Target Space Entanglement and Space-Time Geometry

Sandip Trivedi
Tata Institute of Fundamental Research,
Mumbai, India.

Target Space Entanglement and Space-Time Geometry

S. Das, S. Liu, A. Kaushal, G. Mandal, ST, hep-th/2011.13857

S. Das, A. Kaushal, G. Mandal, ST, hep-th/2004.00613

Outline

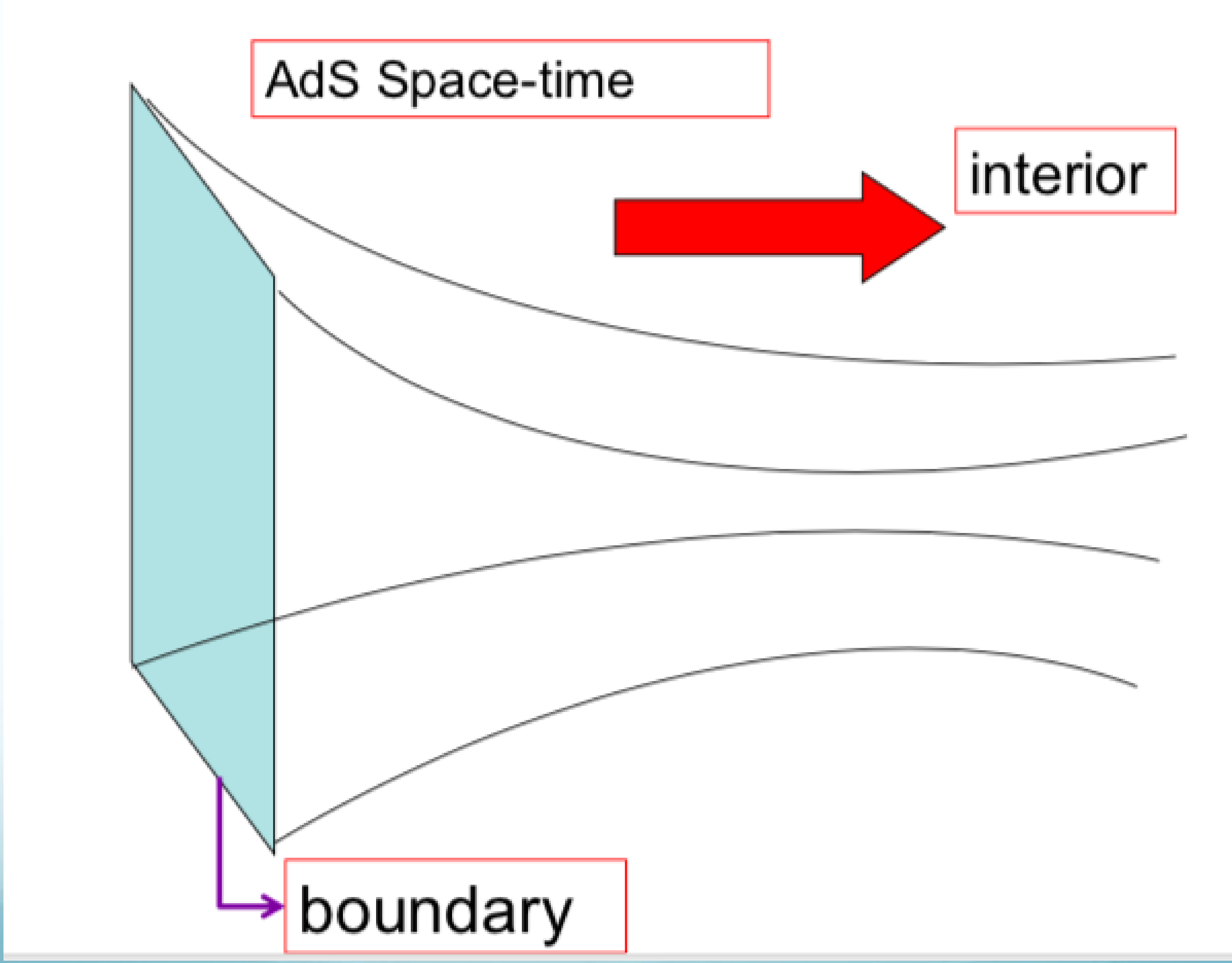
- Introduction
- Target Space Entanglement: Definition
- Space-Time Entanglement: Proposal
- Conclusions

Introduction

Key Question: How does a smooth space-time emerge from underlying degrees of freedom?

AdS/CFT: CFT provides the underlying description - J. Maldacena

(although perhaps better understood as a duality between string theory and CFT)


AdS Space-time
interior
boundary

Smooth AdS: Large N, large coupling

Correspondence can be extended, by considering near horizon geometries of Dp-branes, to non-AdS spacetimes.

E.g., near horizon geometries of Dp-branes.

Bulk not AdS and boundary not CFT

We will in particular be interested in the case of D0 branes where the boundary is quantum mechanics.

A simple context.
Impressive numerical progress.

Caterall & Wiseman; *Hanada, Hyakutake, Ishiki & Nishimura*; *Berkowitz, Rinaldi, Hanada, Ishiki, Shimasaki & Vranas*

(Our results will be more general and also extend to field theories)

One Motivation:

To understand the connections between entanglement and smooth spacetime geometry better.

Confluence of theoretical and Numerical ideas and calculations holds great promise.

