ICTS-RRI Maths Circle, Saturday 26th July 2025
Geometry and Topology-II

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We will continue to explore topology through graph theory. We start with the Königsberg problem, understand the Euler characteristic and apply this knowledge to a problem in geometry. These are subjects that can be appreciated at an elementary school level, but connect to very advanced areas of Mathematics and Physics. This theme will continue over the next three sessions.

Like last time, we will spend some time discussing mathematics in general: its scope, traditions, history, and applications. This part will consist of some exposition followed by a general discussion in which we will all participate. To prepare, you could read up on different traditions of mathematics across the world, and the history of mathematics. It would be nice to share our collective knowledge of this subject. Last time you solved a problem in mathematical cryptography by deciphering Egyptian and Mesopotamian relics. Was this easier or harder than regular cryptography? Why?

After the tea-break we will start with a puzzle which will introduce the subject of topology. While this subject is very intuitive, it is not taught in schools at all. Indeed, formalising the intuition took many years. It was only in the twentieth century that the subject came into its own. We will study the problem of 'The bridges of Königsberg'.

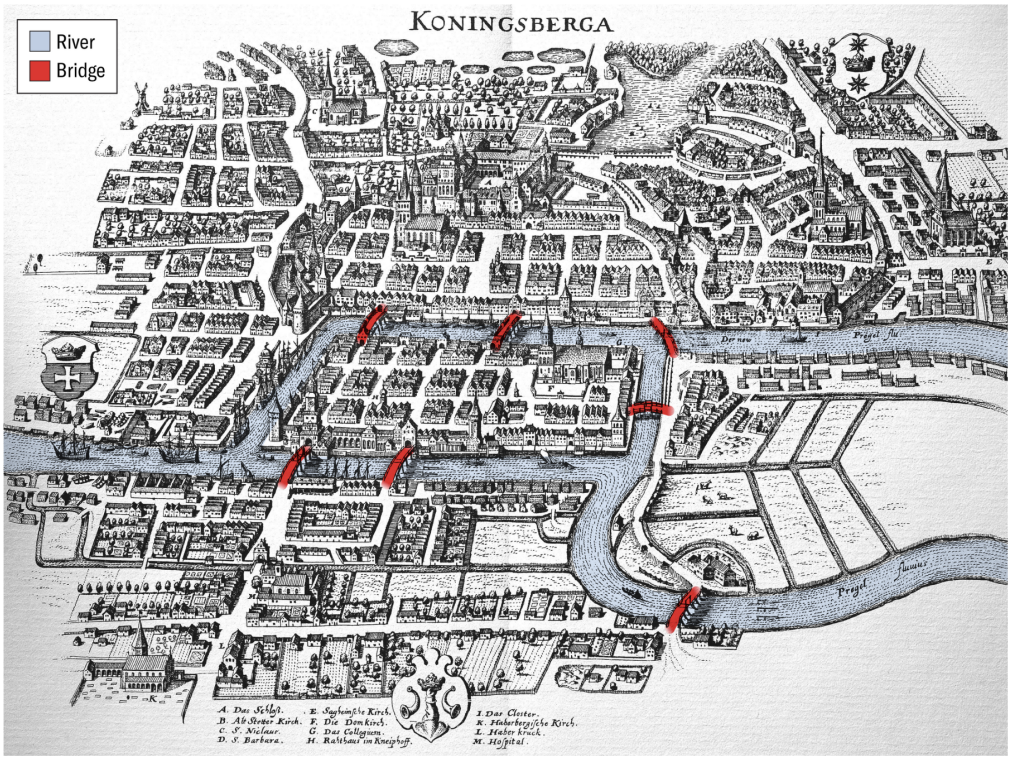

Figure 1: The Prussian city of Königsberg, now Kaliningrad in Russia, in the Middle Ages. The 18th-century city map is a digitally enhanced reproduction that was modified to highlight the river and bridges.Credit: Alamy Stock Photo (map); Amanda Montañez (highlighting), from Scientific American.

The residents of Königsberg, had a curious, unsettled question. The River Pregel, which ran through the city, had two islands, and there were seven bridges connecting the various land masses (as shown in the figure). The question was: Is it possible to walk through the city traversing each bridge exactly once? Opinion was divided: some said it couldn't be done. Others were not so sure. Finally, someone wrote to Leonhard Euler, the great Swiss mathematician. Euler was initially

irritated with the question, deeming it to be outside the realm of mathematics, which dealt with questions of quantity like numbers, lengths, weights, areas and angles. When his irritation subsided, Euler answered this question in the negative, in the process, sowing the seeds of two *new* branches of mathematics: topology and graph theory. In a previous session with Prof. Jaikumar you learned about graphs and the matching problem. These subjects deal with the mathematics of quality rather than quantity. They are not only interesting in their own right; topology and graph theory can be used to answer questions in geometry.

Last time we discussed the problem of supplying Gas, Water and Electricity from three stations to three houses. The constraint was that the supply lines should not cross, and each house should be supplied directly from each station with all three utilities. Think about this on the plane, the sphere and on a Vada (doughnut) shaped Earth. Last time you drew a plan of your house using a graph. Now, design a house, whose graph can be drawn on a plane and has at least two faces. Next, design a house whose graph cannot be drawn on a plane.

The ancient Greeks knew five Platonic solids (regular solids). You will prove using graph theory that there *are* only five.

Note for Maths circle old timers: Some of you may have already been exposed to this subject. If so, please hold your peace so that the others have the pleasure of discovery.