Combinatorical and symplectic invariants of the moduli space of vector bundles

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(joint work with Pieter Belmans and Sergey Galkin)

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Brief outline

1. The main player is the moduli space $M_C^{\epsilon}(2, L)$ of rank two bundles on a smooth curve C with fixed determinant L. ($\epsilon = \deg(L)$) and its parabolic analogs to construct degenerations.
2. We want to analyze toric degenerations X_{C_0} of the above as the curve degenerates maximally C_0.
3. Use these degenerations produce (weak)LG-mirrors of $M_C^{odd}(2, L)$ and compute some Gromov-Witten numbers.
4. Recover the toric variety X_{C_0} from the maximally degenerate curve C_0.
5. We want to study "cluster like" structures by comparing two maximal degenerations.

Character Variety for $SU(2)$

Let K be a compact, group and let G be a complexification of K. Let Σ be a surface of genus $g \geq 2$. Consider the character variety

$$\mathcal{N}_g^0 = \mathrm{Hom}(\pi_1(\Sigma), K)//K$$

We can also consider a variant $\mathcal{N}_g$ for a punctured surface $\Sigma - p$ with prescribed monodromy of a loop around the point p.

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Character varieties for n-punctured sphere with small monodromies recover Kapovich-Millson polygon spaces which will be discussed later.

Refined Invariants: Conformal Blocks

Fix a complex, simple, Lie algebra $\mathfrak{g}$ and $\ell \in \mathbb{Z}_{>0}$ and $\lambda_1, \dots, \lambda_n$ dominant integral weights $(\lambda, \theta) \leq \ell$.

- Let $z_1, \dots, z_n$ are n-distinct complex numbers consider the subspace $\mathbb{V}_{\vec{\lambda}}(\mathfrak{g}, \ell, \vec{z})$ of $\mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}, \mathbb{C})$ on which $T_{\vec{z}}^{\ell}$ acts trivially, where $T_{\vec{z}} = \sum_{i=1}^{n} z_i X_{\theta}$. This operator is nilpotent.

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- if $n = 3$ and $\mathfrak{g} = \mathfrak{sl}(2)$, the space of conformal blocks equals $\mathfrak{sl}(2)$ invariants forms on $\mathrm{Sym}^a \mathbb{C}^2 \otimes \mathrm{Sym}^b \mathbb{C}^2 \otimes \mathrm{Sym}^c \mathbb{C}^2$ if
  - $a + b + c$ is even.
  - (triangle inequality) $|a - c| \leq b \leq |a + c|$ and same with the a and c.
  - (perimeter) $a + b + c \leq 2\ell$ and zero otherwise.

B side: Symplectic invariants for X

Let X be a smooth Fano manifold with cyclic Picard group. Let $X_{0,k,m}$ denote the Kontsevich moduli of stable maps f from a rational curve with k marked points and $\deg f^*(-K_X) = m$.

- A point is of the form $[C; p_1, \ldots, p_n; f : C \to X]$, where C is a rational curve.
- The moduli space space of stable maps $X_{0,k,m}$ has virtual dimension $\dim X - 3 + m + k$. The evaluation map

$$\mathrm{ev}_i : X_{0,k,m} \to X$$

  sends a point $[C; p_1, \ldots, p_n; f : C \to X]$ to $f(p_i) \in X$.
- There is a tautological line bundle Ψ_i whose fiber at $[C; p_1, \ldots, p_n; f : C \to X]$ is $T^*_{p_i} C$.

cont

Definition

The $m \geq 2$-th descendent Gromov Witten number

$$p_m = \int_{X_{0,1,m}} \psi^{m-2}\, \mathrm{ev}_1^{-1}([pt]),$$

where ψ is the *Psi* class on $X_{0,1,m}$ and $\mathrm{ev}_1 : X_{0,1,m} \to X$.

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Compute

$$\sum_{m \geq 0} p_m t^m \quad \text{for } X; \, p_0 = 1, \; p_1 = 0.$$

Finding mirror potentials W

Question

Construct $W : (\mathbb{C}^{\times})^{\dim X} \to \mathbb{C}$, such that

$$\frac{1}{m!}\text{Coefficient of Constant Term}(W^m) = p_m.$$

and find an efficient way to compute it.

- (Hori-Vafa, Givental) The quantum periods of X is a smooth toric Fano or Fano complete intersection in a toric variety.
- (Coates-Corti-Galkin-Kasprzyk) If X is a smooth Fano three fold, then quantum periods are known.
- Many other results due to works of Batyrev-Ciocan-Fontanine-Kim-van-Straten, Bondal-Galkin, Coates, Przyalkowski,...

Polytopes

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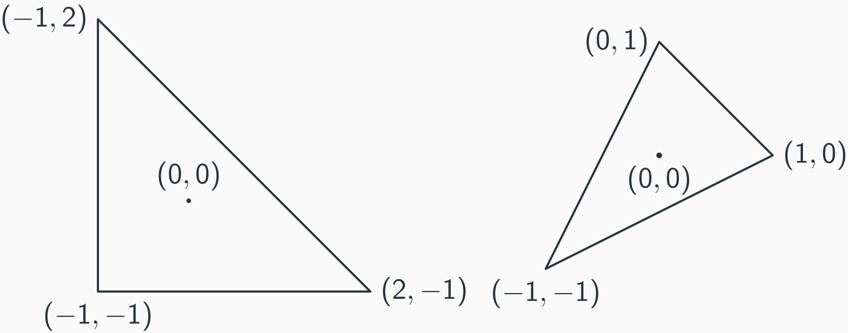
- **Polar Dual:** If $P = \bigcap_{F \in Facets} \{x \in L | \langle u_F, x \rangle \geq -1\}$ where u_F in the dual lattice of L. The polar dual

$$P^0 = Conv\{u_F \mid F \in \text{Facets of P}\}.$$

Examples

$$P = \{(x, y) | x \geq -1, y \geq -1, -x - y \geq -1\}.$$



Newton Polynomial of P^0 equals $x + y + x^{-1}y^{-1}$.