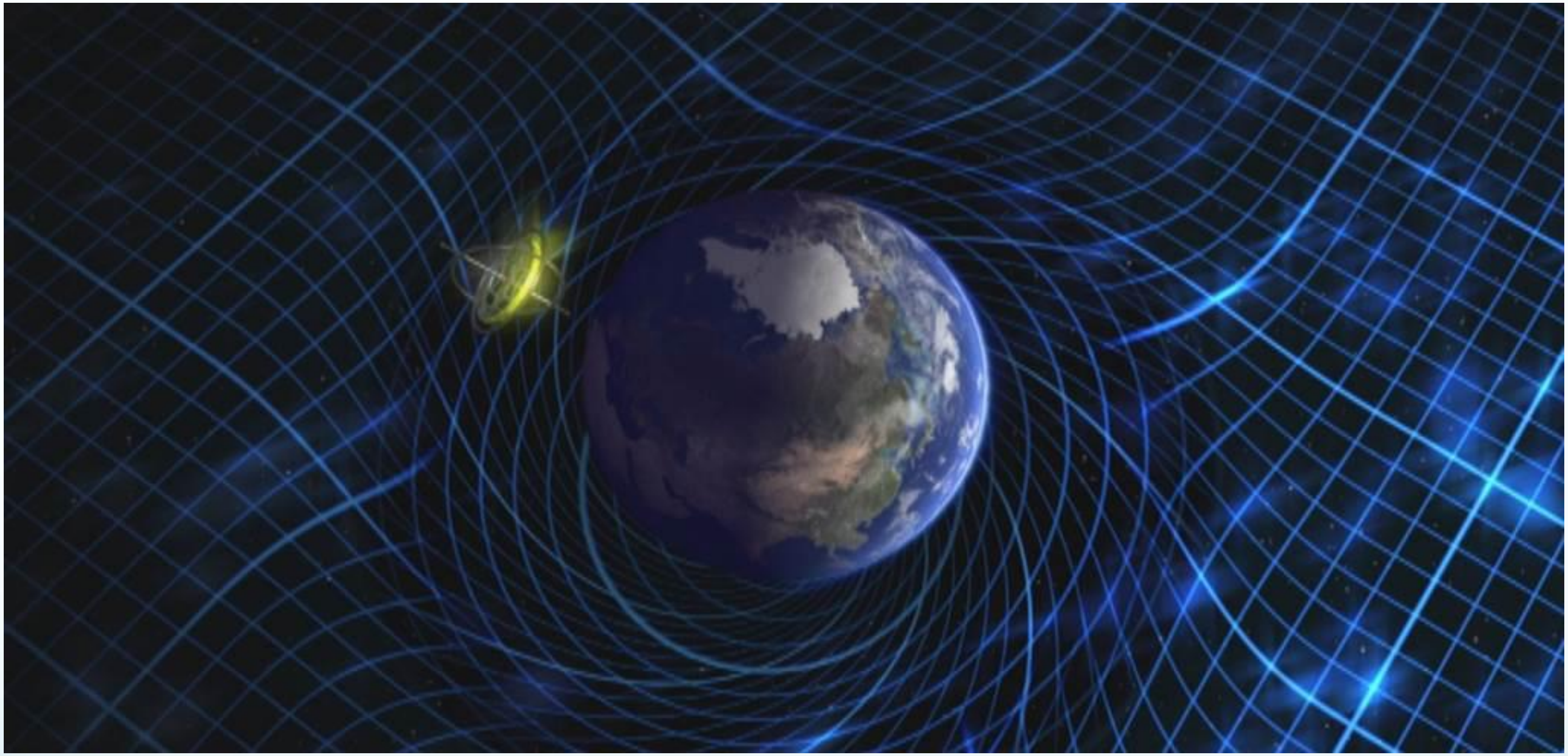
Near horizon Dp branes especially D0 brane geometry.

Entanglement

There is increasing evidence (and some hype!) that entanglement is in fact tied to the emergence of a smooth spacetime geometry.

The intrinsically quantum features of the wavefunction give rise to the ``fabric” for a smooth space- time!

(Swingle, Van Raamsdonk, …)



Entanglement ``is” the fabric of space-time

Entanglement

Quantum correlations have properties which cannot be mimicked by classical systems.

Entanglement refers to those aspects of the correlations which cannot arise in classical theory even when we allow for extra hidden local variables. (Bell 1960s)

More precisely:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\rho = |\psi >< \psi|$$

$$\rho_A = Tr_B(\rho)$$

$$S_{EE} = -Tr_A(\rho_A \log \rho_A)$$

Von Neumann Entropy of Density Matrix

(Some subtleties for gauge theories,
Casini Huerta Rosabal; Ghosh, Soni, ST)

Another Motivation

Is the Beckenstein Hawking Entropy formula more general?

For a Black Hole:

$$S_{BH} = \frac{A_H}{4G_N}$$

A_H : Area of Black Hole Horizon

For quantum fields (in bulk) entanglement also depends on Area:

X-axis

$$S_{EE} = \frac{A_T}{\delta^2}$$

Transverse area

Cut-off

In a theory of gravity is $\delta \to l_{Pl}$

$$S_{EE} = \frac{A_T}{4G_N} \quad ? \qquad (1)$$

Motivated by the Beckenstein-Hawking formula.

(Jacobson; *Bianchi & Myers)*

Both sides in eq (1) renormalised in same way.

Many questions remain. Especially : bulk locality itself is approximate. So what does this formula precisely mean?

Third Motivation:

Extend the beautiful work in 1+1 dimensional string theory.

Showing that Bulk Entanglement is finite at finite N. *S.R.Das (1995)*

To more general settings.

(See also *S. Hartnoll and E. Mazenc (2015)*)

Key Results:

- In this talk we will give a precise definition of entanglement entropy in the boundary theory. This will be called target space entanglement.

- Propose that it is related to bulk entanglement (for a smooth classical geometry):

$$S_{EE} = \frac{A_T}{4G_N}$$

Key Results:

- The Definition of Target Space Entanglement we give is precise.
- The connection with Beckenstein-Hawking formula in bulk is a proposal which might need to be``fine tuned" further.

Near-horizon D0 branes (10 dim)

$$ds^2_{string} = -H_0(r)^{-1/2}dt^2 + H_0(r)^{1/2}[dx_1^2 + \cdots + dx_9^2]$$

$$H_0(r) = \frac{R^7}{r^7}$$

$$r^2 = x_1^2 + x_2^2 + \cdots x_9^2$$

$$e^{-2\phi} = g_s^{-2} H_0(r)^{-3/2}$$

$$R^7 = \frac{(2\pi)^7}{7\Omega_8} l_s^7 (g_s N).$$

IIA Supergravity solution valid when

$$(N \gg 1)$$

$$\boxed{g_s^{1/3} N^{1/7} \ll r/l_s \ll (g_s N)^{1/3}}$$

$$\frac{g_s^{1/3} N^{1/7}}{l_s} \ll E \ll \frac{(g_s N)^{1/3}}{l_s}$$

$$(\quad E \sim r/l_s^2 \,)$$

Boundary Quantum Mechanics

$$S = \frac{N}{2(g_s N) l_s} \int dt [\sum_{i=1}^{9} Tr(D_t X^I)^2 - \frac{1}{l_s^4} \sum_{I \neq J = 1}^{9} Tr([X^I, X^J]^2)]$$

+ fermions

$$D_t X^I = \partial_t X^I + i[A_t, X^I]$$

$X^1, X^2 \cdots, X^9$: NXN matrices

$O(N^2)$: degrees of freedom.

Boundary Quantum Mechanics

$U(N)$ Gauge Theory

Invariant under $X^I \to U X^I U^{-1}$

Boundary Quantum Mechanics
Characterised by one energy scale:
Λ

$$H = \frac{(g_s N)^{1/3}}{2 l_s} \mathrm{Tr} \left[\frac{1}{N} \sum_{I=1}^{9} (\tilde{P}^I)^2 + N \sum_{I \neq J = 1}^{9} [\tilde{X}^I, \tilde{X}^J]^2 \right] + \text{fermions}$$

$$X^I = (g_s N)^{1/3} l_s \tilde{X}^I \qquad P^I = \frac{1}{(g_s N)^{1/3} l_s} \tilde{P}^I$$

$$\Lambda = \frac{(g_s N)^{1/3}}{l_s}$$

Duality

Matrix theory weakly coupled when

$$E \gg \Lambda$$

Gravity theory is highly curved in this region

Similarly when gravity theory weakly coupled, matrix theory strongly coupled, $E \ll \Lambda$.

