Complex Langevin simulations of the matrix model for superstrings

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Talk at "Nonperturbative and Numerical Approaches to Quantum Gravity, String Theory and Holography",

ICTS, Bangalore, India, January 18-22, 2021

Ref.) J.N. and Asato Tsuchiya, JHEP 1906 (2019) 077 [arXiv:1904.05919 [hep-th]]
Anagnostopoulos-Azuma-Hatakeyama-Hirasawa-Ito-J.N.-Tsuchiya-Papadoudis, work in progress

IKKT matrix model

Ishibashi, Kawai, Kitazawa, Tsuchiya, 1996

a conjectured nonperturbative formulation of superstring theory

$$S_{\mathrm{b}} = -\frac{1}{4g^2}\mathrm{tr}([A_\mu, A_\nu][A^\mu, A^\nu])$$

$$S_{\mathrm{f}} = -\frac{1}{2g^2}\mathrm{tr}(\Psi_\alpha(\mathcal{C}\,\Gamma^\mu)_{\alpha\beta}[A_\mu, \Psi_\beta])$$

SO(9,1) symmetry

$N \times N$ Hermitian matrices

$A_\mu \quad (\mu = 0, \cdots, 9)$ Lorentz vector

$\Psi_\alpha \quad (\alpha = 1, \cdots, 16)$ Majorana-Weyl spinor

Lorentzian metric $\eta = \mathrm{diag}(-1, 1, \cdots, 1)$ is used to raise and lower indices.

Wick rotation $(A_0 = -iA_{10}\ , \quad \Gamma^0 = i\Gamma_{10})$

⇒ Euclidean matrix model SO(10) symmetry

Anagnostopoulos, et al. JHEP 06 (2020) 069 , arXiv: 2002.07410 [hep-th]

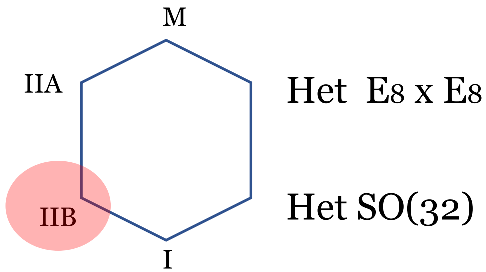
Crucial properties of the IKKT matrix model

as a nonperturbative formulation of superstring theory

- The connection to perturbative formulations can be seen manifestly by considering type IIB superstring theory in 10d.

  worldsheet action, light-cone string field Hamiltonian, etc.

- It is expected to be a nonperturbative formulation of <u>the unique theory</u> underlying the web of string dualities.



- The model has $10D\ \mathcal{N}=2$ SUSY , which cannot be realized in quantum field theories without gravity.
  The low energy effective theory should inevitably include quantum gravity !

  In the SUSY algebra, translation is realized as $A_\mu \mapsto A_\mu + \alpha_\mu \mathbf{1}$,

  which suggests that the space-time is represented as the eigenvalue distribution of A_μ .

Geometry emerges from matrix degrees of freedom dynamically in this approach .

Plan of the talk

0. Introduction
1. Definition of the Lorentzian IKKT matrix model
2. Complex Langevin method
3. Emergence of (3+1)-dim. expanding behavior
4. Emergence of a smooth space-time in a new phase
5. Relationship of the new phase to the Euclidean model
6. Summary and discussions

1. Definition of the Lorentzian IKKT matrix model

Lorenzian v.s. Euclidean

The reason why no one dared to study the Lorentzian model for many years:

$$S_{\text{b}} \propto \text{tr}\,(F_{\mu\nu}F^{\mu\nu}) = -2\,\text{tr}\,(F_{0i})^2 + \text{tr}\,(F_{ij})^2$$

$$F_{\mu\nu} = -i[A_\mu, A_\nu]$$

opposite sign

ill defined as it is !

Once one Euclideanizes it by $A_0 = -iA_{10}$,

$$S_{\text{b}} \propto \text{tr}\,(F_{\mu\nu})^2$$

positive semi-definite!

The flat direction ($[A_\mu, A_\nu] \sim 0$) is lifted due to quantum effects. (Aoki-Iso-Kawai-Kitazawa-Tada '99)

Euclidean model is well defined without any need for cutoffs.

Krauth-Nicolai-Staudacher ('98), Austing-Wheater ('01)

Partition function of the Lorentzian type IIB matrix model

Kim-J.N.-Tsuchiya PRL 108 (2012) 011601 [arXiv:1108.1540]

partition function

$$Z = \int dA\, d\Psi\, e^{i(S_{\mathrm{b}}+S_{\mathrm{f}})} = \int dA\, e^{iS_{\mathrm{b}}} \mathrm{Pf}\mathcal{M}(A)$$

This seems to be natural from the connection to the worldsheet theory.

$$S = \int d^2\xi \sqrt{g} \left(\frac{1}{4} \{X^\mu, X^\nu\}^2 + \frac{1}{2} \bar{\Psi} \gamma^\mu \{X^\mu, \Psi\} \right)$$

$$\xi_0 \equiv -i\xi_2$$

(The worldsheet coordinates should also be Wick-rotated.)

Regularizing the Lorentzian model

- Unlike the Euclidean model, the Lorentzian model is NOT well defined as it is.

$$Z = \int dA\, d\Psi\, e^{i(S_{\rm b}+S_{\rm f})} = \int dA\, e^{iS_{\rm b}}\, \mathrm{Pf}\mathcal{M}(A)$$

pure phase factor

polynomial in A

(which is real,
unlike the Euclidean case)

We definitely need some sort of regularization :
IR cutoffs in both temporal and spatial directions

- Difficult to study by Monte Carlo methods due to the sign problem.

We use the complex Langevin method,
which has developed significantly in recent years.

IR cutoffs as a regularization

- First we generalize the model by introducing two parameters.

$$Z = \int dA \, e^{-S(A)} \mathrm{Pf} \mathcal{M}(A)$$

$$S(A) = N\beta \, e^{-i\frac{\pi}{2}(1-s)} \left\{ \frac{1}{2} \mathrm{tr} \, [A_0, A_i]^2 - \frac{1}{4} \mathrm{tr} \, [A_i, A_j]^2 \right\}$$

"*s*" : Wick rotation parameter on the worldsheet

$$A_0 \mapsto e^{-ik\pi/2} A_0$$

A_0: Hermitian

$(s, k) = (0, 0)$ corresponds to the Lorentzian model.

"*k*" : Wick rotation parameter in the target space

- Introduce the IR cutoffs so that
the extent in temporal and spatial directions become finite.

$$\frac{1}{N} \mathrm{tr} \, (A_0)^2 = \kappa L^2$$

$$\frac{1}{N} \mathrm{tr} \, (A_i)^2 = L^2$$

In what follows, we set $L = 1$ without loss of generality.

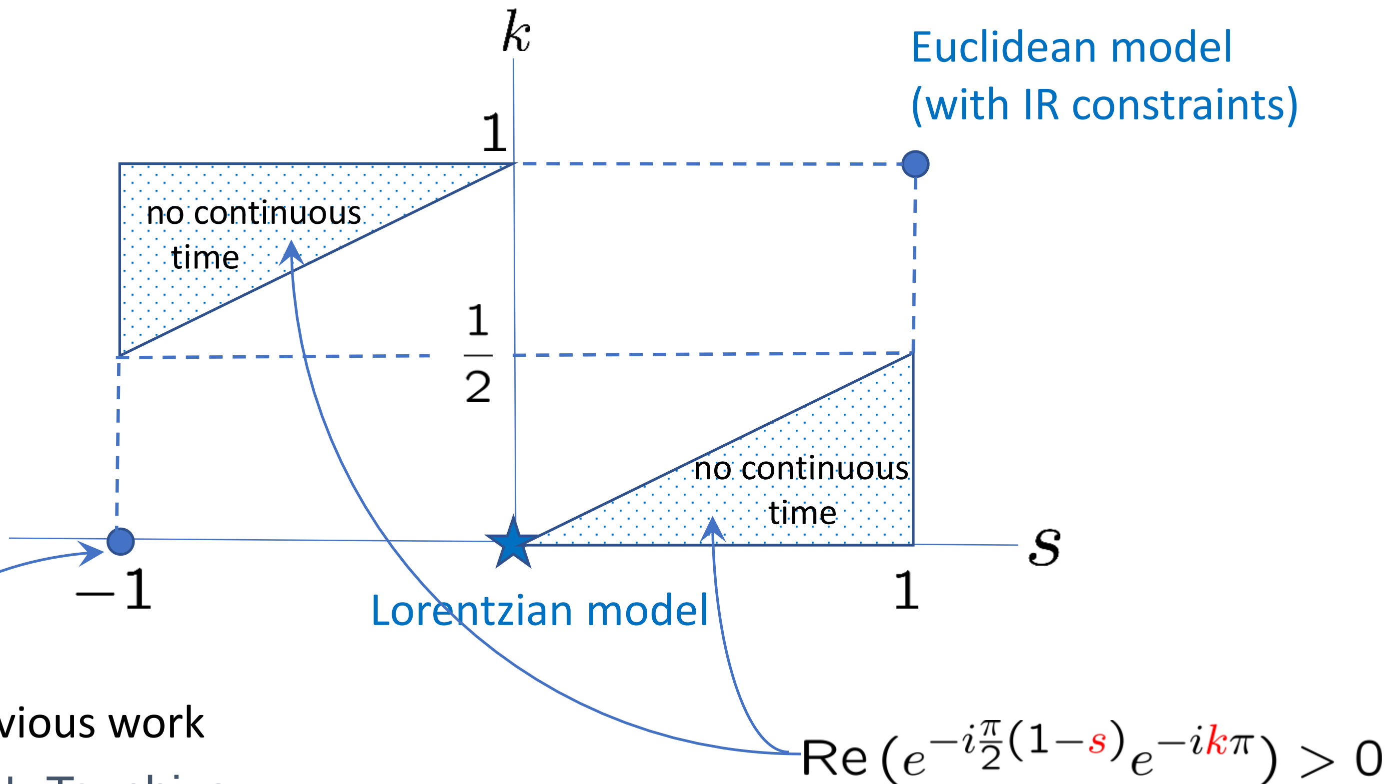
The phase diagram we consider in this talk

$$S = N\beta\, e^{-i\frac{\pi}{2}(1-s)} \left\{ \frac{1}{2} e^{-ik\pi} \text{tr}\, [A_0, A_i]^2 - \frac{1}{4} \text{tr}\, [A_i, A_j]^2 \right\}$$

IR constraints

$$\frac{1}{N}\text{tr}\,(A_0)^2 = \kappa$$

$$\frac{1}{N}\text{tr}\,(A_i)^2 = 1$$



Our previous work

Kim-J.N.-Tsuchiya,
PRL 108 (2012) 011601

2. Complex Langevin method

The complex Langevin method

Parisi ('83), Klauder ('83)

$$Z = \int dx\, w(x) \qquad x \in \mathbb{R}$$

complex

MC methods inapplicable due to sign problem !