B side: Eguchi-Hori-Xiong-Givental proposal

- Degenerate a smooth Fano variety X to a toric Fano variety X_0 over a base. Let T_{X_0} be the torus acting on X.
- The toric Fano variety should have "nice singularities" and look at the moment map $\Phi : X_0 \to P$.
- Example: The only lattice points of P^0 are the origin and the vertices

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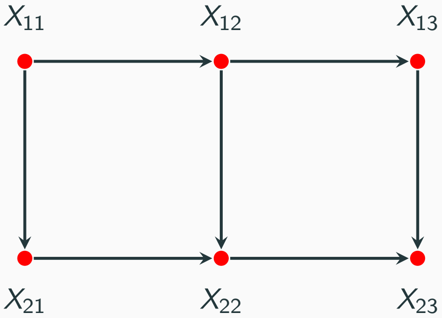
The above is now a Theorem due to **Nishinou-Nohara-Ueda** (under assumption that X_0 has small resolution), **Belmans-Galkin-M**, combining **Tonkonog, Bondal-Galkin.**

EHX: Edge potential for Gr(2, 5)

The proposed Newton polynomial for $Gr(2,5)$ is the following:

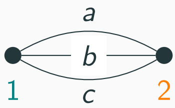


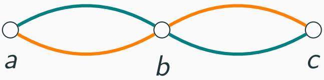
$$X_{11} + \frac{X_{12}}{X_{11}} + \frac{X_{21}}{X_{11}} + \frac{X_{22}}{X_{21}} + \frac{X_{22}}{X_{12}} + \frac{X_{13}}{X_{12}} + \frac{X_{23}}{X_{22}} + \frac{X_{23}}{X_{13}} + qX_{23}^{-1}$$

Toric degenerations of M_C^ϵ

Let $\widetilde{C}$ be a maximally degenerate nodal curve of g:

- The dual graph (Γ, V, E) of $\widetilde{C}$ (V corresponds to components and E correspond to intersection of components) is trivalent.





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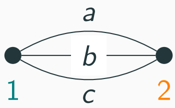
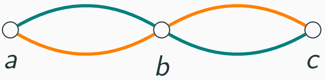
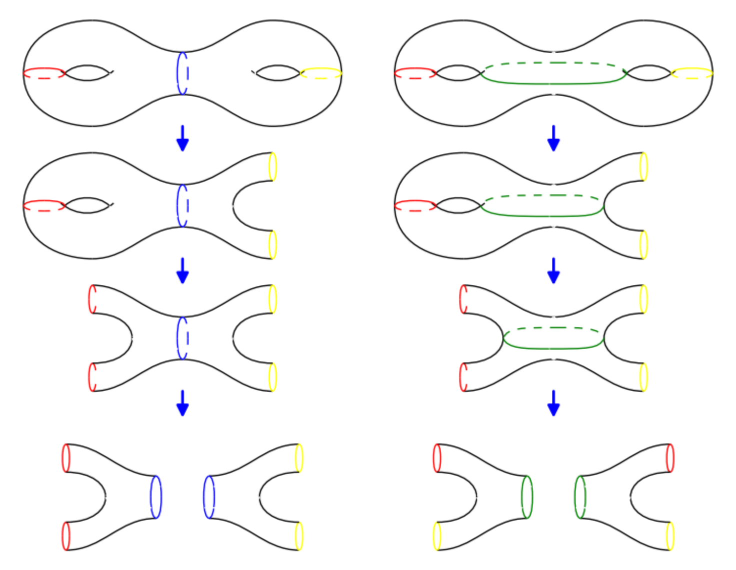




- Choose $3g-3$ disjoint circles in C and cut them such that C decomposes into a pair of pants.

Pants decomposition



Trivalent graphs v/s pants decomposition

Trivalent Graphs

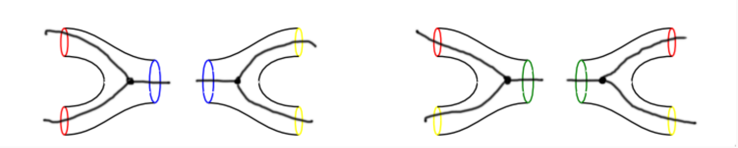
A trivalent graph is a graph where each vertex have degree 3.

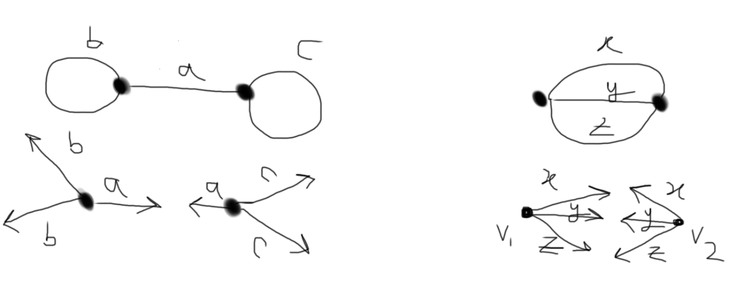
Pants decomposition from trivalent graph

Give a pair of pants decomposition

- Assign a trinion to each pair of pants in the decomposition.
- Connect the half edges if the pair of pants is obtained by cutting along a circle from the original surface.
- These graphs also correspond to the deepest strata in the moduli space of curves.

