Zero velocity Lagrangians in Higgs bundle moduli spaces

Geometric Structures and Stability
ICTS Bengaluru

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Introduction

- X - compact Riemann surface of genus $g \geq 2$;
- $\mathcal{N}$ - moduli space of semistable rank n vector bundles over X;
- $T^*\mathcal{N} \subset \mathcal{M}$ - moduli space of semistable $\mathrm{SL}(n, \mathbb{C})$-Higgs bundles over X, equipped with natural symplectic structure;

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- In some simple cases, these can be realized as conormal bundles of submanifolds of other Lagrangians.
  - $\{E\} \subset \mathcal{N} \rightsquigarrow T^*_E\mathcal{N} \hookrightarrow \mathcal{M}$.

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  - $\{E\} \subset \mathcal{N} \rightsquigarrow T^*_E\mathcal{N} \hookrightarrow \mathcal{M}$.
- **"Infinitesimal conormal principle":** A Lagrangian L in $\mathcal{M}$ and a submanifold S of L should give rise to another Lagrangian by integrating the conormal spaces $N^*_p S \subset T^*_p L$, with $p \in S$.

Fixed points of the $\mathbb{C}^*$-action

- The group $\mathbb{C}^*$ acts on $\mathcal{M}$ by

$$t \cdot (E, \Phi) = (E, t\Phi).$$

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$$\Phi_j := \Phi|_{E_j} : E_j \to E_{j+1} \otimes K, \quad \Phi|_{E_l} = 0.$$

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- At a fixed point (E, Φ), we have further decompositions

$$T_{(E,\Phi)}\mathcal{M} = \bigoplus_j T^j_{(E,\Phi)}\mathcal{M}$$

$$\operatorname{End} E = \bigoplus_j \operatorname{End}_j E.$$

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$$0 \to N^*_{(E,\Phi)} F_\alpha \to T^*_{(E,\Phi)} \mathcal{C}_\alpha \to T^*_{(E,\Phi)} F_\alpha \to 0,$$

$$0 \to N^*_{(E,\Phi)} F_\alpha \to \bigoplus_{j \leq 0} \left(T^j_{(E,\Phi)} \mathcal{M} \right)^* \to \left(T^0_{(E,\Phi)} \mathcal{M} \right)^* \to 0.$$

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- We look for a Lagrangian $\mathcal{L}_\alpha$ such that

$$T_{(E,\Phi)} \mathcal{L}_\alpha = T_{(E,\Phi)} F_\alpha \oplus N^*_{(E,\Phi)} F_\alpha = T^0_{(E,\Phi)} \mathcal{M} \oplus \bigoplus_{j \geq 2} T^j_{(E,\Phi)} \mathcal{M},$$

since $\left(T^j_{(E,\Phi)} \mathcal{M} \right)^* = T^{1-j}_{(E,\Phi)} \mathcal{M}$.

Upward flows

- (E, Φ) fixed point,

$$W^+_{(E,\Phi)} = \left\{ q \in \mathcal{M} \,\middle|\, \lim_{t\to 0} t \cdot q = (E, \Phi) \right\}$$

satisfies

$$T_{(E,\Phi)} W^+_{(E,\Phi)} = \bigoplus_{j \geq 1} T^j_{(E,\Phi)} \mathcal{M}.$$

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- BB-slice $(D'' = \bar{\partial}_E + \Phi,\ D' = \partial_E + \Phi^*)$

$$\mathcal{S}^+_{(E,\Phi)} = \left\{ (\beta, \varphi) \in \begin{array}{c} \Omega^{0,1}(\operatorname{End}_+ E) \\ \oplus \\ \Omega^{1,0}(\operatorname{End}_0 E \oplus \operatorname{End}_+ E) \end{array} \,\middle|\, \begin{array}{l} D''(\beta,\varphi) + [\beta,\varphi] = 0 \\ D'(\beta,\varphi) = 0 \end{array} \right\}.$$

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Theorem (Collier and Wentworth 2019; Hausel and Hitchin 2022)

(E, Φ) *stable fixed point* $\implies W^+_{(E,\Phi)} \cong \mathcal{S}^+_{(E,\Phi)}$
$\implies W^+_{(E,\Phi)}$ *is Lagrangian.*

- Around $p = (E, \Phi) \in F_\alpha$, we define a submanifold $\mathcal{L}_p$ modeled on

$$\left\{ (\beta, \varphi) \in \begin{matrix} \Omega^{0,1}(\operatorname{End}_0 E \oplus \operatorname{End}_{\geq 2} E) \\ \oplus \\ \Omega^{1,0}(\operatorname{End}_{-1} E \oplus \operatorname{End}_{\geq 1} E) \end{matrix} \;\middle|\; \begin{matrix} D''(\beta, \varphi) + [\beta, \varphi] = 0 \\ D'(\beta, \varphi) = 0 \end{matrix} \right\}$$

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- Lagrangian with the right tangent space at the fixed point.