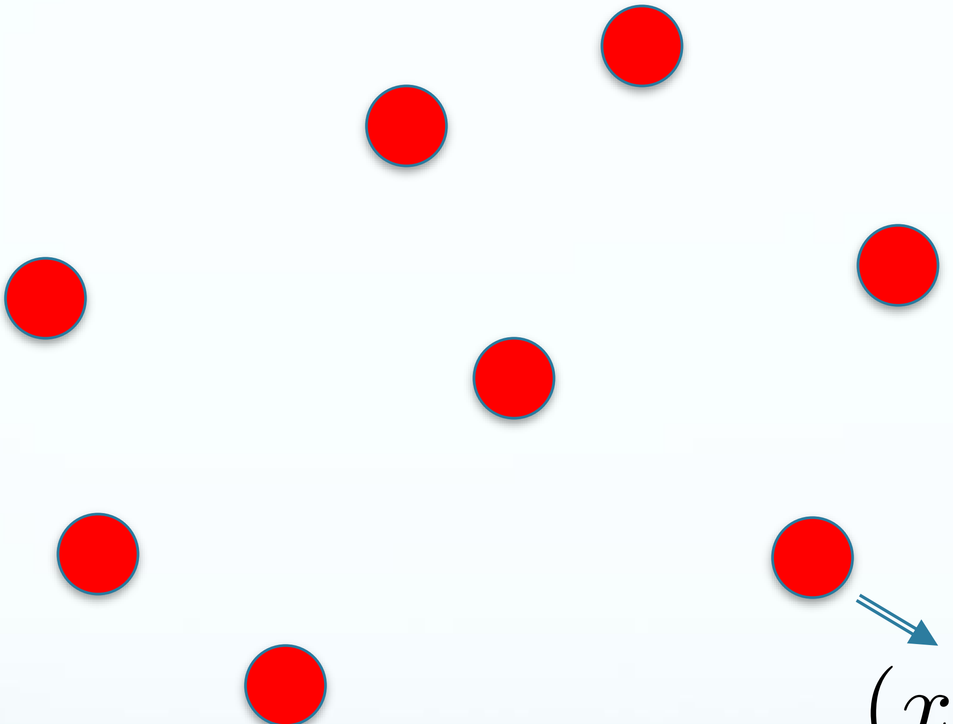
Boundary Quantum Mechanics

16 supersymmetries

Moduli space of vacua (Coulomb branch) : All 9 matrices commute.

$$X^1 = diag(x_1^1, x_2^1, \cdots x_N^1),$$
$$X^2 = diag(x_1^2, x_2^2, \cdots x_N^2),$$

$(x_1^1, x_1^2, \cdots x_1^9)$: location of first D0 brane etc



$$(x_1^1, x_1^2, \cdots x_1^9)$$

D0 branes in Bulk (in Coulomb branch) One to one correspondence with the vacuua

At finite temperature: Black Hole

$$\frac{E}{\Lambda} = 7.41 N^2 \left(\frac{T}{\Lambda}\right)^{14/5}$$

Agrees with Numerics:

Berkowiz, Rinaldi, Hanada, Ishiki, Shimasaki, Vranas, 2016

At finite temperature: Black Hole

$$ds^2_{string} = -H_0(r)^{-1/2} g(r) dt^2 + H_0(r)^{1/2}[\frac{dr^2}{g(r)} + r^2 d\Omega_8^2]$$

$$e^{\phi} = g_s H_0(r)^{3/4} \qquad A_0 = -\frac{1}{2}(H_0^{-1} - 1)$$

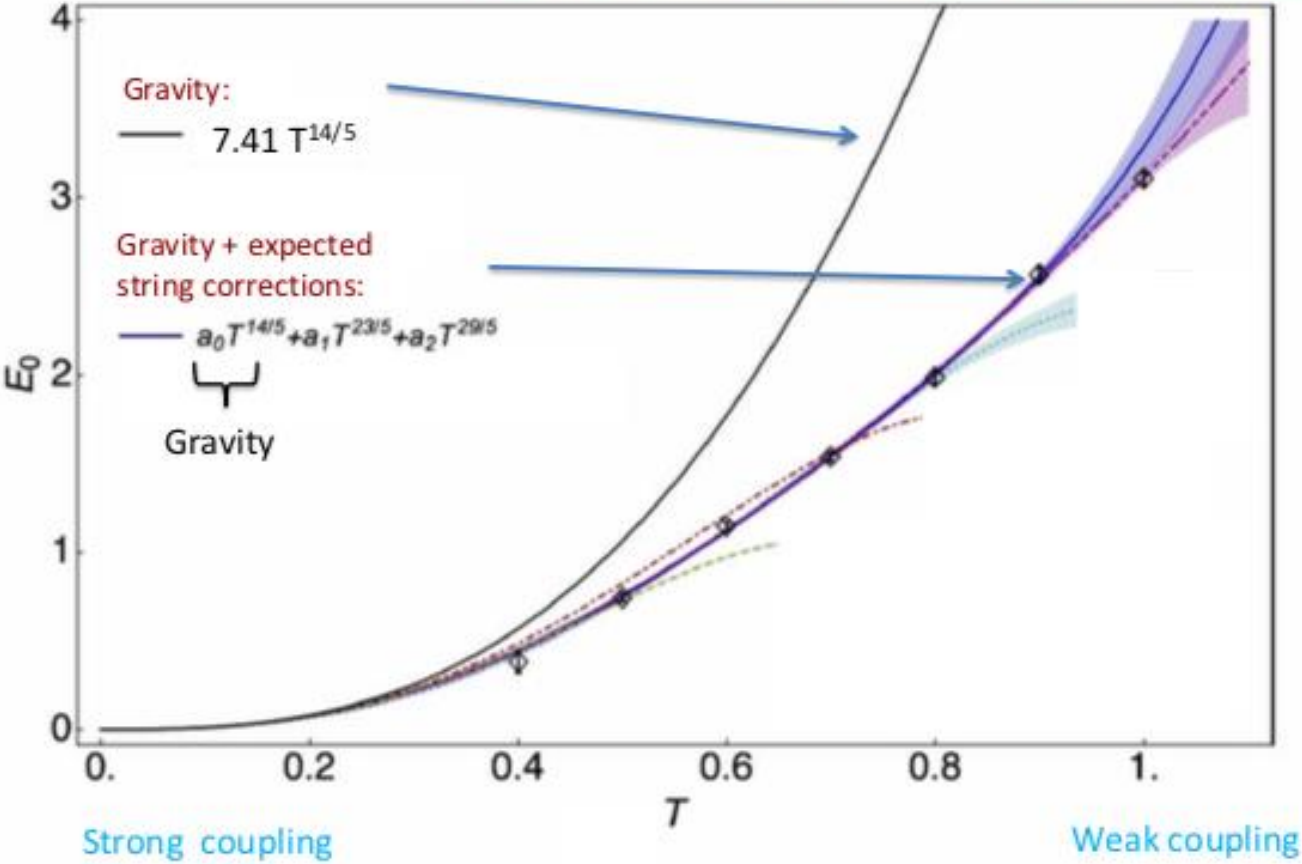
where

$$g(r) = 1 - \left(\frac{r_H}{r}\right)^7 \quad H_0(r) = \frac{R^7}{r^7}, \qquad r^2 = x_1^2 + \cdots x_9^2.$$