Example in genus 2





Theorem: Manon, combined with Belmans-Galkin-M

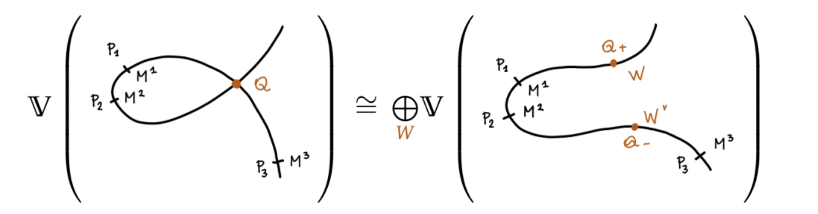
Let (Γ, c) be a trivalent graph with one (zero) colored vertex of genus g. The moduli spaces M_C (M_C^0-even degree determinant) degenerates to a toric variety $X_{\Gamma,c}$. whose moment polytope $P_{\Gamma,c}$ in $\mathbb{R}^{|E|}$ is given by:

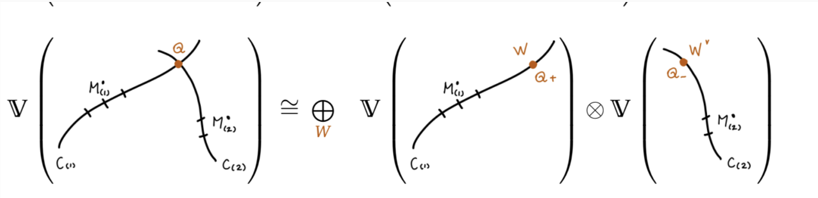
If $c(v) = (-1)^\epsilon$,

- $(-1)^\epsilon(x + y + z) \geq -1$.
- $(-1)^\epsilon(x - y - z) \geq -1$.
- $(-1)^\epsilon(-x - y + z) \geq -1$.
- $(-1)^\epsilon(-x + y - z) \geq -1$.

with respect to a lattice L_Γ in $L = \mathbb{Z}^{|E|}$ of index 2^g.

Degeneration of curves and factorization of conformal blocks





The toric degeneration

- Consider the section ring $R_C := \bigoplus_{i \geq 0} H^0(M_C^{\epsilon}, \Theta^{\otimes i})$ and consider it as the algebra of conformal blocks constructed as invariants of integrable representations of affine Kac-Moody algebras.
- Degenerate the curve C to C_0 to degenerate R_C and use factorization theorem to relate the degeneration of R with the section ring R_{par} of parabolic rank two bundles on on the normalization $\widetilde{C}_0$ which is a product of $\mathbb{P}^1$ with 3 marked points.
- Relate the ring R_{par} to tensor product of SL(2) representations as described earlier using the Clebsch-Gordan identities.

Singularities of $X_{\Gamma,c}$

Theorem: Belmans-Galkin-M

- The variety $X_{\Gamma,c}$ has terminal singularities if the graph Γ has no separating edges.
- Equivalently, the only lattice points of the polar dual $P^0_{\Gamma,c}$ of the moment polytope are the origin and the vertices.

Corollary

Let (Γ, c) be a trivalent graph of genus with no-separating edges and one colored vertex, then let $W_{\Gamma,c}$ be the Newton polynomial (*Graph Potentials*) of the Fan polytope of $X_{\Gamma,c}$. Then $W_{\Gamma,c}$ computes the quantum periods of M_C.

Theorem: Muñoz

The quantum multiplication $\star_0$ by $c_1(M_C)$ on quantum cohomology ring $QH^*(M_C)$ has the following eigen-space decomposition:

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Remark: This decomposition is equivariant with respect to the natural $\mathrm{Sp}(2g)$ action on both sides.

Criticial values of graph potential

Theorem: Belmans-Galkin-M

- All the above eigen-values are criticial values of $W_{\Gamma,c}$ with the dimension of the critical sets begin at least the expected ones.
- If Γ is the Necklace graph, the eigen-values and the critical values match-up and so are the expected dimensions.

Semi-orthogonal decomposition

Theorem: Bondal-Orlov

Let C be a smooth genus two curve, then

$$\mathbf{D}^b(M_C) = \langle \mathbf{D}^b(pt), \mathbf{D}^b(C), \mathbf{D}^b(pt) \rangle$$

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Conjecture : Belmans-Galkin-M; Narasimhan

Let C be a smooth curve of genus g

$$\begin{aligned} \mathbf{D}^b(M_C) = &\langle \mathbf{D}^b(pt), \mathbf{D}^b(C), \cdots, \mathbf{D}^b(\operatorname{Sym}^{g-2} C), \\ &\mathbf{D}^b(\operatorname{Sym}^{g-1} C), \\ &\mathbf{D}^b(\operatorname{Sym}^{g-2} C), \cdots, \mathbf{D}^b(C), \mathbf{D}^b(pt) \rangle. \end{aligned}$$

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This is a theorem of Tevelev and Torres that generalizes older work of Bondal-Orlov, Narasimhan, Belmans-Mukhopadhyay et al.