Complexify the dynamical variables, and consider their (fictitious) time evolution :

$$z^{(\eta)}(t) = x^{(\eta)}(t) + i\, y^{(\eta)}(t)$$

defined by the complex Langevin equation

$$\frac{d}{dt} z^{(\eta)}(t) = v(z^{(\eta)}(t)) + \eta(t)$$

Gaussian noise (real)

probability $\propto \mathrm{e}^{-\frac{1}{4}\int dt\, \eta(t)^2}$

$$\langle \mathcal{O} \rangle \stackrel{?}{=} \lim_{t \to \infty} \langle \mathcal{O}(z^{(\eta)}(t)) \rangle_\eta$$

$$v(x) \equiv \frac{1}{w(x)} \frac{\partial w(x)}{\partial x}$$

Rem 1 : When w(x) is real positive, it reduces to one of the usual MC methods.

Rem 2 : The drift term $v(x) \equiv \frac{1}{w(x)} \frac{\partial w(x)}{\partial x}$ and the observables $\mathcal{O}(x)$ should be evaluated for complexified variables by analytic continuation.

Complex Langevin equation

J.N. and Asato Tsuchiya, JHEP 1906 (2019) 077
[arXiv:1904.05919 [hep-th]]

The effective action

$$S_{\text{eff}} = N\beta\, e^{-i\frac{\pi}{2}(1-s)} \left\{ \frac{1}{2} e^{-ik\pi} \text{tr}\,[A_0, A_i]^2 - \frac{1}{4} \text{tr}\,[A_i, A_j]^2 \right\} + \frac{1}{2} N \text{tr}\,(\mathcal{A}_i)^2 + \frac{1}{2} N \text{tr}\,(\mathcal{A}_0)^2 - \log \Delta(\alpha) - \sum_{a=1}^{N-1} \tau_a$$

$$A_0 = \frac{\sqrt{\kappa}}{\sqrt{\frac{1}{N}\text{tr}\mathcal{A}_0^2}} \mathcal{A}_0$$

$$A_i = \frac{1}{\sqrt{\frac{1}{N}\text{tr}\mathcal{A}_j^2}} \mathcal{A}_i$$

$$A_0 = \text{diag}(\alpha_1, \cdots, \alpha_N)$$

$$\alpha_1 < \alpha_2 < \cdots < \alpha_N$$

$$\begin{aligned} \alpha_1 &= 0 \\ \alpha_2 &= e^{\tau_1} \\ \alpha_3 &= \alpha_2 + e^{\tau_2} \\ &\vdots \\ \alpha_N &= \alpha_{N-1} + e^{\tau_{N-1}} \end{aligned}$$

Complex Langevin equation

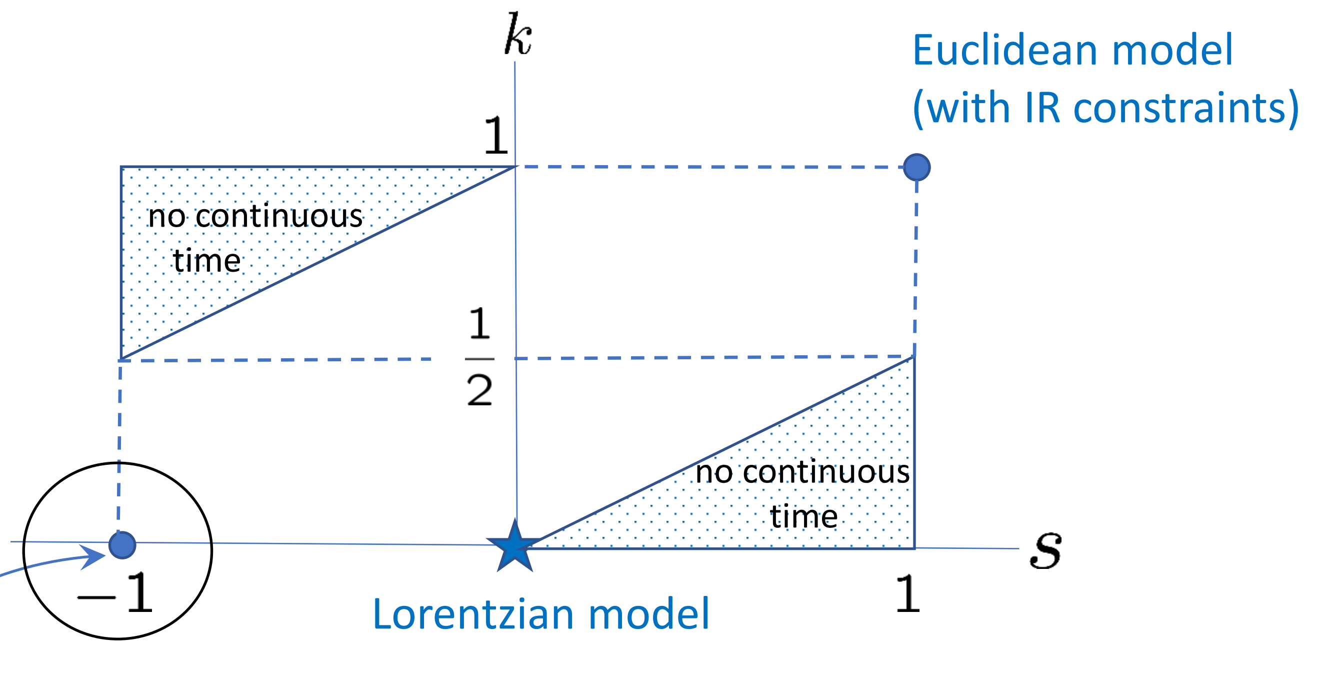
$$\begin{cases} \dfrac{d\tau_a}{dt} = -\dfrac{\partial S_{\text{eff}}}{\partial \tau_a} + \eta_a \\ \dfrac{d(\mathcal{A}_i)_{ab}}{dt} = -\dfrac{\partial S_{\text{eff}}}{\partial (\mathcal{A}_i)_{ba}} + (\eta_i)_{ab} \end{cases}$$

τ_a :complex variables, $\mathcal{A}_i$: general complex matrices.

In this work, we omit the fermionic matrices to reduce computation time → bosonic model

3. Emergence of (3+1)-dimensional expanding behavior

Results at (s,k)=(-1,0)



Our previous work

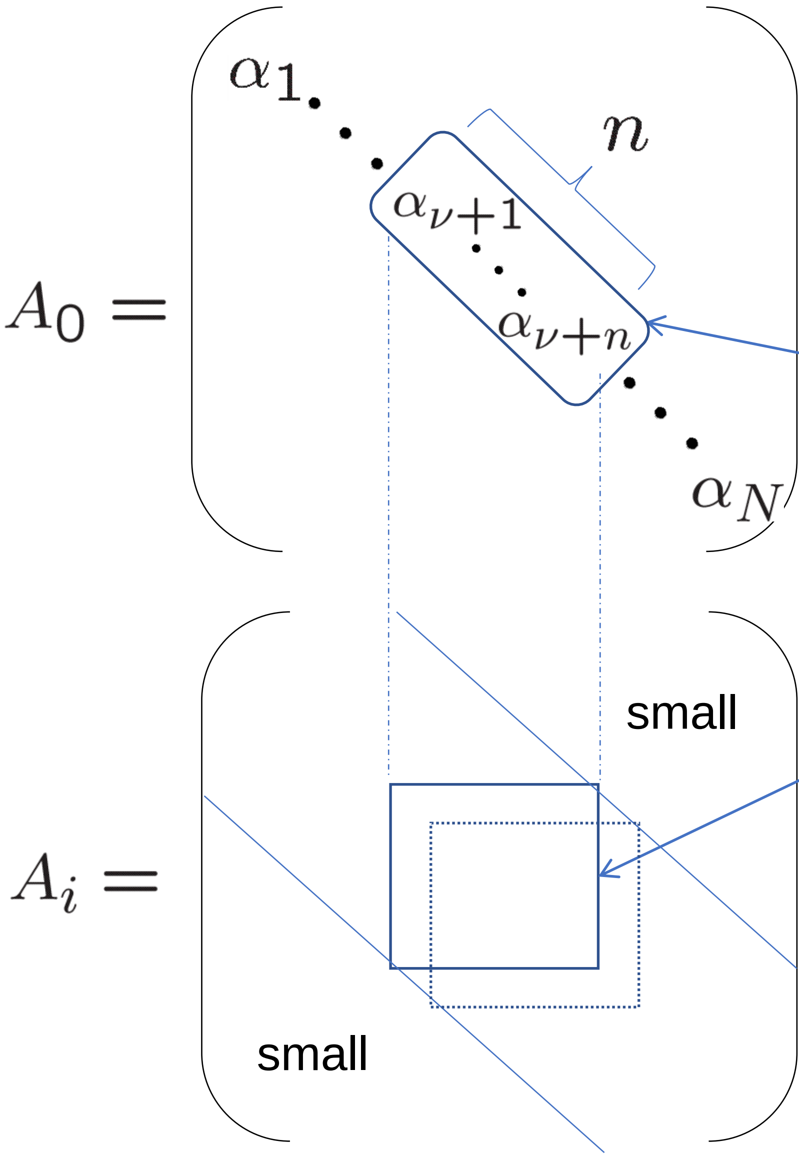
Kim-J.N.-Tsuchiya,
PRL 108 (2012) 011601

$$S = N\beta \left\{ -\frac{1}{2}\mathrm{tr}\,[A_0, A_i]^2 + \frac{1}{4}\mathrm{tr}\,[A_i, A_j]^2 \right\}$$

Boltzmann weight = e^{-S}
⇨ no sign problem
in this case !