$$T = \frac{7}{4\pi R}\left(\frac{r_H}{R}\right)^{5/2}$$

Computation of the free energy in the quantum mechanical mode

Berkowitz, Rinaldi, Hanada, Ishiki, Shimasaki, Vranas. 20



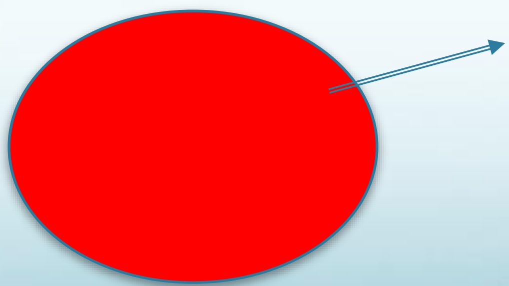
(a_0 in agreement with gravity within the numerical error bars of about 7%)

(slide taken from J. Maldacena, ICTS lecture 2018)

Target Space Entanglement

Often for a field theory in $R^d \times T$
One considers a subregion of R^d

And asks how entangled are the degrees of freedom inside with those outside.

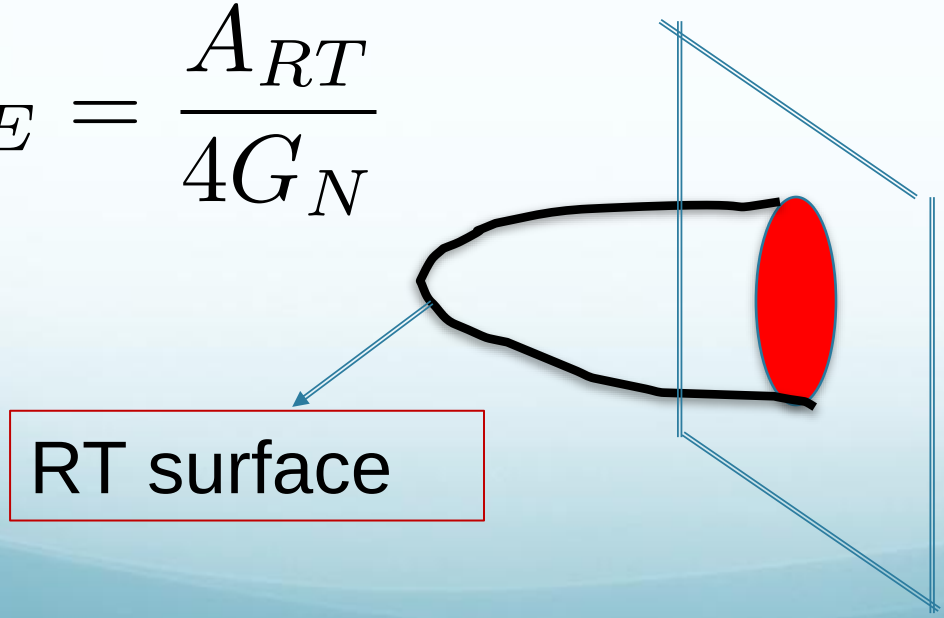


Spatial region in R^d

When field theory is a CFT on the boundary of AdS

This entanglement is given by the Ryu-Takayanagi formula

$$S_{EE} = \frac{A_{RT}}{4G_N}$$



Target Space Entanglement

But what if the boundary Theory is Quantum Mechanics?

Intuitively one expects that the entanglement should be among the colour degrees of freedom – which are of $O(N^2)$.

But how do we make this precise in a gauge invariant way?

Some References:

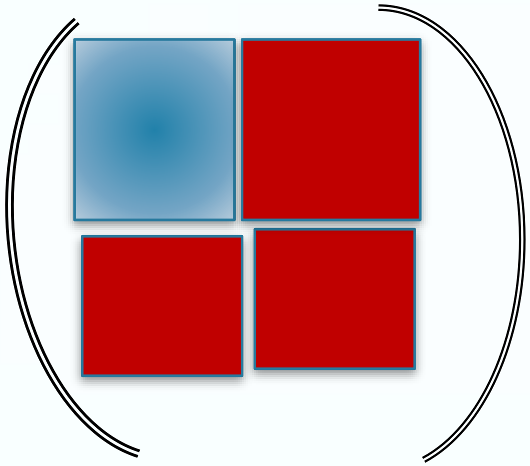
1) S. R. Das, G. Mandal, ST (unpublished)

2) Mazenc and Ranard (hep-th/1910.07449)

3) S. Das, A. Kaushal, G. Mandal, ST, (hep-th/2004.00613)

4) S. Das, S. Liu, A. Kaushal, G. Mandal, ST, (hep-th/2011.13857)

5) H. Hampapura, J. Harper, A. Lawrence, (hep-th/2012.15683)



NXN Matrix

U(N) gauge transformation will mix the blue block with the rest.

Target Space Entanglement

We use the bulk to motivate a definition

Suppose we are interested in the bulk region $x^1 > a$

Look at corresponding target space constraint in boundary $X^1 > a$

More precisely go to a gauge where X^1 is diagonal

$$X^1 = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & & \lambda_N \end{pmatrix}$$

Let's say m eigenvalues meet the condition $\lambda > a$

Fix gauge so that these are the first m eigenvalues

Target Space Entanglement

How entangled are the first m eigenvalues with the rest?

This is a gauge invariant question.

Resulting Von Neumann entropy is the Target Space Entanglement.

It arises due to the colour degrees of freedom.

Target Space Entanglement

In general remaining matrices:

$$X^2, X^3 \cdots X^9$$

Will not be diagonal in this gauge.

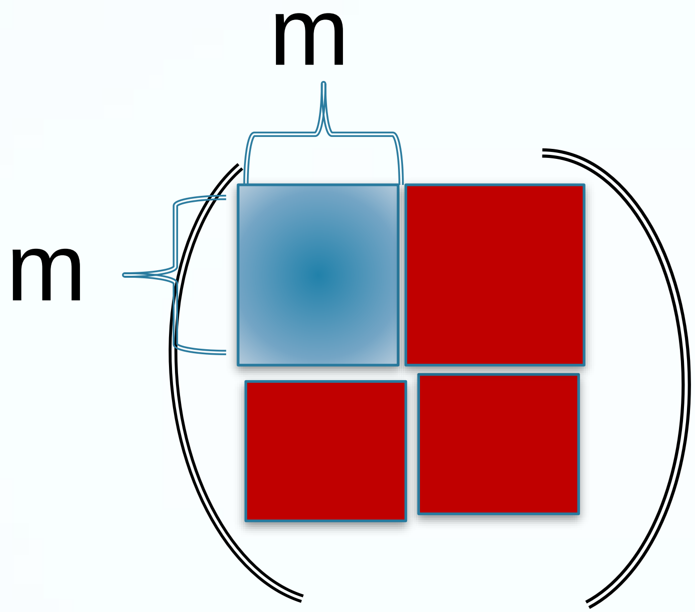
We have two versions of the definition of target space entanglement.