Torelli Theorem

Torelli: Abelian and Non abelian

- If $(Jac(C), \Theta_C)$ is isomorphic to $(Jac(C'), \Theta'_C)$ as polarized abelian varieties, then $C \cong C'$. (Abelian)
- Isomorphism class of the moduli space of semi-stable vector bundles on a smooth projective curve uniquely determines the isomorphism class of the curve and the rank of the vector bundles. (Non abelian)

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Torelli type Theorem: Belmans-Galkin-M

Let (Γ, c) and (Γ', c') be as above with no separating edges, then the toric varieties $X_{\Gamma,c} \cong X_{\Gamma',c'}$ if and only if Γ is isomorphic to Γ' such that the class $[c] \in H^0(\Gamma, \mathbb{F}_2)$ maps to the class $[c']$.

Main Question: Periods of the Newton polynomial

For the Graph potential $W_{\Gamma,c}$ consider the sequence

$$a_m = \text{constant term of}(W_{\Gamma,c})^m$$

Consider the generating series

$$\pi_{W_{\Gamma,c}}(t) = \sum_{m\geq 0} a_m t^m \text{ or } \check{\pi}_{W_{\Gamma,c}}(t) = \sum_{m\geq 0} \frac{a_m}{m!} t^m$$

Question

How to effectively compute a_m for all $m \in \mathbb{Z}_{\geq 0}$ or equivalently find a formula for the generating series?

Periods

Definition

Let $W : (\mathbb{C}^\times)^n \to \mathbb{C}$ be a Laurent polynomial. A classical period of W is the following Laurent series.

$$\pi_W(t) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{|x_1|=\cdots=|x_n|=1} \frac{1}{1 - tW(x_1, \ldots, x_n)} \operatorname{dlog} \vec{x}$$

The (inverse) Laplace transform

$$\check{\pi}_W(t) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{|x_1|=\cdots=|x_n|=1} e^{tW(x_1,\ldots,x_n)} \operatorname{dlog} \vec{x}$$

Trinion Potential

$$W_+ = xyz + \frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy}$$

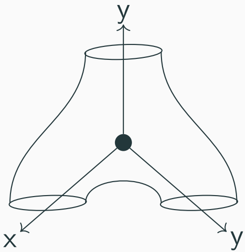


Figure 1: Trinion Graph

Colored Trinion Potential

$$W_- = \frac{1}{xyz} + \frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}$$

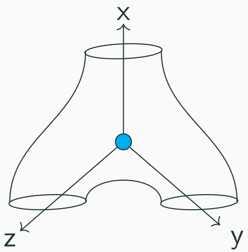


Figure 2: Trinion Graph

Remark

Think of the coloring scheme as taking a variable and replace it by its inverse. Changing it even number of times doesn't change color.

Graph potentials

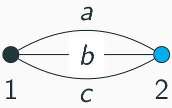
Let (Γ, c) be a colored trivalent graph (loops are allowed) and

$$c : V(\Gamma) \to \{\pm 1\}$$

Definition of graph potential

$$W_{\Gamma,c} := \sum_{v \in V(\Gamma)} W_{v,c(v)}$$

Example $g = 2$

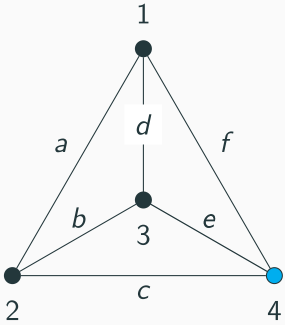


$$\left(abc + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right) + \left(\frac{1}{abc} + \frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c}\right)$$

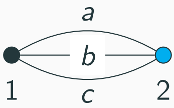
$$b^2 a + \frac{2}{a} + \frac{a}{b^2} + \left(\frac{1}{ac^2} + 2a + \frac{c^2}{a}\right)$$

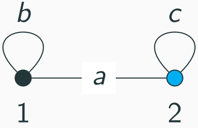
Example $g = 3$

$$adf + \frac{f}{ad} + \frac{a}{df} + \frac{d}{af} +$$
$$bde + \frac{e}{bd} + \frac{b}{ed} + \frac{b}{de}$$
$$+abc + \frac{b}{ac} + \frac{c}{ab} + \frac{a}{bc}$$
$$+\left(\frac{1}{cef} + \frac{ef}{c} + \frac{cf}{e} + \frac{ce}{f}\right)$$



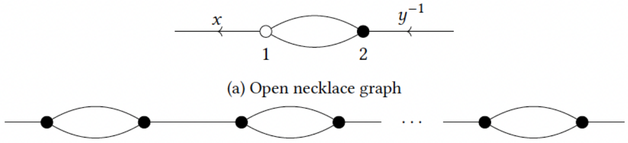
Example g=2





$$\check{\pi}_{W_{\Gamma,c}}(t) = \sum_{n \geq 0} \frac{(2n!)^2}{n!^2} t^{2n}.$$

Periods via the necklace graph



(a) Open necklace graph

(b) Open necklace graph with many beads

$$B(x) := \sum_{m=0}^{\infty} \frac{x^{2m}}{(m!)^2}, \text{ Bessel function,}$$

$$T_1(x,y) := B(t(x+y)B(t(x^{-1}+y^{-1})),$$

Formula for generating functions

Theorem: Belmans-Galkin-M

Let A be the Hilbert-Schmidt operator on $L^2(S^1)$ given by

$$T_1(x, y) = B(t(x + y)B(t(x^{-1} + y^{-1}))$$

and S be the linear operator that sends $x^n \to x^{-n}$. For any trivalent colored graph Γ, c with no half edges and genus $g \geq 2$, the inverse Laplace transform

$$\check{\pi}_{W_{\Gamma,c}}(t) = tr(A^{g-1} S^{\epsilon+g})$$

where ϵ is the parity of the number of colored vertices.