- How do we glue these local Lagrangians?

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- the local Lagrangian pieces all satisfy the same condition - the "zero velocity" condition.

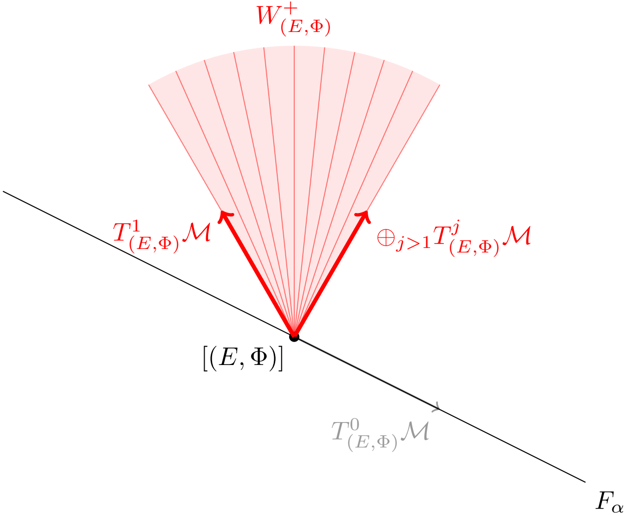
Theorem

Let F_α be a fixed point component inside the moduli space $\mathcal{M}$. There exists a $\mathbb{C}^*$–invariant Lagrangian submanifold $\mathcal{L}_\alpha$ containing F_α such that, for every $(E,\Phi) \in VHS_\alpha$, we have

$$T_{(E,\Phi)}\mathcal{L}_\alpha = T^0_{(E,\Phi)}\mathcal{M} \oplus \bigoplus_{j\geq 2} T^j_{(E,\Phi)}\mathcal{M}.$$

Indeed, we define

$$\mathcal{L}_\alpha := \{q \in W^+_\alpha \mid \gamma'_q(0) = 0\}.$$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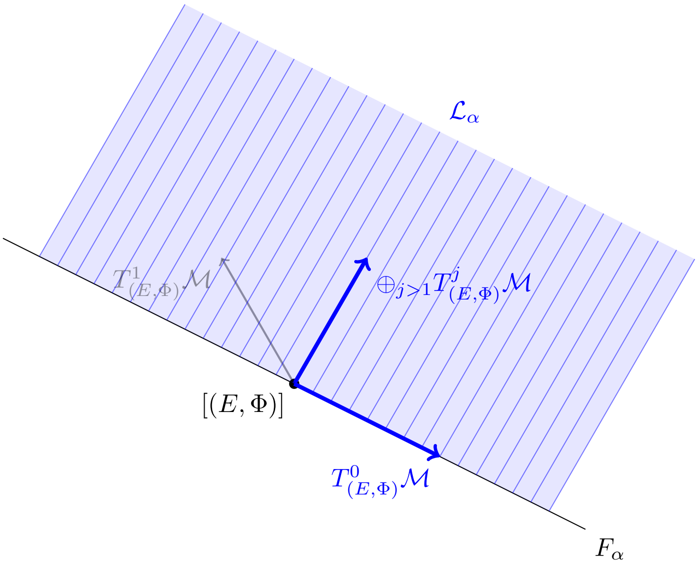
F_α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Example – $\mathrm{SL}(2,\mathbb{C})$

- Fixed point in F_α

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- $\deg L = g - 1 \implies \mathcal{L}_\alpha$ = Hitchin sections.

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Proposition (Schaposnik 2013)

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Corollary

$$|\mathcal{L}_\alpha \cap h^{-1}(a)| = 2^{2g} \binom{4g-4}{2g-2-2\deg L}$$

$$\mathcal{L}_\alpha \cap h^{-1}(a) = \bigcup_{(E,\Phi)\in F_\alpha} \mathcal{L}_\alpha \cap W^+_{(E,\Phi)} \cap h^{-1}(a)$$

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Find the subset $F_\alpha(a)$ of F_α for which $\mathcal{L}_\alpha \cap W^+_{(E,\Phi)} \cap h^{-1}(a) \neq \varnothing$