Extracting time-evolution from the Lorentzian model

Kim-J.N.-Tsuchiya PRL 108 (2012) 011601 [arXiv:1108.1540]



$$\alpha_1 < \cdots < \alpha_N$$

definition of time "t"

$$t = \frac{1}{n} \sum_{i=1}^{n} \alpha_{\nu+i}$$

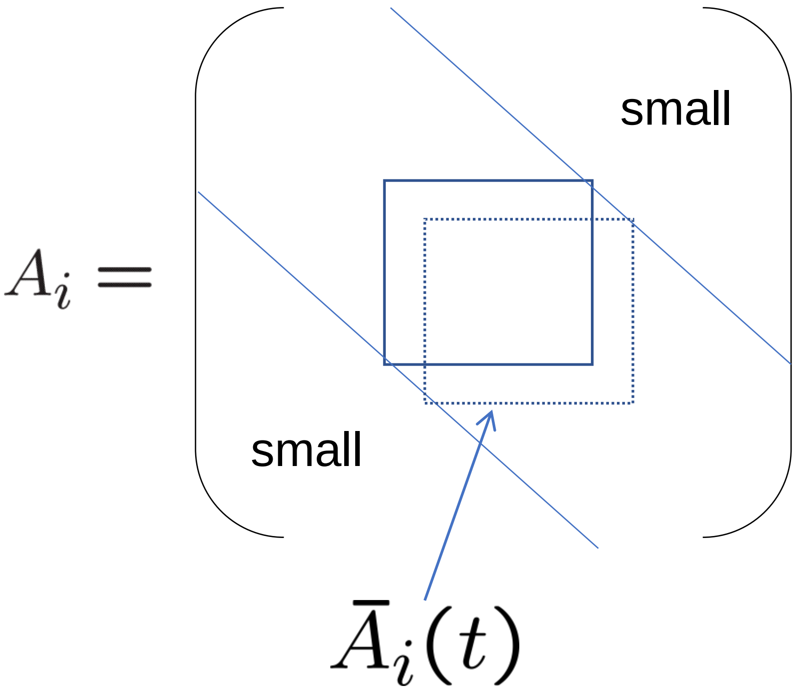
The state of the universe $\bar{A}_i(t)$ at time t

A_i has a band diagonal structure

non-trivial dynamical property

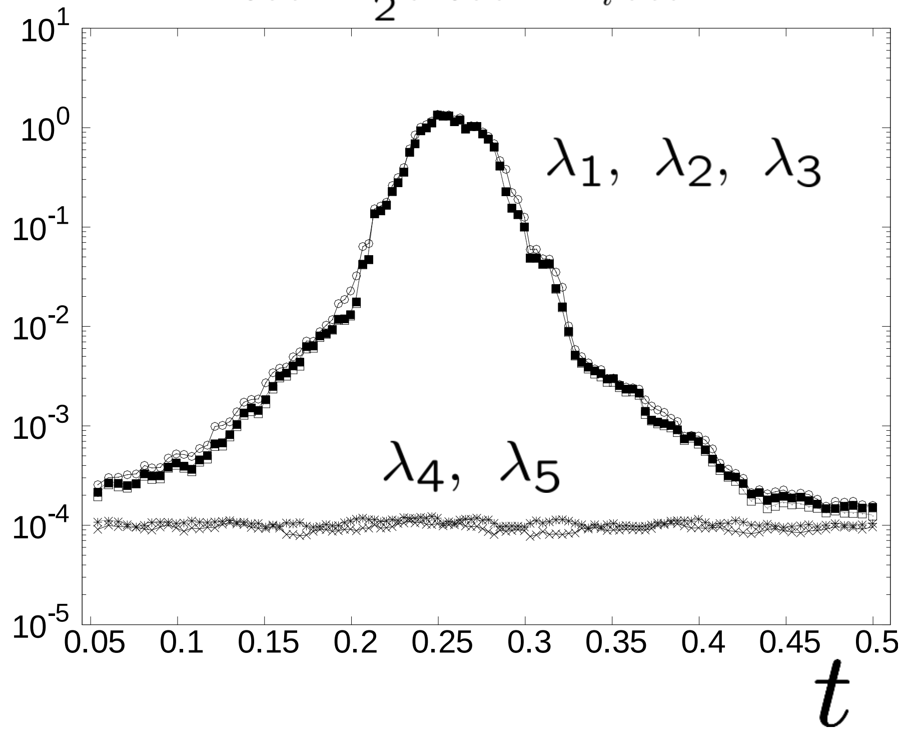
Emergence of (3+1)-dim. expanding behavior

6D bosonic model $N = 128\ ,\quad \kappa = 0.02\ ,\quad \beta = 8\ ,\quad (s,k) = (-1,0)\ ,\quad n = 16$



eigenvalues of $T_{ij}(t) = \frac{1}{n}\mathrm{tr}\left\{X_i(t)X_j(t)\right\}$

$$X_i(t) = \frac{1}{2}(\bar{A}_i(t) + \bar{A}_i^\dagger(t))$$



SSB : SO(5) → SO(3) occurs at some point in time.

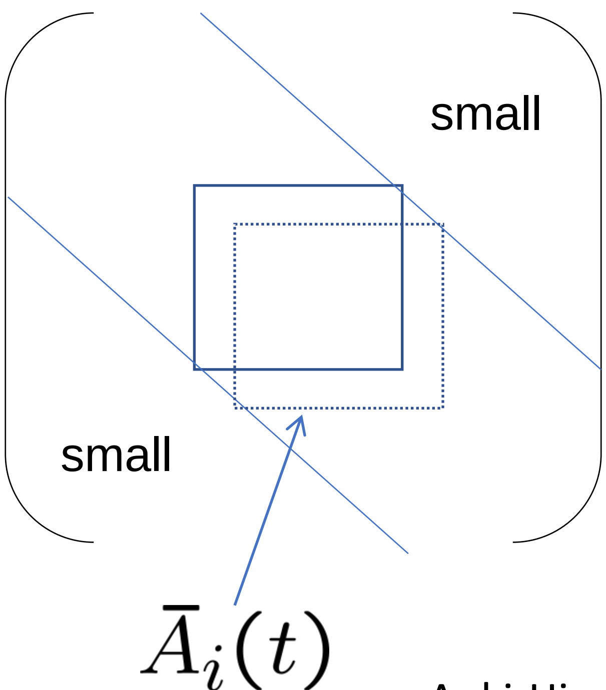
The mechanism of the SSB

$$S = N\beta \left\{ -\frac{1}{2}\mathrm{tr}\,[A_0, A_i]^2 + \frac{1}{4}\mathrm{tr}\,[A_i, A_j]^2 \right\}$$

favors A_j close to diagonal

favors maximal non-commutativity between A_j

$A_i =$



maximize $\mathrm{NC} = -\mathrm{tr}\,[\bar{A}_i(t), \bar{A}_j(t)]^2$
for $\mathrm{tr}\,(\bar{A}_i(t))^2 = \mathrm{const.}$

$$\bar{A}_i(t) \propto \sigma_i \quad \text{for } i = 1, 2, 3$$
$$\bar{A}_i(t) = 0 \quad \text{for } i \geq 4$$

up to SO(5) rotation

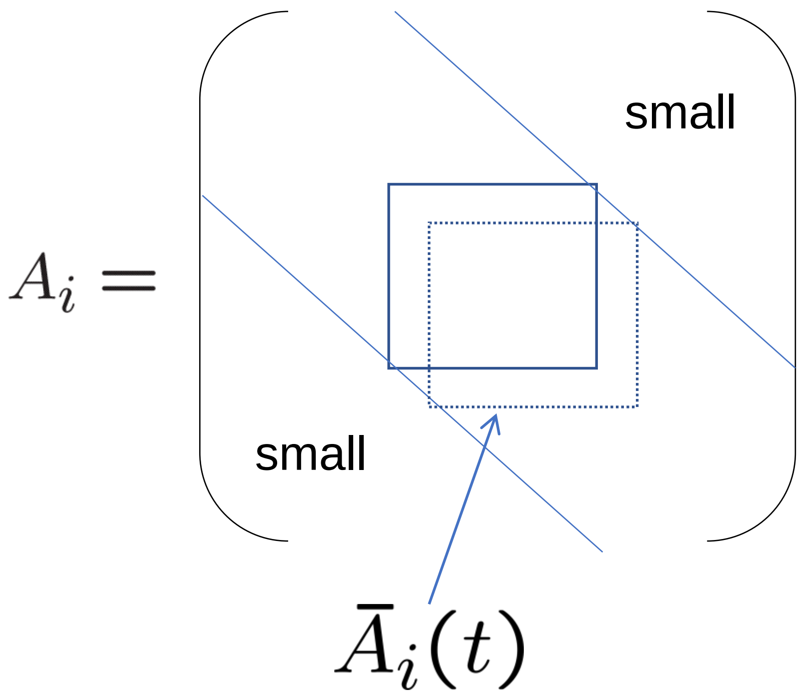
Aoki-Hirasawa-J.N.-Ito-Tsuchiya, PTEP 2019 (2019) 9, 093B03
arXiv:1904.05914 [hep-th]