Computational efficiency

https://github.com/pbelmans/graph-potential-tqft

```
sage: %time load("tqft.sage")
CPU times: user 5.06 s, sys: 6.12 ms, total: 5.07 s
Wall time: 5.07 s
sage: %time regularized_period(3, 1)
CPU times: user 2.48 s, sys: 5.25 ms, total: 2.48 s
Wall time: 2.48 s
1 + 384*t^2 + 23040*t^3 + 3265920*t^4 + 435456000*t^5 + 68263641600*t^6 + 11300889600000*t^7 +
```

```
sage: %time load("naive.sage")
Defining a, b, c, d, e, f
1 + 384*t^2 + 23040*t^3 + 3265920*t^4 + 435456000*t^5 + O(t^6)
CPU times: user 1min 11s, sys: 197 ms, total: 1min 11s
Wall time: 1min 11s
```

Comparison

g	p_0	p_2	p_4	p_6	p_8	p_{10}	p_{12}	p_{14}	p_{16}
2	1	8	216	8000	343000	16003008	788889024	40424237568	2131746903000
3	1	0	384	23040	3265920	435456000	68263641600	11300889600000	1984905402480000
4	1	0	576	11520	8769600	1175731200	445839609600	115772770713600	41211916193448000
5	1	0	768	0	16853760	928972800	1378578432000	295708763750400	237075779068128000
6	1	0	960	0	27518400	232243200	3112327680000	299893321728000	795162277629720000
7	1	0	1152	0	40763520	0	5892216422400	133905855283200	2006716647119184000
8	1	0	1344	0	56589120	0	9963493478400	22317642547200	4248683870158728000
9	1	0	1536	0	74995200	0	15571407667200	0	7983708676751808000
10	1	0	1728	0	95981760	0	22961207808000	0	13760135544283128000

Table 1: Period sequence for the odd graph potential

g	p_0	p_2	p_4	p_6	p_8	p_{10}	p_{12}	p_{14}	p_{16}	p_{18}
2	1	0	384	0	645120	0	1513881600	0	4132896768000	0
3	1	0	576	0	6350400	0	136604160000	0	3976941969000000	0
4	1	0	576	0	12640320	0	805929062400	0	80306439693480000	0
5	1	0	768	0	18144000	0	1915060224000	0	401643111149280000	0
6	1	0	960	0	27518400	0	3418888704000	0	1062973988196120000	0
7	1	0	1152	0	40763520	0	5953528627200	0	2211592605702480000	0
8	1	0	1344	0	56589120	0	9963493478400	0	4323671149117320000	0
9	1	0	1536	0	74995200	0	15571407667200	0	7994421145174464000	0
10	1	0	1728	0	95981760	0	22961207808000	0	13760135544283128000	0

Table 2: Period sequence for the even graph potential

Some combinatorial observations

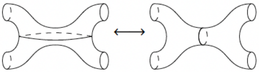
Theorem: Belmans-Galkin-M

- Any odd indexed coefficient i.e. $[W^{2k+1}_{\Gamma,c}]_0$ vanishes.
- Let $g \geq 2$ and $k < 2g - 2$. Then $[W^{k}_{\Gamma,c}]_0$ is independent of the coloring. i.e. these numbers associated to $M^{\epsilon}_{C}(2, L)$ does not depend on ϵ.
- If $g \geq 2$, and $k \geq 0$, if $4k + 2 < 2g - 2$, then $4k + 2$-th period of the graph potential in the odd case vanishes. In the even case $4k + 2$-th period is always zero.

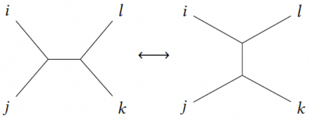
Question

What is a geometric interpretation of the above?

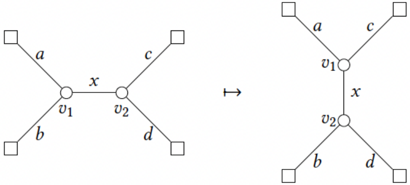
Graph mutations and cluster charts



(a) Elementary transformation on a surface



(b) Elementary transformation on the dual graph



cont..

$$
\begin{aligned}
W_\gamma^{\text{mut}} &= xcd + \frac{x}{cd} + \frac{c}{dx} + \frac{d}{cx} + abx + \frac{a}{bx} + \frac{x}{ab} + \frac{b}{ax} \\
&= \frac{1}{x}\left(\frac{a}{b} + \frac{b}{a} + \frac{c}{d} + \frac{d}{c}\right) + x\left(cd + \frac{1}{cd} + \frac{1}{ab} + ab\right).
\end{aligned}
$$

Denote

$$
\begin{aligned}
\mu &:= \frac{1}{abcd}(ad+bc)(ac+bd) \\
\nu &:= \frac{1}{abcd}(1+abcd)(cd+ab)
\end{aligned}
$$

Then we get

$$
W_\gamma^{\text{mut}} = \frac{\mu}{x} + \nu x
$$

cont..