Target Space Entanglement

Version 1: We keep the first $m \times m$ blocks of the $X^2, X^3 \cdots X^9$ matrices.

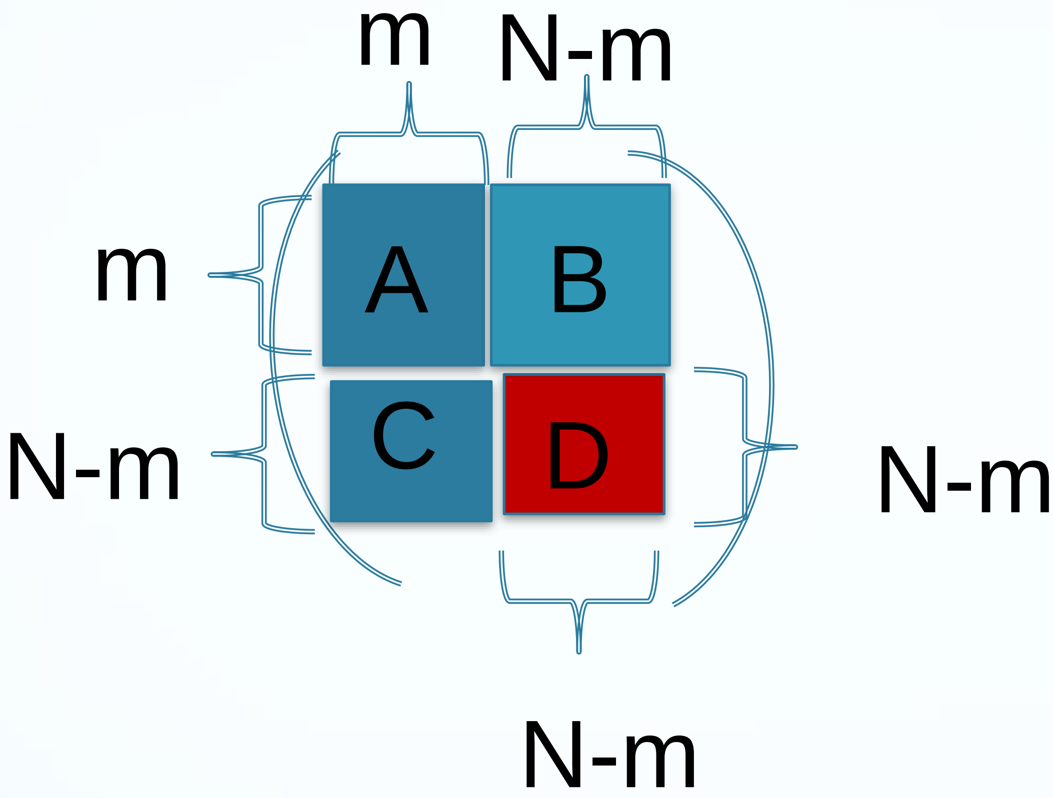
Ask how entangled are the first m eigenvalues of X^1 and the $m \times m$ blocks of the remaining matrices with the rest? (Also include fermions)

(This is a gauge invariant question)



Version 1

Keep $m \times m$ block for all matrices and ask what is the entanglement entropy of the resulting density matrix?



Version 2

Keep blocks A, B, C and remove D – which is of size (N-m)X(N-m)- for all matrices.

For either version we now have to sum over the various sectors: $m = 0,1, \ldots, N$

Full Target Space Entanglement is the sum over the contributions from the various sectors.

Entanglement in the m th sector:

$$S_m = -Tr_{(m)}\rho_m log(\rho_m)$$

Full entanglement given by a sum over all sectors

Target space entanglement

$$S_T = \sum_{m=0}^{m=N} S_m$$

Include possibility that no eigenvalues lies in region of interest

Target Space Entanglement

Definition can be generalised for any constraint involving the matrices

$$X^I, I = 1 \cdots 9$$

As operators X^I commute, any constraint of the form $F(X^I) > 0$ can be dealt with by diagonalising the constraint and then obtaining the entanglement sector by sector.

E.g. $$F(X^I) = \sum_{I=1}^{9} (X^I)^2 > R^2$$

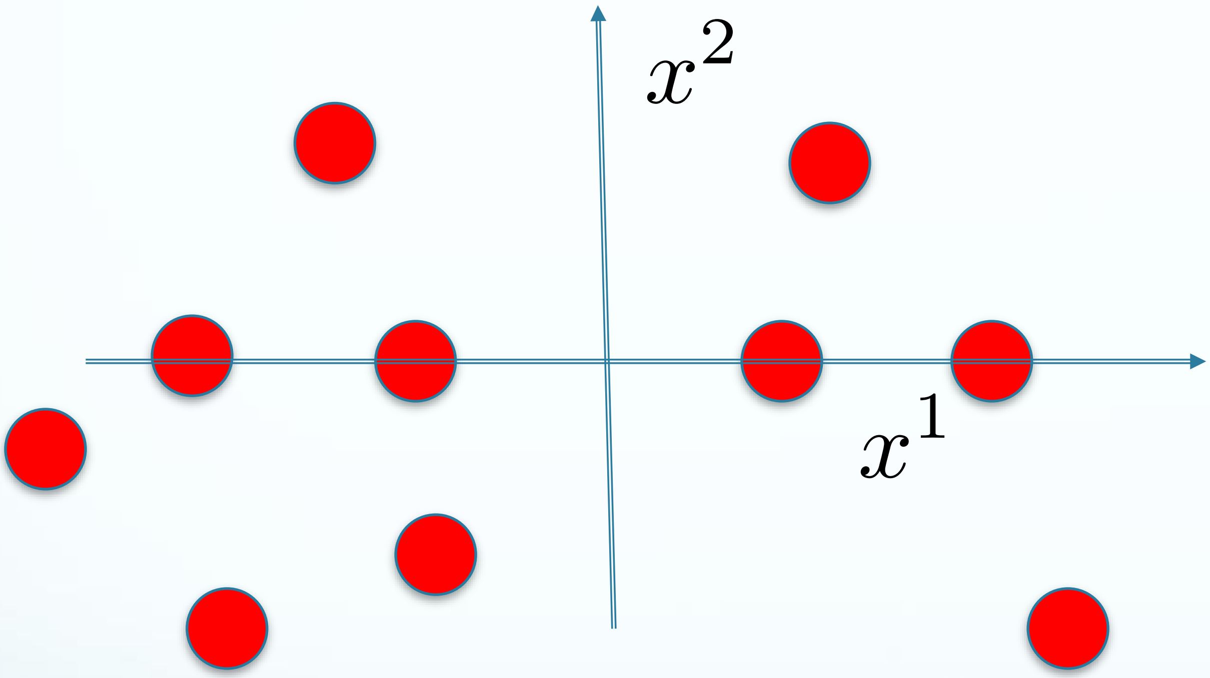
Target Space Entanglement

Definition can be extended to a general density matrix (instead of a pure state)