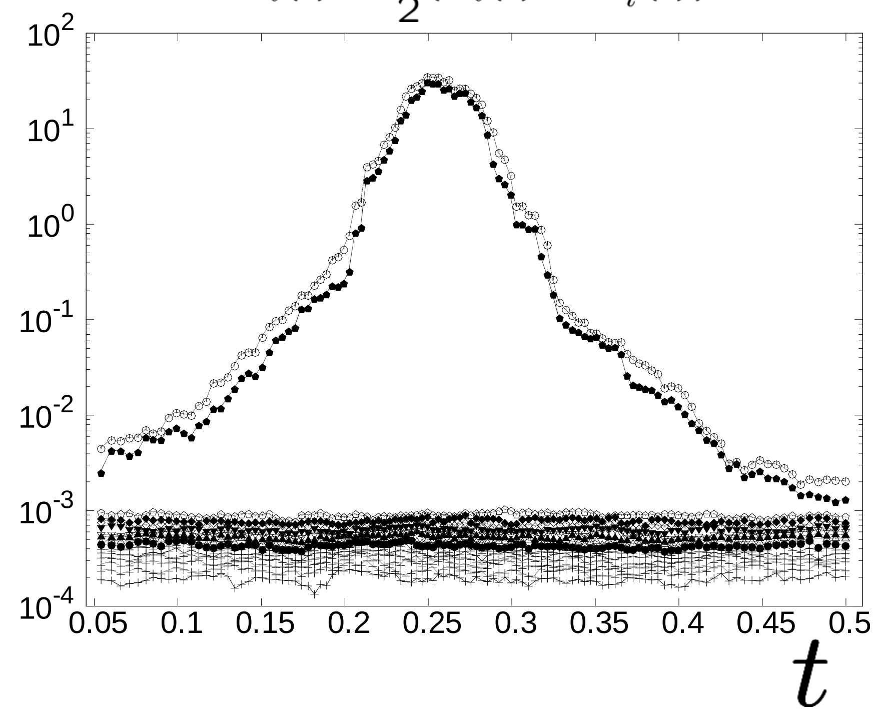
Confirmation of the mechanism

6D bosonic model $N = 128\ ,\quad \kappa = 0.02\ ,\quad \beta = 8\ ,\quad (s,k) = (-1,0)\ ,\quad n = 16$

eigenvalues of $Q = \sum_{i=1}^{5} \left\{ X_i(t) \right\}^2$

$X_i(t) = \frac{1}{2}(\bar{A}_i(t) + \bar{A}_i^\dagger(t))$

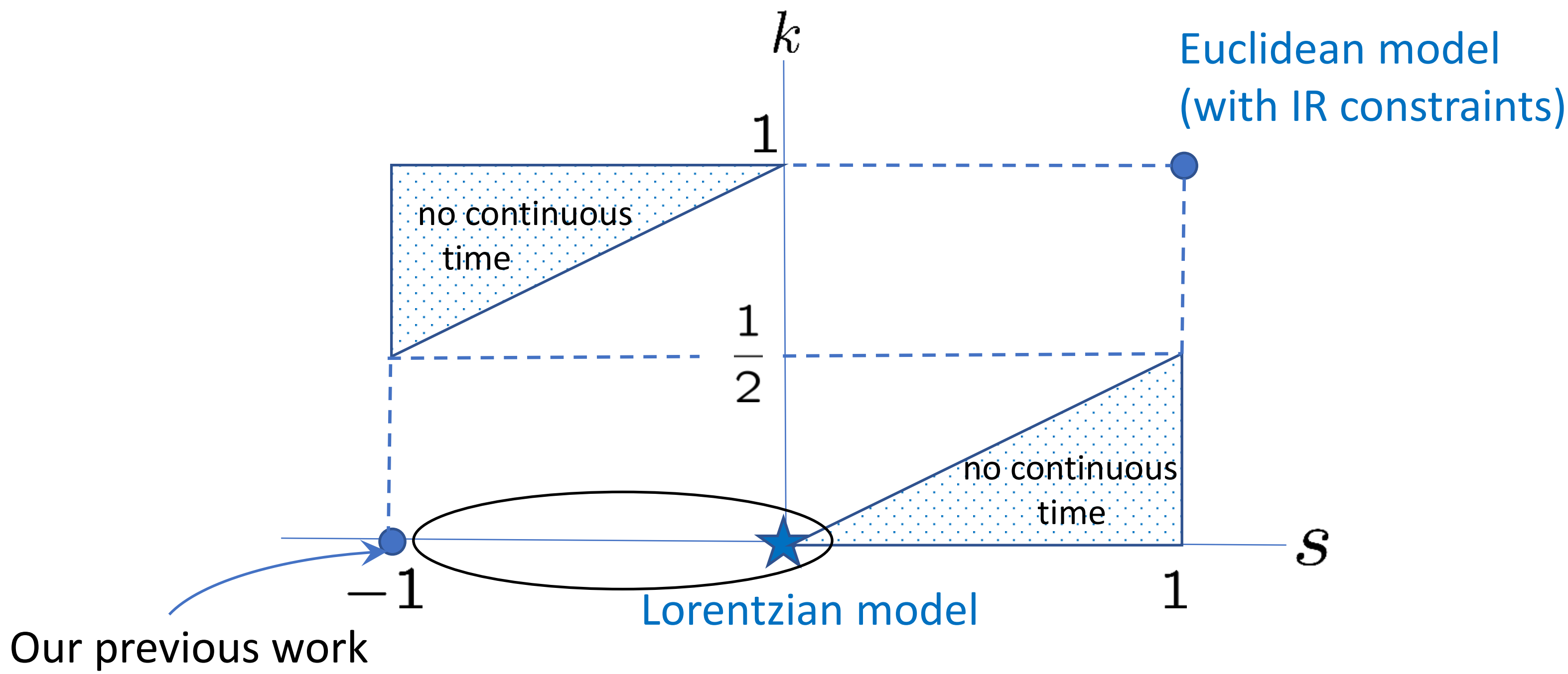




Only 2 Evs of Q become large suggesting the Pauli-matrix structure.

4. Emergence of a smooth space-time in a new phase

Exploring the phase diagram towards $(s,k)=(0,0)$



$$S = N\beta e^{-i\frac{\pi}{2}(1-s)}\left\{\frac{1}{2}e^{-ik\pi}\mathrm{tr}\,[A_0, A_i]^2 - \frac{1}{4}\mathrm{tr}\,[A_i, A_j]^2\right\}$$

Real part changes sign at $s=0$.

Can we obtain (3+1)-dim. expanding behavior
with a smooth space-time structure ?

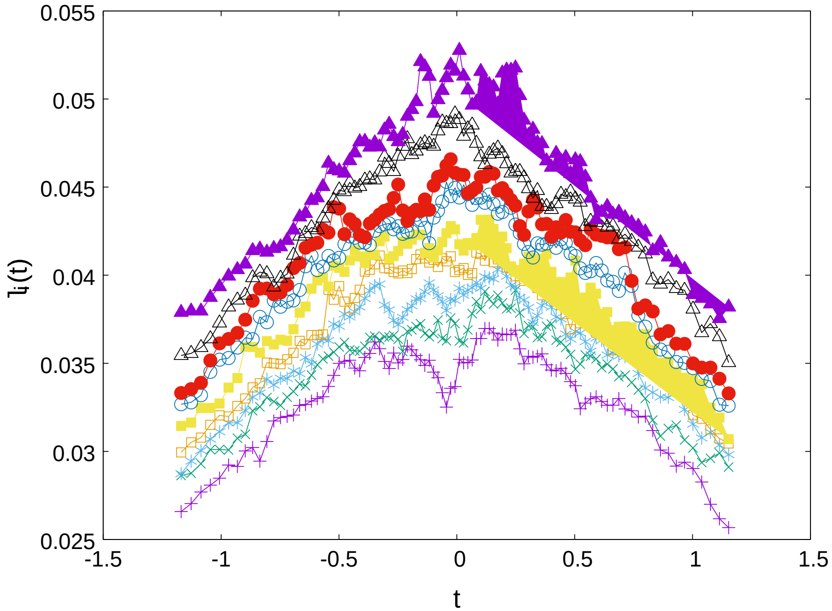
Note: Pauli-matrix structure is obtained by maximizing tr $(F_{ij})^2$!

A new phase appears at $-1 < s \leq 0$!

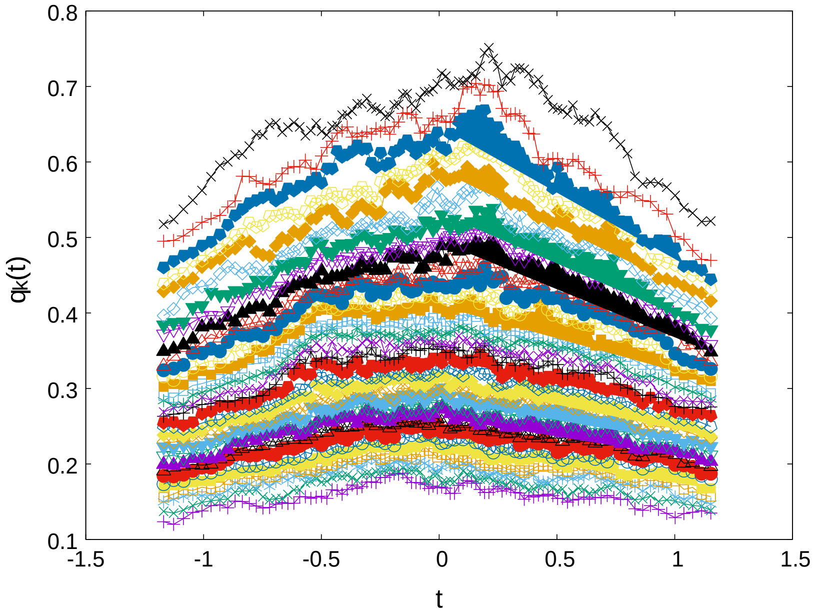
10D bosonic model $N = 128\ , \quad \beta = 2.5\ , \quad \kappa = 0.8\ , \quad (s,k) = (-0.8, 0)$