$$\begin{aligned} W_{\gamma'}^{\text{mut}} &= x'bd + \frac{x'}{bd} + \frac{b}{dx'} + \frac{d}{bx'} + acx' + \frac{a}{cx'} + \frac{c}{ax'} + \frac{x'}{ac} \\ &= \frac{1}{x'}\left(\frac{b}{d} + \frac{d}{b} + \frac{a}{c} + \frac{c}{a}\right) + x\left(bd + \frac{1}{bd} + ac + \frac{1}{ac}\right). \end{aligned}$$

Denote

$$\begin{aligned} \mu' &:= \frac{1}{abcd}(ab + cd)(ad + bc) \\ \nu' &:= \frac{1}{abcd}(1 + abcd)(ac + bd) \end{aligned}$$

Then we get

$$W_{\gamma'}^{\text{mut}} = \frac{\mu'}{x'} + \nu' x'$$

Key Observation: $\mu'\nu' = \mu\nu$

Cluster like mutations

Theorem: Belmans-Galkin-M

$W^{\text{mut}}_{\gamma'}$ and W^{mut}_{γ} are related by a rational change of coordinates i.e.

$$\frac{\mu'}{\mathbf{x}'} + \nu'\mathbf{x}' = \frac{\mu}{\mathbf{x}} + \nu\mathbf{x},$$

where $z = \nu x$ and and then $z = \frac{\mu'}{x'}$. Using the relation $\mu\nu = \mu'\nu'$, we get the equality.

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Corollary

We have a function $W : \bigcup_{\Gamma,c}(T_{\Gamma,c}^{\vee}) \longrightarrow \mathbb{C}$, where the two tori in the union are identified by a rational change of coordinates as above.

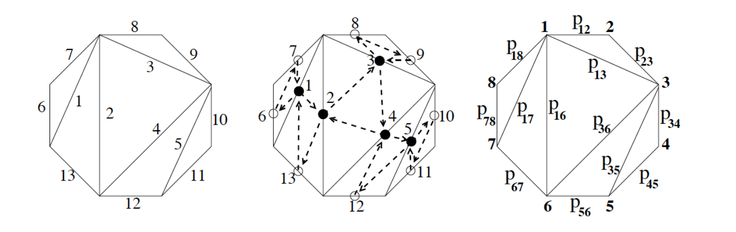
This is “like” the upper cluster variety and is an open part of the mirror of M_C in the sense of Homological Mirror Symmetry.

Cluster Structures on Gr$(2, n)$

The homogenous coordinate ring $\mathbb{C}[C(\mathrm{Gr}(2,n))]$ is generated by the Plücker coordinates p_{ij} with relations:

$$p_{ik}p_{jl} = p_{ij}p_{kl} + p_{jk}p_{il}, \text{for } 1 \leq i < j < k < l \leq n$$

The Plücker coordinates are in bijection with the set sides and diagonal of a polygon with n-sides.



Triangulations of a hexagon

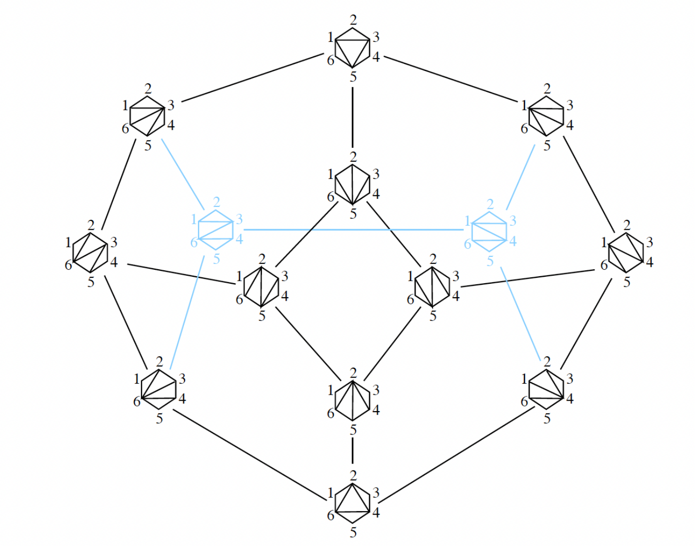
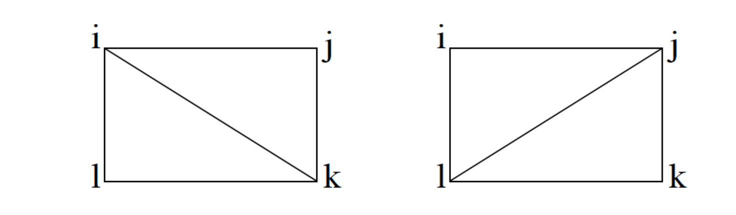


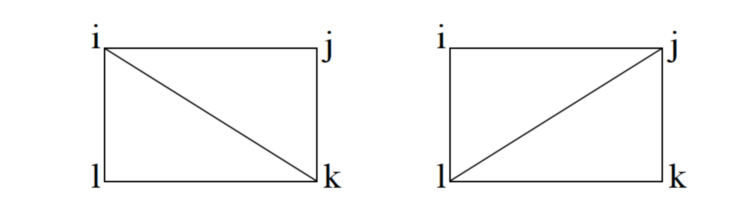
Figure 3: Picture:Laurent Williams

Plücker as cluster mutation



$$p_{jl} = \frac{p_{ij} p_{kl} + p_{jk} p_{il}}{p_{ik}}$$

Plücker as cluster mutation



$$p_{jl} = \frac{p_{ij}p_{kl} + p_{jk}p_{il}}{p_{ik}}$$

If we assign variables p_{ij}, to the n sides of a n-gon and to the diagonals. Let T_1 and T_2 be triangulation that differs at one diagonal say p_{ik} gets replaced by p_{jl}. Then the above is just the exchange relation/seed mutation of a cluster structure