In particular to thermal density matrix

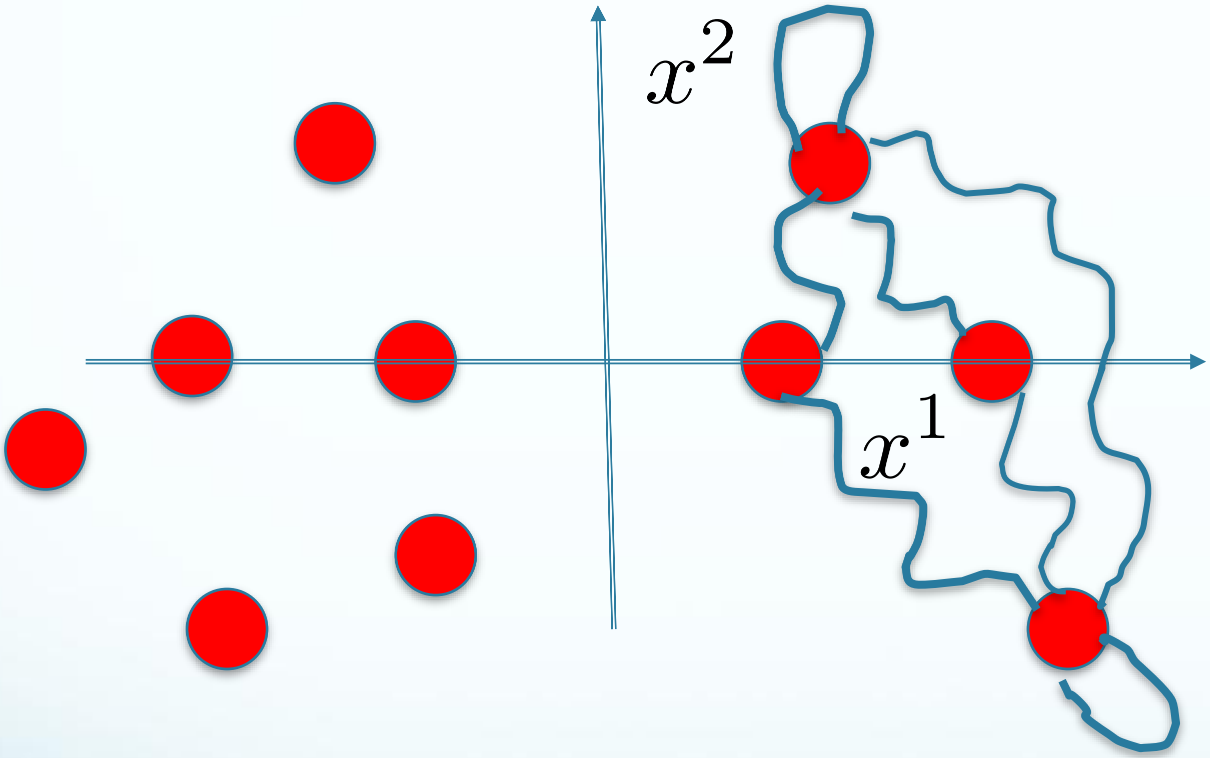
$$\rho = \frac{e^{-\beta H}}{Tr(e^{-\beta H})}$$

Physical Picture: $x^1 > 0$



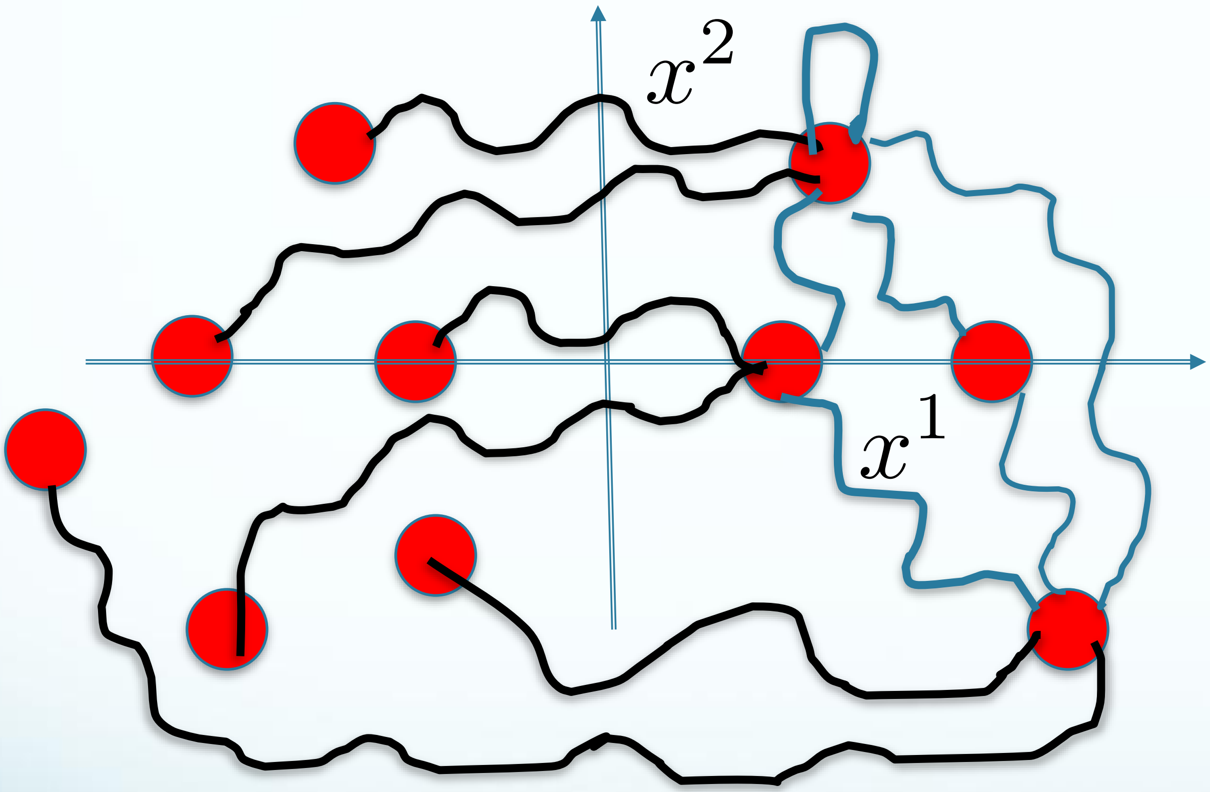
4 D0 branes to meet constraint.
(m=4 sector)

Version 1: $x^1 > 0$



4 D0 branes meet constraint. Keep all degrees of freedom associated with them denoted by blue strings.

Version 2: $\quad x^1 > 0$



In addition also keep black strings running between the 4 D0 branes and the rest.