eigenvalues of $T_{ij}(t) = \frac{1}{n}\mathrm{tr}\left\{X_i(t)X_j(t)\right\}$

eigenvalues of $Q = \sum_{i=1}^{9}\left\{X_i(t)\right\}^2$



no SSB of SO(9) symmetry

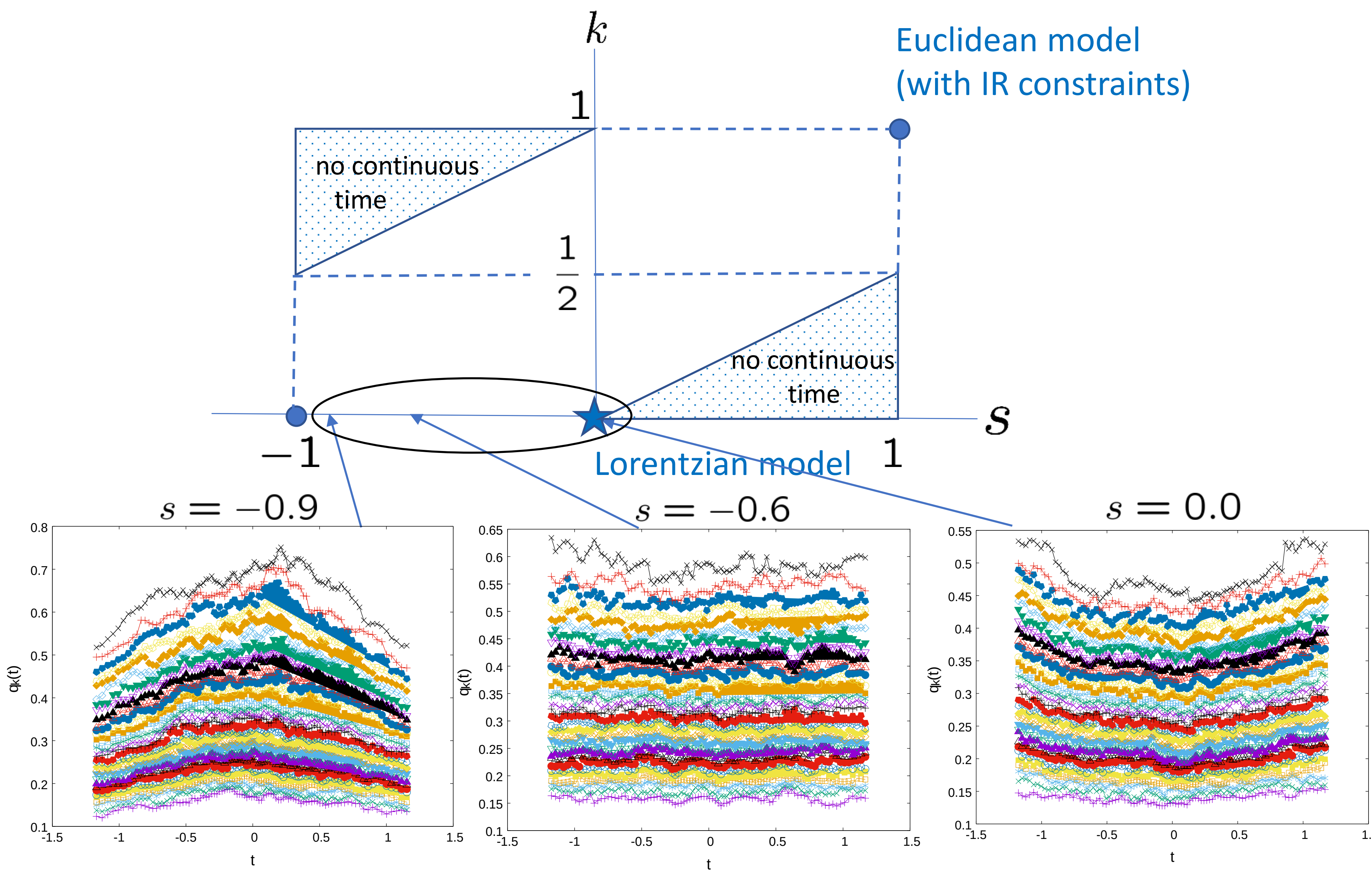


continuous space-time

(weak expanding behavior, though)

Rem.) Pauli-matrix-like phase can also be realized by using matrices thermalized at *s = -1* as the initial configuration.

“expanding” behavior changes qualitatively with *s*



10D bosonic model

$N = 128\ ,\quad \beta = 2.5\ ,\quad \kappa = 0.8\ ,\quad k = 0$

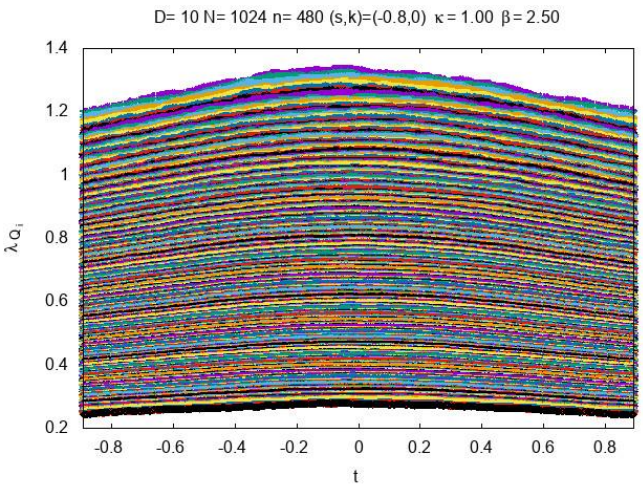
Results at larger N

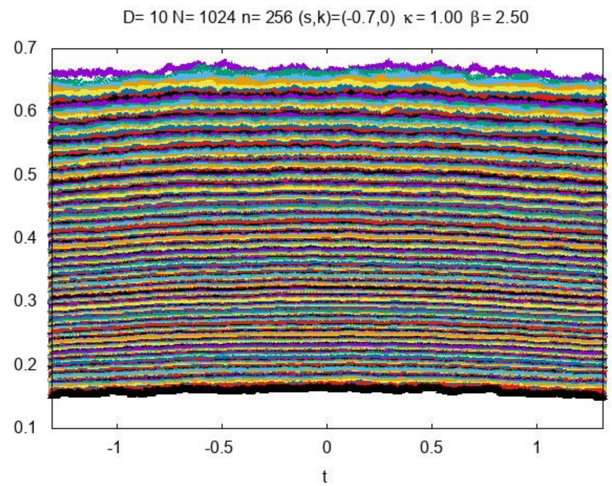
10D bosonic model

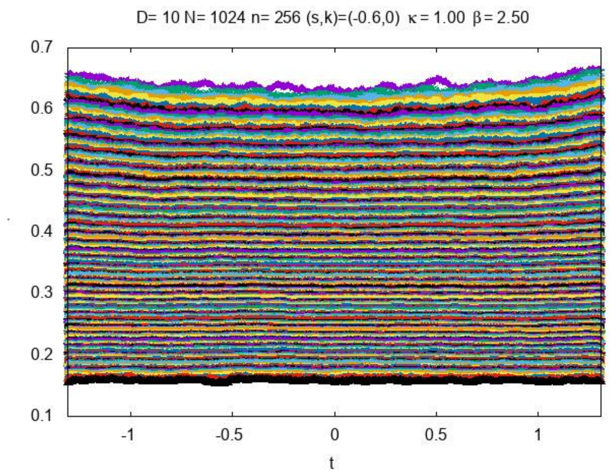
$$N = 1024\ ,\quad \beta = 2.5\ ,\quad \kappa = 1.0\ ,\quad k = 0$$

eigenvalues of $Q = \sum_{i=1}^{9} \left\{ X_i(t) \right\}^2$

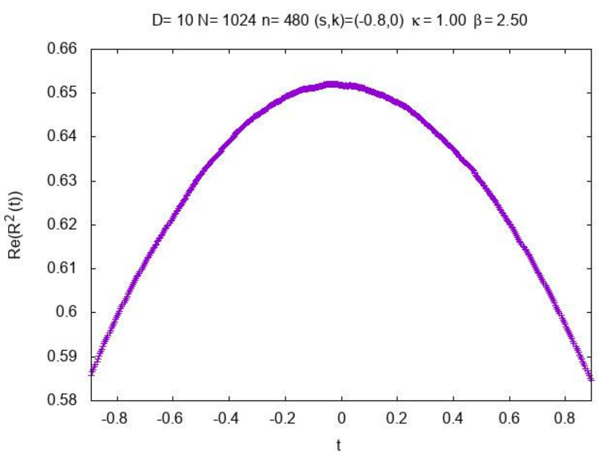
$s = -0.8$ $\qquad$ $s = -0.7$ $\qquad$ $s = -0.6$