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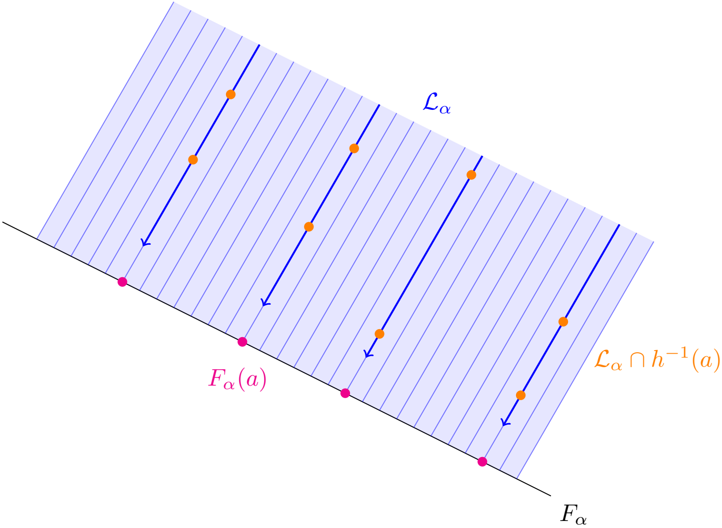

$\mathcal{L}_\alpha$
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Problem 1

A Higgs field of a point in $\mathcal{L}_\alpha$ takes the form

$$\Phi = \begin{pmatrix} 0 & & & & & & & \\ \Phi_1 & \ddots & & & & * & & \\ & \ddots & 0 & & & & & \\ & & \Phi_{i-1} & 0 & & & & \\ & & & \Phi_i & 0 & & & \\ & & & & \Phi_{i+1} & \ddots & & \\ & & 0 & & & \ddots & 0 & \\ & & & & & & \Phi_{n-1} & 0 \end{pmatrix}.$$

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At a point $c \in \operatorname{div} \Phi_i$,

$$\Phi_c = \left(\begin{array}{cccc|cccc} 0 & & & & & & & \\ \Phi_1 & \ddots & * & & & * & & \\ & \ddots & 0 & & & & & \\ & & \Phi_{i-1} & 0 & & & & \\ \hline & & & 0 & 0 & & & \\ & & & & \Phi_{i+1} & \ddots & * & \\ & & 0 & & & \ddots & 0 & \\ & & & & & & \Phi_{n-1} & 0 \end{array}\right),$$

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so the characteristic polynomial of Φ at c must split into two other "traceless" polynomials

$$\lambda^n + a_2\lambda^{n-2} + \cdots + a_n = (\lambda^i + b_2\lambda^{i-2} + \cdots + b_i)(\lambda^{n-i} + c_2\lambda^{n-i-2} + \cdots + c_{n-i}).$$

We get a system of $n-1$ polynomial equations in $i-1+(n-i-1)=n-2$ variables, which has a solution if and only if its resultant $R_{n,i}(a_2,\ldots,a_n)$ vanishes (Gelfand, Kapranov, and Zelevinsky 1994).

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Easy observations and examples:

- $R_{n,i} = R_{n,n-i}$
- $R_{n,1} = a_n$
- $R_{4,2} = a_3$
- $R_{5,2} = a_2a_3a_5 - a_3^2a_4 - a_5^2$
- $R_{6,2} = a_2a_3a_5^2 + a_3^3a_6 - a_3^2a_4a_5 - a_5^3$
- $R_{6,3} = a_2^2a_6 - a_2a_3a_5 + a_3^2a_4 - 4a_4a_6 + a_5^2$

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				2				
			3		3			
		4		3		4		
	5		10		10		5	
6		15		10		15		6

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$$n^{2g} \prod_{i=1}^{\frac{n-1}{2}} \binom{\binom{n}{i}(2g-2)}{m_i, m_{n-i}}, \quad \text{if } n \text{ odd.}$$

Problem 2

Theorem (Hausel and Hitchin 2022)

Let $\mathcal{E}$ be a very stable fixed point. Let $a \in \mathcal{A}$ be such that the spectral curve $X_a \subset K$ is smooth and $\sum_i \operatorname{div} \Phi_i$ avoids the ramification divisor of $\pi : X_a \to X$.