Version 1

m values

N-m values

Wave function $\Psi(x^a, x^\alpha, Y_{ab}, Y_{a\alpha}, Y_{\alpha a}, Y_{\alpha\beta})$

$$\rho_m(x^a, Y_{ab}, x'^a, Y'_{ab}) =$$

$$\int_{\bar{A}} \Psi^*(x^a, x^\alpha, Y_{ab}, Y_{a\alpha}, Y_{\alpha a}, Y_{\alpha\beta})\Psi(x'^a, x^\alpha, Y'_{ab}, Y_{a\alpha}, Y_{\alpha a}, Y_{\alpha\beta}) dx^\alpha dY_{a\alpha} dY_{\alpha a} dY_{\alpha\beta}$$

m th sector density matrix

$\bar{A}$ Complement of the region of interest, $x^1 < 0$

Gauge Invariant Description

Another way to think about entangelment entropy:

Associated with a sub-algebra of observers.

Density matrix lies in subalgebra

Gives the correct expectation value of all operators in subalgebra.

Gauge Invariant Description

$$|\psi>=\frac{1}{\sqrt{2}}[|+->-|-+>]$$

Instead of tracing over 2nd spin.
Think of Subalgebra:

$$\sigma_i \otimes I \text{ , } \quad I \otimes I$$

$$\rho = (\frac{1}{2})I \otimes I)$$

For target space entanglement

Define projector:

$$P^1 = \int_{x>0} dx \ \delta(x\mathbf{I} - X^1)$$

For target space constraint $x^1 > 0$

Version 1:

Project all operators by acting with this projector.

Then take a trace to obtain gauge invariant operators.

$$X^I \to (X^I)^{P_1} = P^1 X^I P^1, \; \Pi_J \to (\Pi_J)^{P_1} = P_1 \Pi_J P^1$$

$$Tr((X^I)^{P_1}, ...(\Pi_J)^{P_1}, \cdots)$$

Version 1:

This gives rise to a sub algebra.

Entangelment entropy is associated with this sub algebra.

Similarly for version 2.

Subalgebra defined With an appropriately chosen projector.

Bulk Entangelment : Proposal

Bulk Entangelment Entropy for region

$$f(x^i) > 0$$

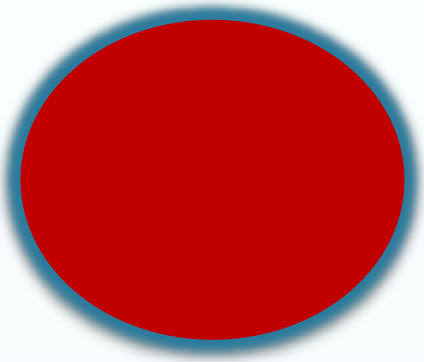
Is given by Target space entanglement associated with constraint $f(X^I) > 0$

And this bulk entanglement saturates the Beckenstein-Hawking bound (for a general surface).

Proposal for Bulk Entanglement

And $$S_{Bulk} = \frac{A_\partial}{4G_N} = S_T$$

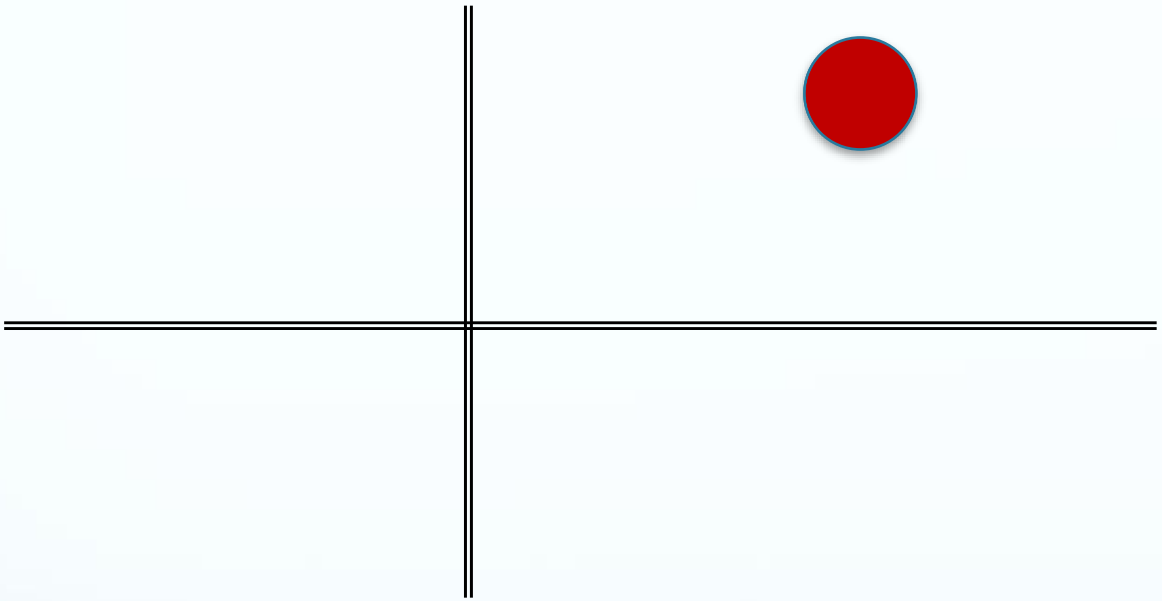
Target space entanglement



Boundary with area A_∂

(In one of the two versions of our proposal for S_T)

Some Evidence: Coulomb Branch:



Moduli space of vacua of Quantum Mechanics agrees with Bulk

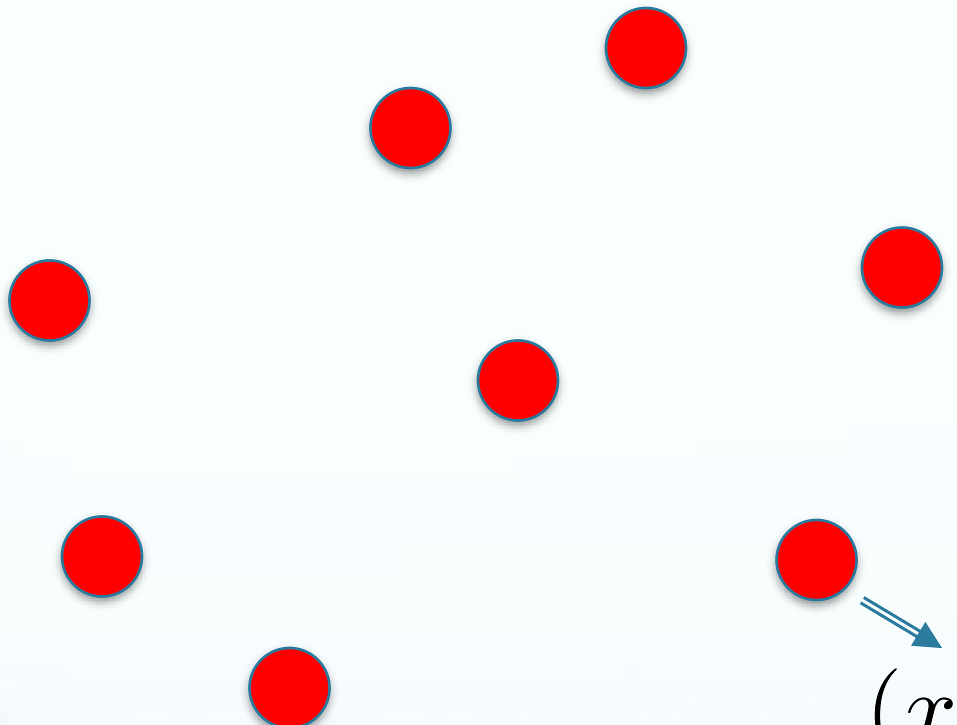
Boundary Quantum Mechanics

16 supersymmetries

Moduli space of vacua (Coulomb branch) : All 9 matrices commute.