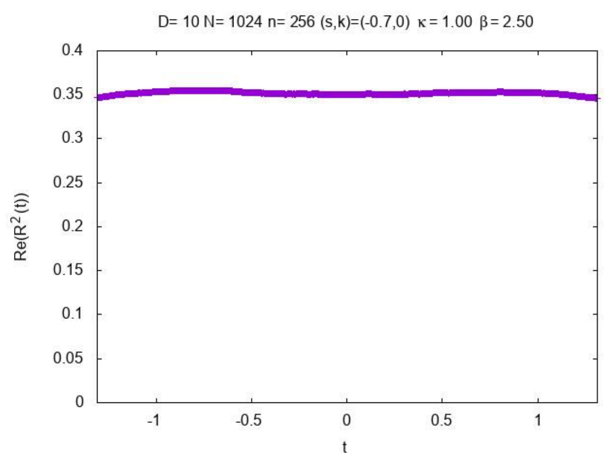


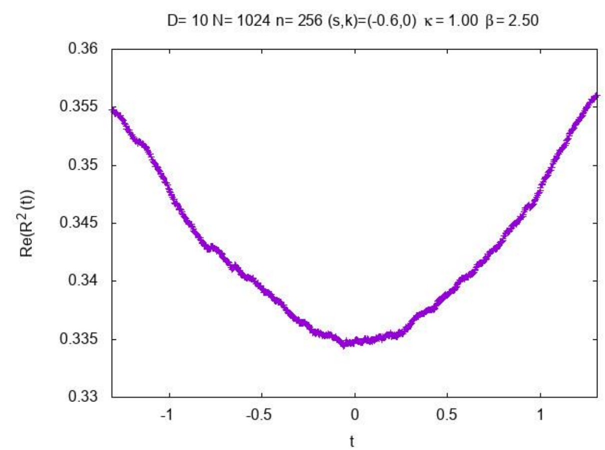




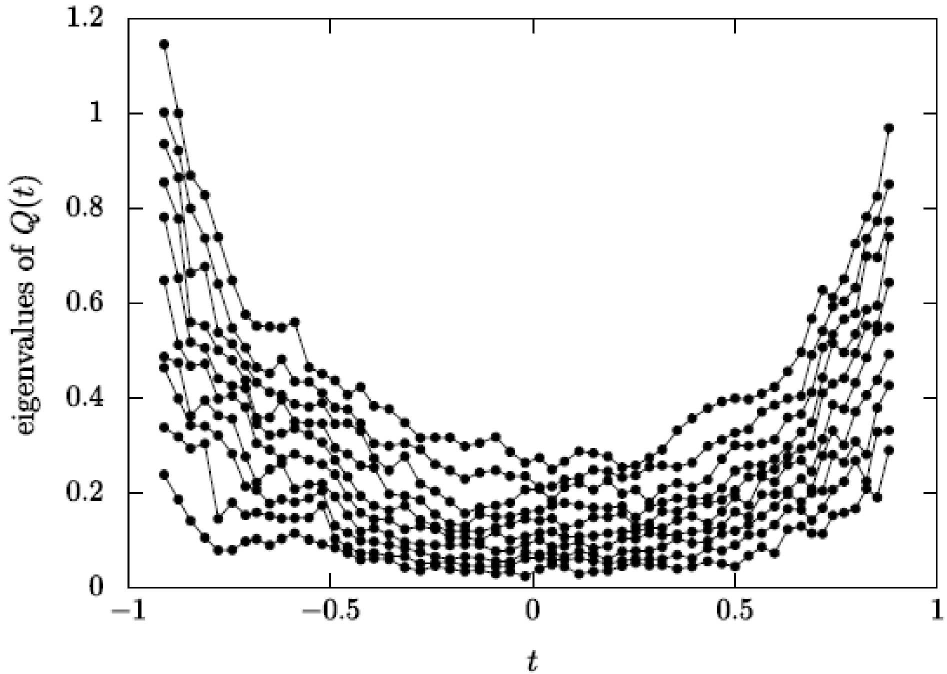
$$R^2(t) \equiv \frac{1}{N} \mathrm{tr} \bar{A}_i(t)^2$$





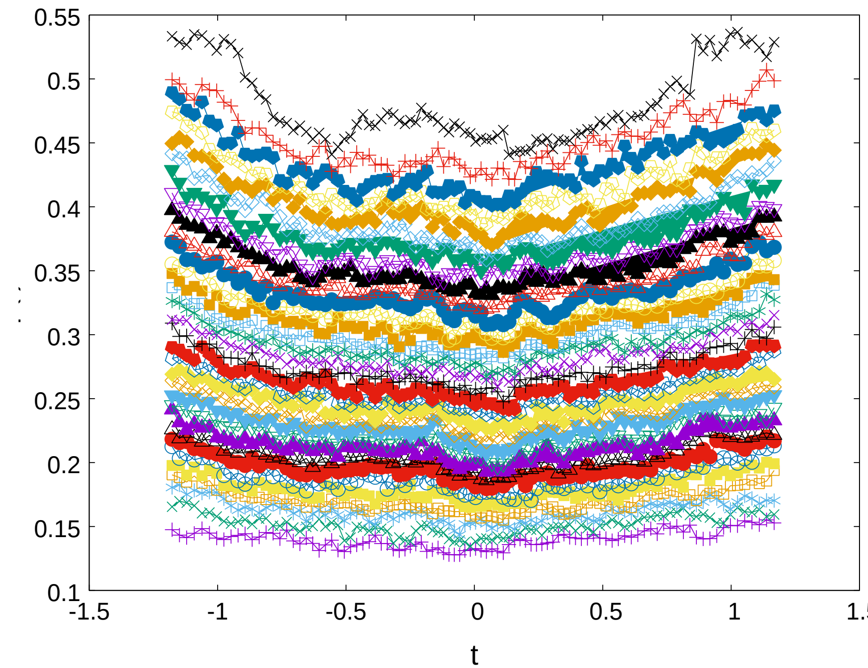


Expanding behavior at (s,k)=(0,0) resembles that of typical classical solutions



typical classical solutions
of the Lorentzian bosonic model
generated numerically

Hatakeyama-Matsumoto-J.N.-Tsuchiya-Yosprakob, *PTEP* 2020 (2020) 4, 043B10



10D bosonic model $(s, k) = (0, 0)$

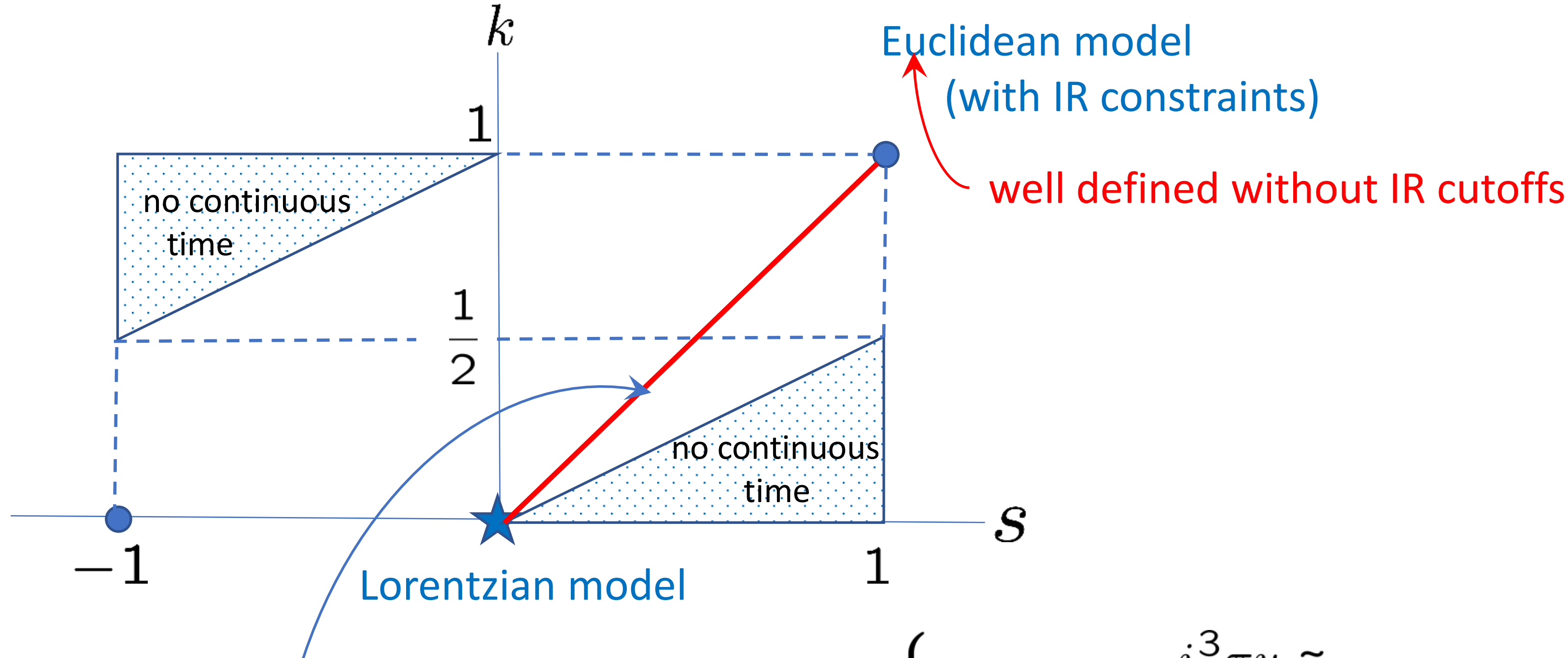
$N = 128\ ,\quad \beta = 2.5\ ,\quad \kappa = 0.8$

Stronger expansion may be obtained by tuning β and κ properly and/or by introducing SUSY.

5. Relationship of the new phase to the Euclidean model

The new phase is smoothly connected to the Euclidean model

$$S = N\beta\, e^{-i\frac{\pi}{2}(1-s)} \left\{ \frac{1}{2} e^{-ik\pi} \mathrm{tr}\,[A_0, A_i]^2 - \frac{1}{4} \mathrm{tr}\,[A_i, A_j]^2 \right\}$$



If there were no IR constraints,
one can make analytic continuation.

(pointed out by Yuhma Asano)

$$\begin{cases} A_0 = \mathrm{e}^{-i\frac{3}{8}\pi u} \tilde{A}_0 \\ A_i = \mathrm{e}^{i\frac{1}{8}\pi u} \tilde{A}_i \end{cases}$$

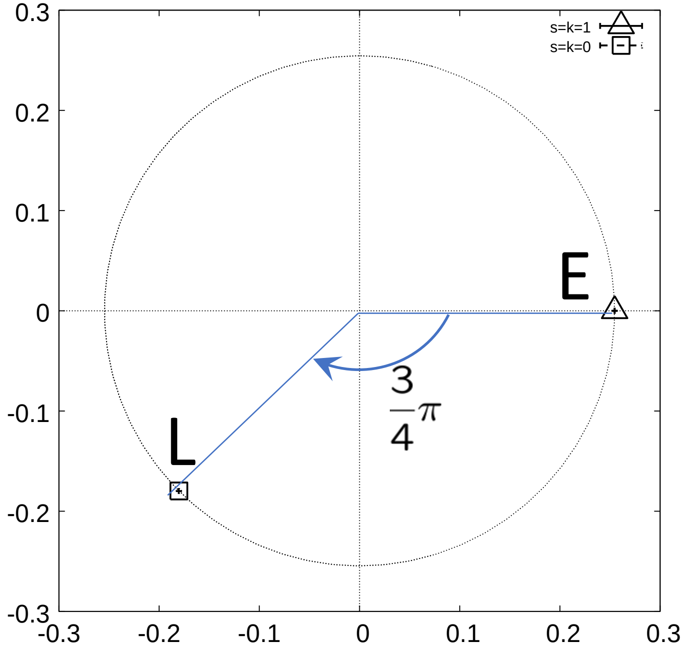
$$u = 0 \mapsto u = 1$$

Lorentzian Euclidean

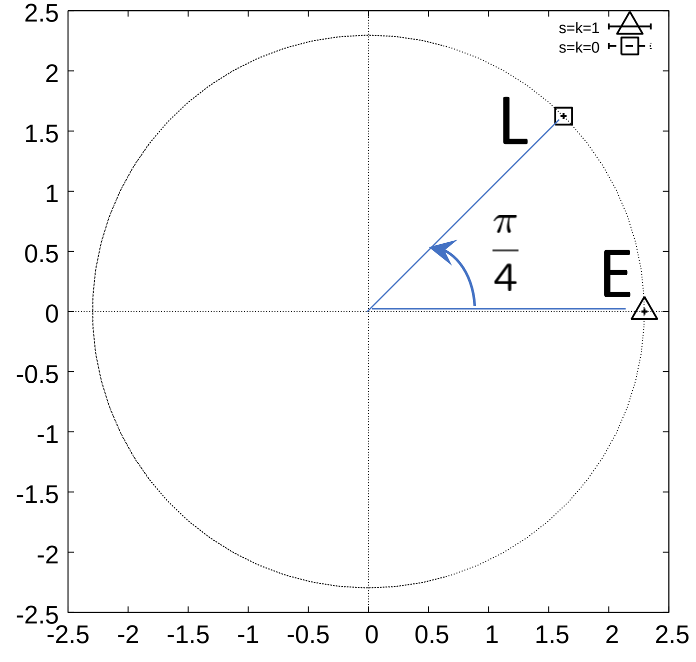
Confirmation of continuity by CL simulation <u>without IR constraints</u>

