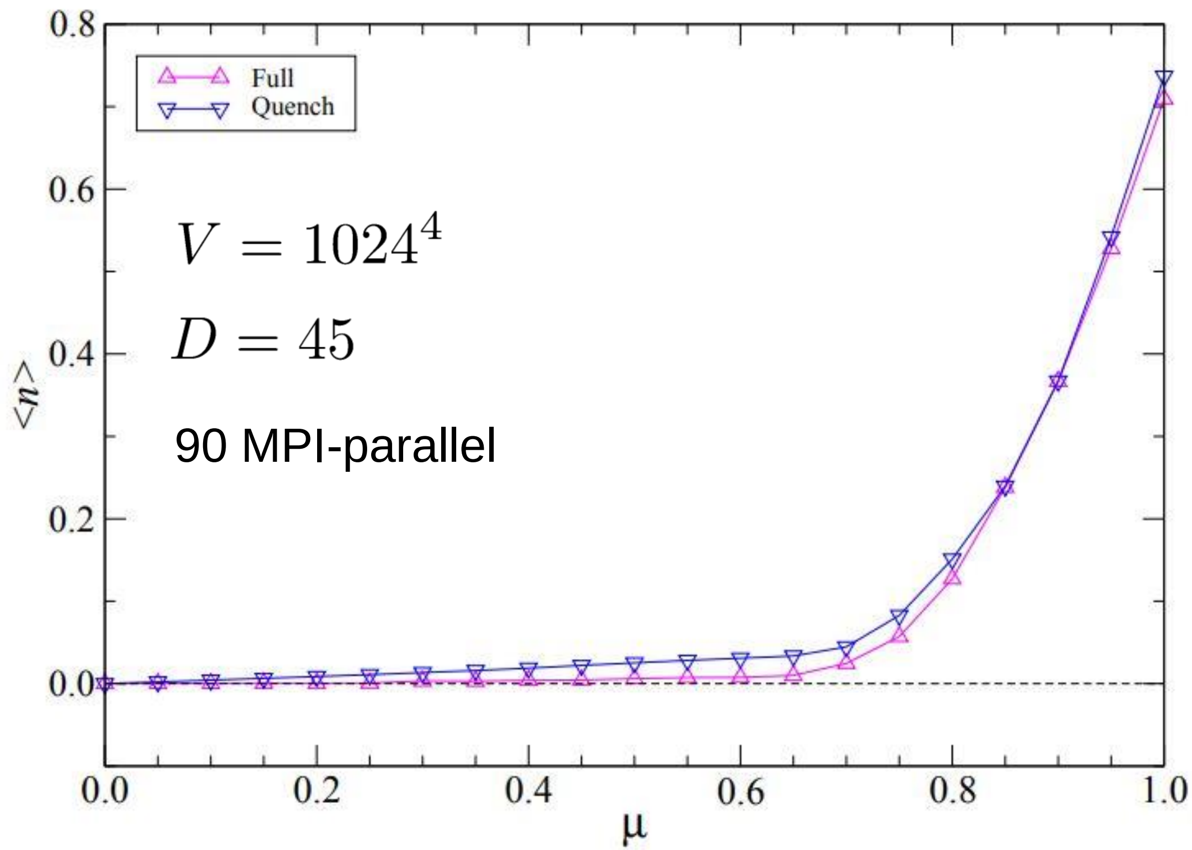
  Bad: cost is high for $d \geq 3$ and large internal DoF.

- We find that the Gaussian quadrature works for scalar theory.

- The triad networks are effective in higher dimensions (at least, in 3 dimensions).

4d complex scalar theory at finite density

Akiyama, DK, Kuramashi, Yamashita, Yoshimura. JHEP 2020



$\lambda = 1$

$m^2 = 0.01$

$K = 64$

The Silver braze is obtained for 4d complex scalar theory at finite density using the parallel computation of ATRG with a supercomputer.

future outlook

- fermions

  The Grassmann-TRG [Gu et al.,2010] is reformulated in [Akiyama-Kadoh,2020].

- lattice gauge theory

  character expansion of gauge group

- further improvements of triad TRG

  many kinds of triad → many variants of the RG

- high cost for large internal DoF (such as large N gauge theories)

  quantum computer?