$$X^1 = diag(x_1^1, x_2^1, \cdots x_N^1),$$
$$X^2 = diag(x_1^2, x_2^2, \cdots x_N^2),$$

$(x_1^1, x_1^2, \cdots x_1^9)$: location of first D0 brane etc



$$(x_1^1, x_1^2, \cdots x_1^9)$$

D0 branes in Bulk (in Coulomb branch) One to one correspondence with the vacuua

Force on a moving brane can be calculated in quantum mechanics by doing an effective action calculation for gauge invariant operators. Agrees with the bulk description.

Same effective action is obtained for the projected gauge invariant operators when brane lies in region of interest.

Testing the proposal:

Region $\quad x^1 > a$

Gravity side:

$$S(a,T) - S(a,T') = B_0\ N^2 a_0^{-5/2} \left[(T_0)^{14/5} - (T_0')^{14/5} \right]$$

$$a_0 = \frac{a}{\Lambda l_s^2} \qquad \Lambda = (g_s N)^{1/3}/l_s \qquad B_0 \simeq 260.502$$

$$T_0 = T/\Lambda \qquad T_0' = T'/\Lambda$$

[This should agree with Version 1) or 2).]

Testing The Proposal

Calculation in Matrix Quantum Mechanics should reproduce this result.

Including N^2 factor

(for $T_0, T_0' \ll 1$)

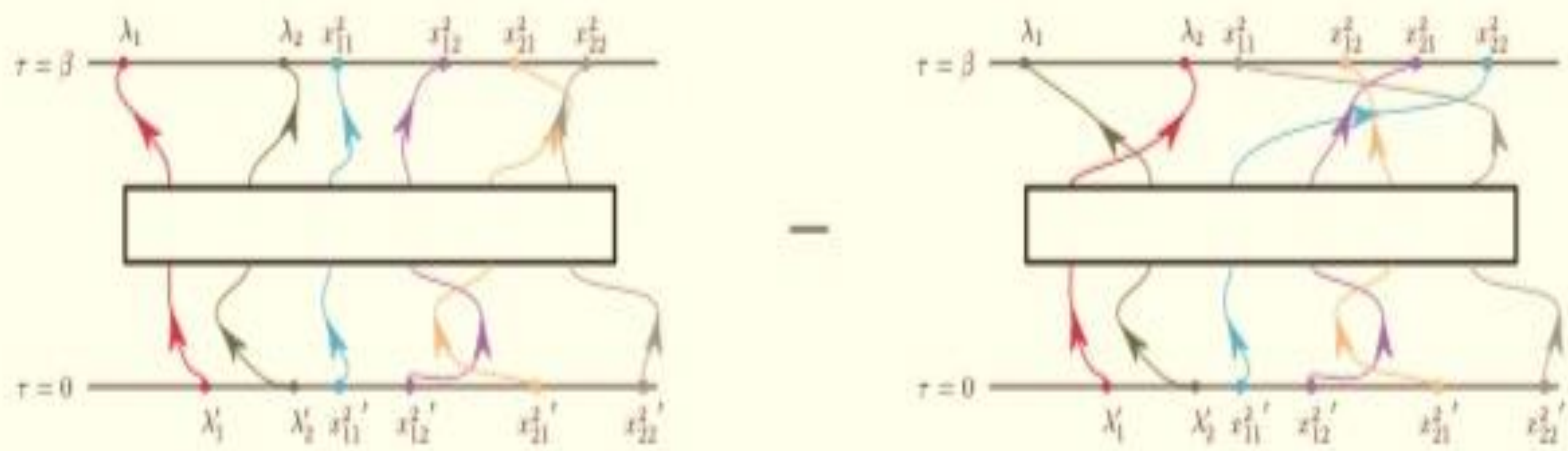
Note: Encouraging that gravity result is expressible in terms of quantities in gauge theory. Not true if $S_{Bulk} \sim A/l_s^8$

How can one calculate target space entanglement?

Path integral methods can be used to calculate density matrix (at Finite Temperature)

$$\langle \lambda_i, X^L_{ij} | \hat{\rho}_0 | \lambda'_i, (X^L_{ij})' \rangle$$

$$= \frac{1}{N!} \sum_{\sigma \in S(N)} (-)^\sigma \int_{\lambda_i(0)=\lambda'_i}^{\lambda_i(\beta)=\lambda_{\sigma(i)}} \mathcal{D}\lambda_i(\tau) \int_{X^L_{ij}(0)=(X^L_{ij})'}^{X^L_{ij}(\beta)=X^L_{\sigma(i)\sigma(j)}} \mathcal{D}X^L_{ij}(\tau) \exp[-S_\beta]$$



Trajectories contributing to density matrix

By tracing over appropriate degrees of freedom the reduced density matrix can then be calculated.

Summary:

Numerical methods have allowed impressive checks of AdS/CFT

Hopefully they will also allow us to understand the connection between target space entanglement entropy in boundary theory and entanglement in the bulk.

Summary:

We have given a precise definition of target space entanglement.

(Definition can be generalised for higher dimensional situations where we are dealing with a field theory with a target space.)

Proposed that the target space entanglement equals the Beckenstein- Hawking area formula for a general region in bulk.

Summary:

Our proposal is not precise (two versions).

Numerical work could help decide which is correct.

It might need further sharpening.

E.g. bulk condition specifying region $f(x^i) > 0$

May not map directly to target space constraint

$$f(X^i) > 0$$

Especially at finite temperature.

The map can perhaps be figured out numerically by calculating the force on a probe brane in bulk and matching with boundary theory. (Rinaldi et. at.)

Summary:

One needs a better understanding of what bulk operators are included in the projected target space operators.

i.e. What kinds of bulk measurements are accounted for by the target space entanglement.

The bulk Hamiltonian is not included (for the spherical region).
Chowdhury, Papadoulaki & Raju

Simple examples are a good way to begin.
E.g. Two matrix bosonic theory.

When is target space entanglement $O(N^2)$
and when $O(N^0)$?

Is there a transition possible? Analogue of
Hawking page transition.

Summary:

Other open questions:

- Does gravity provide a code subspace for encoding target space information in an entangled form in target space? (Pastawski, Yoshida, Harlow, Preskill)
- What is the role that extremal surfaces play in this whole discussion? (*Anous, Karczmarek, Mintun, van Raamsdonk & Way*)

Thank You!

Banyan Tree In TIFR Mumbai

Thank you!