10D bosonic model $\beta = 1$

$$\left\langle \frac{1}{N}\mathrm{tr}(A_0)^2 \right\rangle_{\mathrm{L}} = e^{-\frac{3\pi}{4}i} \left\langle \frac{1}{N}\mathrm{tr}(\tilde{A}_0)^2 \right\rangle_{\mathrm{E}}$$



$$\left\langle \frac{1}{N}\mathrm{tr}(A_i)^2 \right\rangle_{\mathrm{L}} = e^{\frac{\pi}{4}i} \left\langle \frac{1}{N}\mathrm{tr}(\tilde{A}_i)^2 \right\rangle_{\mathrm{E}}$$



$$\begin{cases} A_0 = \mathrm{e}^{-i\frac{3}{8}\pi u}\tilde{A}_0 \\ A_i = \mathrm{e}^{i\frac{1}{8}\pi u}\tilde{A}_i \end{cases}$$

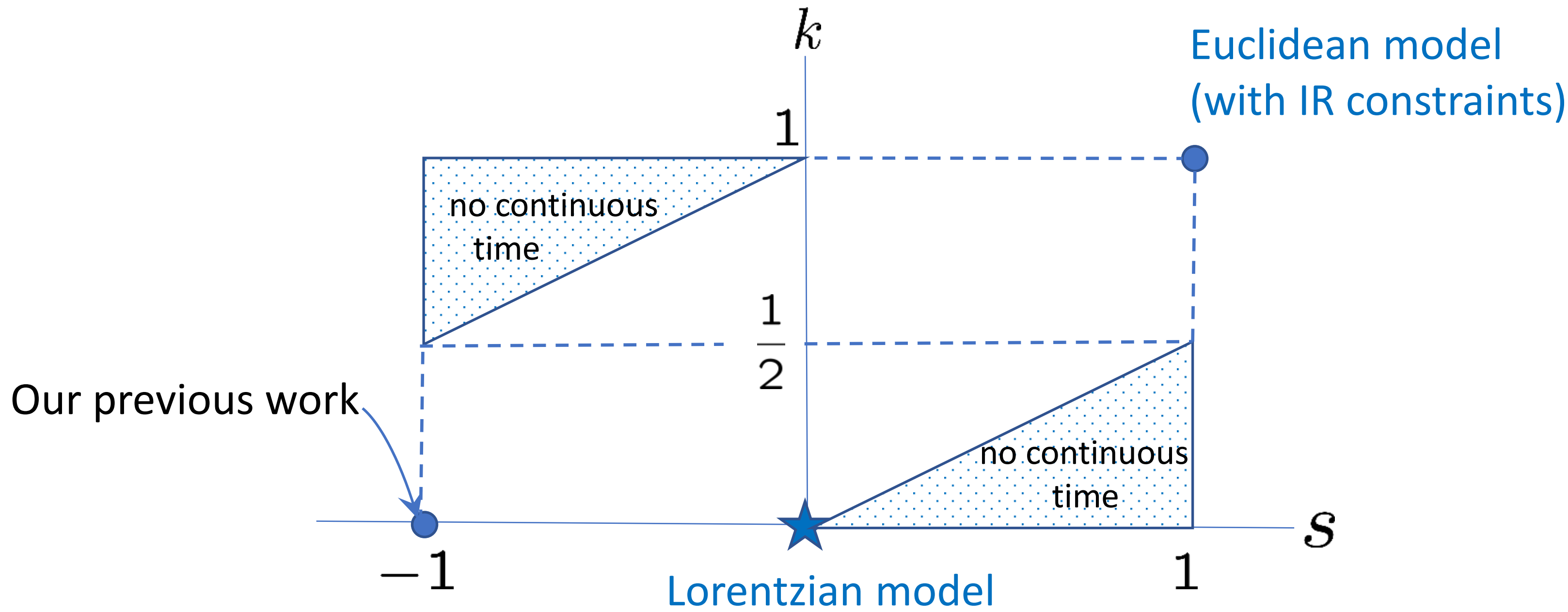
$u = 0 \mapsto u = 1$

Lorentzian Euclidean

IR constraints force $\left\langle \frac{1}{N}\mathrm{tr}(A_0)^2 \right\rangle$, $\left\langle \frac{1}{N}\mathrm{tr}(A_i)^2 \right\rangle$ to be real positive.

6. Summary and Discussions

Summary



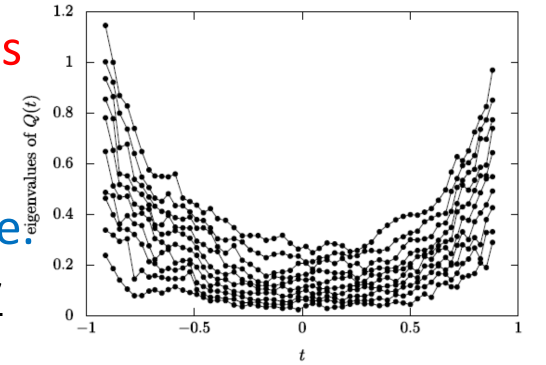
- Expanding behavior of SO(3) symmetric space observed at *(s,k)=(-1,0),* but the expansion is due to 2 eigenvalues (Pauli-matrix like structure).

- We find a new phase for s=-0.9,…,0.0, with continuous space and no SSB. Weak expanding behavior changes qualitatively around s=-0.7.

This phase is connected smoothly to the Euclidean model, which is well defined without IR cutoffs.
IR constraints force the space-time to be real.

Discussions

- The expanding behavior in the new phase at (s,k)=(0,0) is reminiscent of that of classical solutions.



This suggests that a classical solution is dominating there.
If so, solving the classical eq. of motion is a sensible way to explore the late time behavior of this model.

Possible emergence of the Standard Model from the intersecting branes in the extra dimensions.

Chatzistavrakidis-Steinacker-Zoupanos (2011)
Aoki-J.N.-Tsuchiya (2014),
Hatakeyama-Matsumoto-J.N.-Tsuchiya-Yosprakob, *PTEP* 2020 (2020) 4, 043B10

- Effects of the fermionic matrices ➡ stronger expansion with SSB ?

Not straightforward due to the “singular-drift problem” in the CLM caused by the near-zero eigenvalues the Dirac operator.

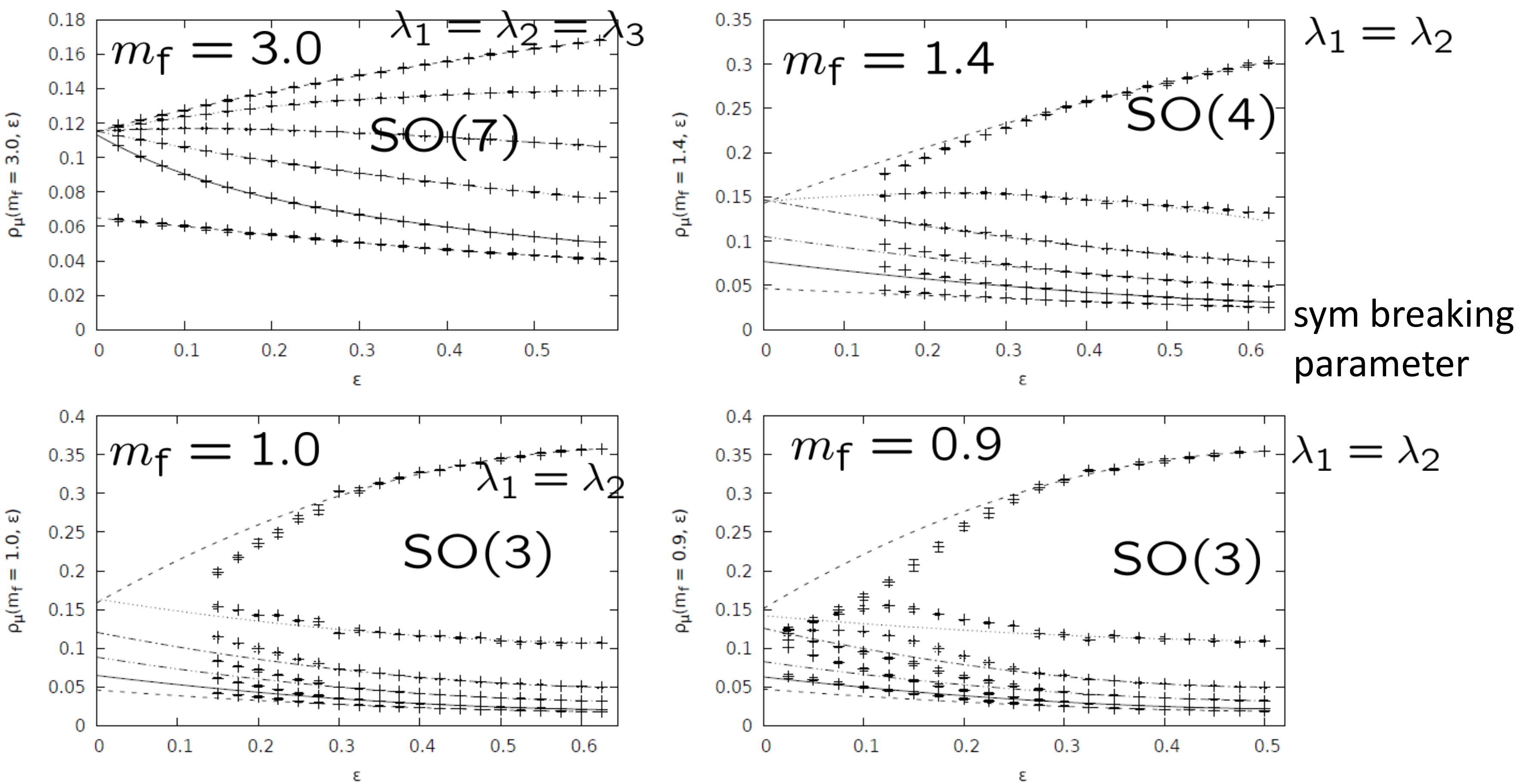
Deformation of the Dirac operator (and extrapolations) may be needed.