Polygon spaces

Let $\mathbf{r} = (r_1, \ldots, r_n)$ be a sequence of integers satisfying $r_i < r_1 + \cdots + r_{i-1} + r_{i+1} + \cdots + r_n$. Then the polygon space $\mathcal{M}_{\mathbf{r}}$.

$$\{(x_1 \ldots, x_n) \in \prod_{i=1}^{n} S^2(r_i) | x_1 + \cdots + x_n = 0\}/SO(3)$$

It is isomorphic to $(\mathbb{CP}^1)^n//_{\mathbf{r}}PGL(2)$.

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It is isomorphic to $(\mathbb{CP}^1)^n//_{\mathbf{r}}PGL(2)$.

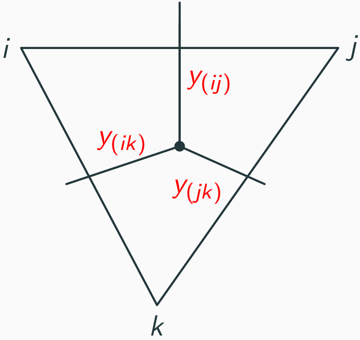
The set of triangulations of a convex polygon P is in natural one-to-one correspondence with the set of trivalent trees with n leaves by sending a triangulation to its dual graph Γ.

Theorem: Nishinou-Nohara

For any triangulation Γ of the reference polygon, there exists a toric degeneration of $\mathcal{M}_{\mathbf{r}}$.

Nohara-Ueda potential

Nohara–Ueda construct a potential for the Euclidean polygon spaces with n sides by exploiting their relation $\mathrm{Gr}(2,n)$. Their key idea is to construct a mirror to $\mathrm{Gr}(2,n)$, equivariantly with respect to $\mathrm{U}(1)^n$, to obtain mirrors to the Euclidean polygon spaces.

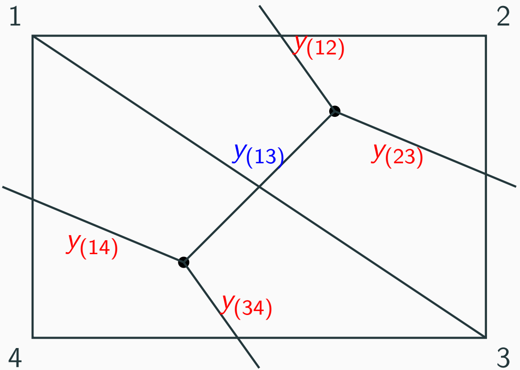


$$\frac{y_{(ij)}\,y_{(jk)}}{y_{(ik)}}+\frac{y_{(jk)}\,y_{(ik)}}{y_{(ij)}}+\frac{y_{(ik)}\,y_{(ij)}}{y_{(jk)}}$$

The trinion potential is

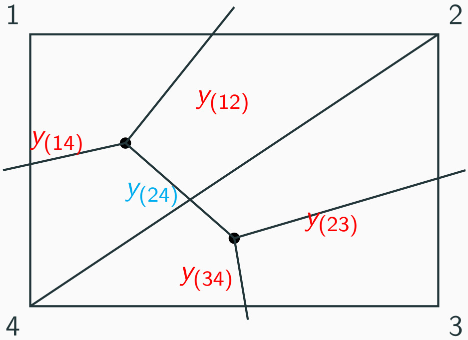
$$\frac{y_{(ik)}}{y_{(ij)}y_{(jk)}}+\frac{y_{(ij)}}{y_{(jk)}y_{(ik)}}+\frac{y_{(jk)}}{y_{(ik)}y_{(ij)}}$$
$$+y_{(ij)}y_{(jk)}y_{(ik)}$$

Nohara Ueda potential for $n = 4$ polygon space and $Gr(2,4)$



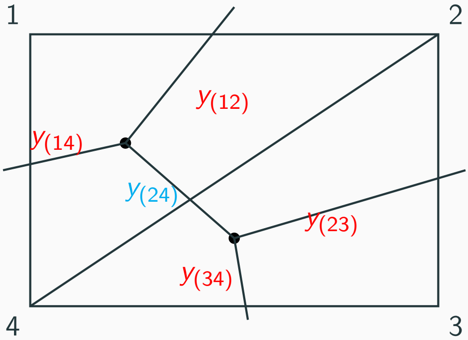
$$y_{(13)}\left(\frac{y_{(14)}}{y_{(34)}}+\frac{y_{(34)}}{y_{(14)}}+\frac{y_{(12)}}{y_{(23)}}+\frac{y_{(23)}}{y_{(12)}}\right)+\frac{1}{y_{(13)}}\left(y_{(14)}y_{(34)}+y_{(12)}y_{(23)}\right)$$

The other triangulation and the rational transformation



$$y_{(24)}\left(\frac{y_{(12)}}{y_{(14)}}+\frac{y_{(14)}}{y_{(12)}}+\frac{y_{(23)}}{y_{(34)}}+\frac{y_{(34)}}{y_{(23)}}\right)+\frac{1}{y_{(24)}}\left(y_{(23)}y_{(34)}+y_{(14)}y_{(12)}\right).$$