1. *To every $(E, \Phi) \in W^+_{\mathcal{E}} \cap h^{-1}(a)$ corresponds a choice of $n - i$ preimages in X_a of π over each zero of Φ_i in X. More precisely, at $c \in \operatorname{div} \Phi_i$, we choose those eigenvalues of Φ_c which are not eigenvalues of the restriction of Φ_c to $(E_i)_c$.*
2. *To every choice of $n - i$ preimages over each zero of Φ_i corresponds a point in $W^+_{\mathcal{E}} \cap h^{-1}(a)$, i.e., the above is a bijection.*
3. $$|W^+_{\mathcal{E}} \cap h^{-1}(a)| = \prod_{i=1}^{n-1} \binom{n}{i}^{m_i}.$$

Rank 3

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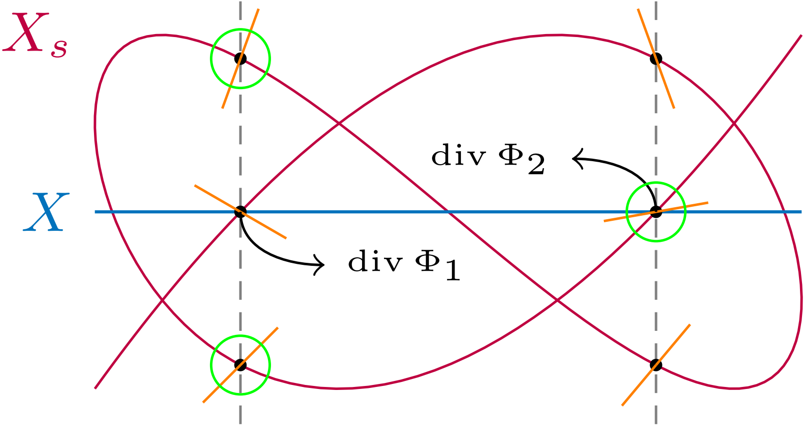

X_s
X
$\operatorname{div}\Phi_2$
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Again following Hausel and Hitchin 2022, we use Hecke transformations to show equality.

Thank you!
Obrigado!

- For general rank, since $\operatorname{div}\Phi_i \subset \operatorname{div} R_{n,i}(a)$, over each zero of Φ_i, there is a partition of the eigenvalues into two blocks of size i and $n-i$, both summing to zero $\implies$ we pick the block of size $n-i$
- for a generic, there should be only one such partition, hence there is only 1 choice if $i \neq n/2$ and 2 choices if $i = n/2$
- We thus get a bound on the possible number of intersections

$$n^{2g} \prod_{i=1}^{\frac{n-1}{2}} \binom{\binom{n}{i}(2g-2)}{m_i, m_{n-i}}$$

$$n^{2g} \binom{\binom{n}{n/2}(g-1)}{m_{n/2}} 2^{m_{n/2}} \prod_{i=1}^{\frac{n}{2}-1} \binom{\binom{n}{i}(2g-2)}{m_i, m_{n-i}, -}$$

If $(E,\Phi) \in \mathcal{L}_\alpha$, its $\bar{\partial}$-operator is of the form

$$\begin{pmatrix} \bar{\partial}_1 & 0 & & * \\ 0 & \bar{\partial}_2 & \ddots & \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \bar{\partial}_n \end{pmatrix}$$

Letting

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E$$

be the Simpson filtration of (E,Φ), we get

$$E_i/E_{i-2} = E_{i-1}/E_{i-2} \oplus E_i/E_{i-1} = L_{i-1} \oplus L_i$$

- Given a very stable fixed point $\mathcal{E}$ and a choice of $n - i$ pre-images of π over each zero of Φ_i, we need to produce a point in $W^+_{\mathcal{E}} \cap h^{-1}(a) \cap \mathcal{L}_\alpha$
- We follow the strategy of Hausel and Hitchin 2022 - starting with a point in the Hitchin section over a we add the zeroes of each Φ_i one by one via Hecke transforms at the eigenspaces corresponding to the choice of $n - i$ eigenvalues
- We need to show that, with the eigenvalue choices that correspond to $\mathcal{L}_\alpha$, the zero velocity condition is preserved by the Hecke transforms

- $c \in \operatorname{div} \Phi_i$
- V - sum of the $n-i$ eigenspaces associated to the choice of $n-i$ eigenvalues; $V_k = V \cap E_k$
- E'_k - Simpson filtration of the Hecke transformed bundle
- L'_k - $\mathbb{C}^*$-limit to 0 of the Hecke transformed bundle

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E'_{k-2} & \longrightarrow & E' & \longrightarrow & L'_{k-1} \oplus L'_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_{k-2} & \longrightarrow & E_k & \xrightarrow{p} & L_{k-1} \oplus L_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (E_{k-2})_c/V_{k-2} & \longrightarrow & (E_k)_c/V_k & \longrightarrow & (L_{k-1} \oplus L_k)_c/p(V_k) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Thank you!
Obrigado!

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