Successful in Euclidean IKKT matrix model ---> next page
Anagnostopoulos-Azuma-Ito-JN-Okubo-Papadoudis,
JHEP 06 (2020) 069 , arXiv: 2002.07410 [hep-th]

Results for SUSY 10D model in the Euclidean case

SSB of SO(10) observed by decreasing the deformation parameter m_{f}.

ten eigenvalues of $T_{\mu\nu}$ after $N \to \infty$



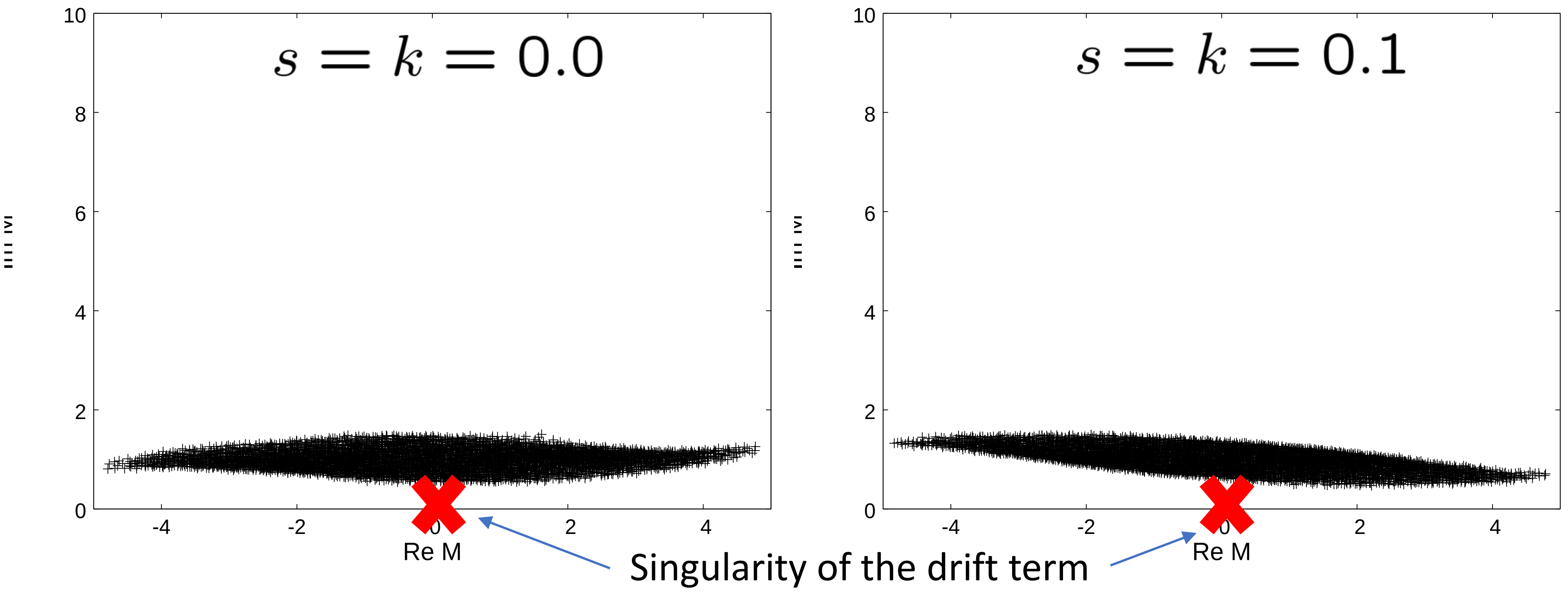
Anagnostopoulos-Azuma-Ito-JN-Okubo-Papadoudis,
JHEP 06 (2020) 069 , arXiv: 2002.07410 [hep-th]

In the Lorenzian case, adding a small "mass" term in the Dirac operator can cure the problem !

The Dirac operator in the Lorentzian model has real eigenvalues for Hermitian configurations !

6D SUSY model with deformation parameter $m_{\mathrm{f}} = 1.0$

$$N = 32 \ , \beta = 1.4 \ , \kappa = 1.0$$



6. Backup slides

How to introduce the "time ordering"

$$Z = \int dA_0 \, dA_i \, e^{-S} = \int d\alpha \, dA_i \, \Delta(\alpha) \, e^{-S}$$

$$A_0 = \text{diag}(\alpha_1, \cdots, \alpha_N)$$

$$\alpha_1 < \alpha_2 < \cdots < \alpha_N$$

$$\Delta(\alpha) = \prod_{a>b} (\alpha_a - \alpha_b)^2$$: van der Monde determinant

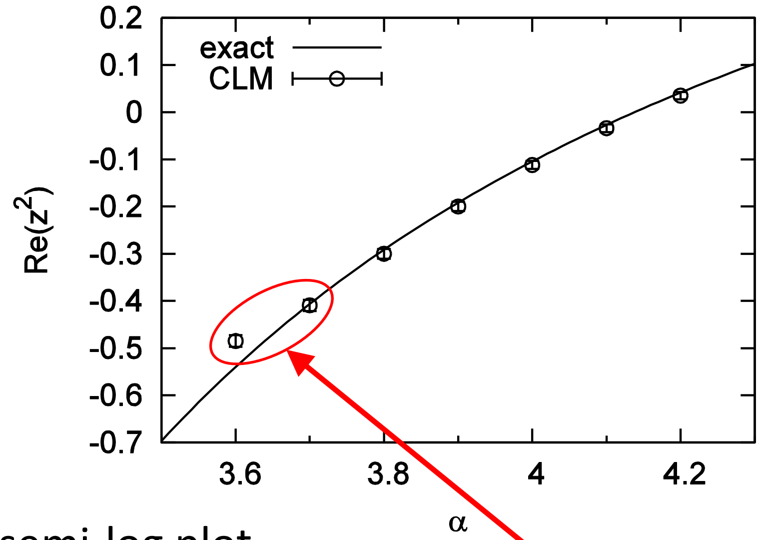
Before complexification, we make the change of variables

$$\alpha_1 = 0 \, , \quad \alpha_2 = e^{\tau_1} \, , \quad \alpha_3 = e^{\tau_1} + e^{\tau_2} \, , \quad \cdots \, , \quad \alpha_N = \sum_{a=1}^{N-1} e^{\tau_a} \, ,$$

in order to introduce the "time ordering".

Then we complexify $\tau_a \ (a = 1, \cdots, N-1)$.

Recent development : the condition for correct convergence

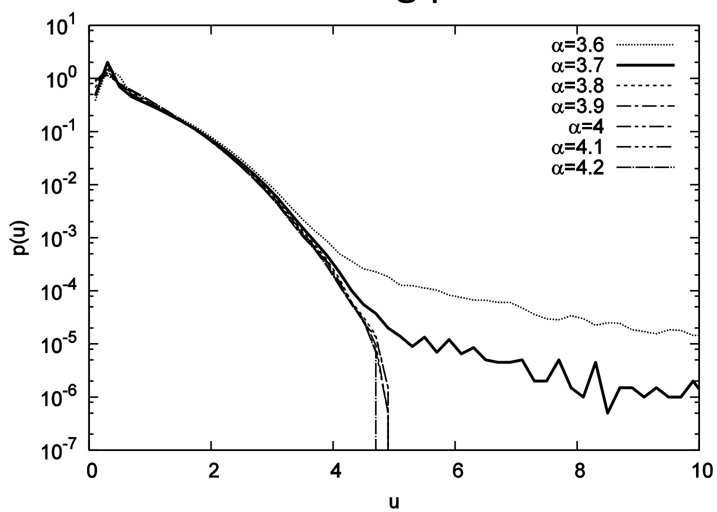


$$Z = \int dx\, w(x)$$
$$w(x) = (x+i\alpha)^p\, \mathrm{e}^{-x^2/2}$$

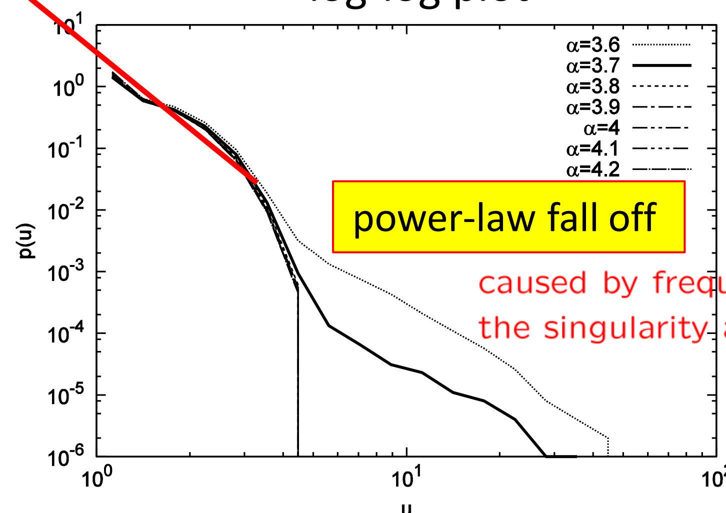
$$p = 4$$

In this model, CLM fails at $\alpha \lesssim 3.7$.

semi-log plot



log-log plot



caused by frequent visits to the singularity at $z = -i\alpha$

The probability distribution of the magnitude of the drift term $u \equiv |v(z)| = \left|\frac{p}{z+i\alpha} - z\right|$ should be suppressed exponentially in order for the method to be justified.

Nagata-J.N.-Shimasaki, Phys.Rev. D94 (2016) no.11, 114515.