The other triangulation and the rational transformation



$$y_{(24)}\left(\frac{y_{(12)}}{y_{(14)}}+\frac{y_{(14)}}{y_{(12)}}+\frac{y_{(23)}}{y_{(34)}}+\frac{y_{(34)}}{y_{(23)}}\right)+\frac{1}{y_{(24)}}\left(y_{(23)}y_{(34)}+y_{(14)}y_{(12)}\right).$$

$$y_{(13)}y_{(24)}=\frac{y_{(12)}y_{(23)}y_{(34)}y_{(14)}}{\left(y_{(12)}y_{(34)}+y_{(23)}y_{(14)}\right)}$$

Recovering Noda-Ueda from Graph potentials

$$(x, y, z) \to \left(\frac{\tau}{X}, \frac{Y}{\tau}, \frac{Z}{\tau}\right).$$

$$\begin{aligned} W(x, y, z) &= xyz + \frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} \\ &= \tau^{-1}\left(\frac{YZ}{X} + \tau^4 \frac{1}{XYZ} + \frac{XY}{Z} + \frac{ZX}{Y}\right). \end{aligned}$$

$$\lim_{\tau \to 0} \tau W(x, y, z) = \frac{YZ}{X} + \frac{XY}{Z} + \frac{ZX}{Y}.$$

cont...

$$\tau \widetilde{W}_{\gamma_4,0} = y_{(13)} \left(\frac{y_{(14)}}{y_{(34)}} + \frac{y_{(34)}}{y_{(14)}} + \frac{y_{(12)}}{y_{(23)}} + \frac{y_{(23)}}{y_{(12)}} \right) + \frac{1}{y_{(13)}} \left(y_{(14)} y_{(34)} + y_{(12)} y_{(23)} + \frac{\tau^4}{y_{(14)} y_{(34)}} + \frac{\tau^4}{y_{(12)} y_{(23)}} \right),$$

$$\tau \widetilde{W}_{\gamma_4',0} = y_{(24)} \left(\frac{y_{(12)}}{y_{(14)}} + \frac{y_{(14)}}{y_{(12)}} + \frac{y_{(23)}}{y_{(34)}} + \frac{y_{(34)}}{y_{(23)}} \right) + \frac{1}{y_{(24)}} \left(y_{(23)} y_{(34)} + y_{(14)} y_{(12)} + \frac{\tau^4}{y_{(23)} y_{(34)}} + \frac{\tau^4}{y_{(14)} y_{(12)}} \right).$$

The rational change of coordinates

$$y_{(13)} = \frac{1}{y_{(24)}} \frac{y_{(12)} y_{(23)} y_{(34)} y_{(14)} + \tau^4}{\left(y_{(12)} y_{(34)} + y_{(23)} y_{(14)} \right)}.$$

Taking limit $\tau \to 0$ recovers the Nohara-Ueda relations.

Plücker Relations

Nohara-Ueda considers the following change of coordinates:

$$y_{(12)} = y_{12}^{\frac{1}{2}},\ y_{(13)} = \frac{y_{13}}{y_{12}^{\frac{1}{2}} y_{23}^{\frac{1}{2}}},\ y_{(23)} = y_{23}^{\frac{1}{2}},$$

$$y_{(34)} = y_{34}^{\frac{1}{2}},\ y_{(24)} = \frac{y_{24}}{y_{23}^{\frac{1}{2}} y_{34}^{\frac{1}{2}}}, y_{(14)} = \frac{y_{14}}{y_{12}^{\frac{1}{2}} y_{23}^{\frac{1}{2}} y_{34}^{\frac{1}{2}}}.$$

The Nohara-Ueda mutation in $y_{(ij)}$ takes the form:

$$\mathbf{\frac{1}{y_{13} y_{24}} = \frac{1}{y_{23} y_{14}} + \frac{1}{y_{12} y_{23} y_{24}}}.$$

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$$y_{(34)} = y_{34}^{\frac{1}{2}},\ y_{(24)} = \frac{y_{24}}{y_{23}^{\frac{1}{2}} y_{34}^{\frac{1}{2}}}, y_{(14)} = \frac{y_{14}}{y_{12}^{\frac{1}{2}} y_{23}^{\frac{1}{2}} y_{34}^{\frac{1}{2}}}.$$

The Nohara-Ueda mutation in $y_{(ij)}$ takes the form:

$$\frac{\mathbf{1}}{\mathbf{y_{13}y_{24}}} = \frac{\mathbf{1}}{\mathbf{y_{23}y_{14}}} + \frac{\mathbf{1}}{\mathbf{y_{12}y_{23}y_{24}}}.$$

Further changing coordinates $\{p_{ij}\}$ with the following:

$$y_{12} = \frac{p_{23}}{p_{12}},\ y_{13} = \frac{p_{34}}{p_{13}},\ y_{23} = \frac{p_{34}}{p_{23}},\ y_{24} = y_{14}\frac{p_{14}}{p_{24}},\ y_{34} = y_{14}\frac{p_{14}}{p_{34}}.$$

With these new coordinates, the above relation transforms to the Plücker relation

$$\mathbf{p_{13}p_{24} = p_{12}p_{34} + p_{23}p_{14}}.$$

Deformed Plücker relations

Our change or variables with τ as the parameter gives

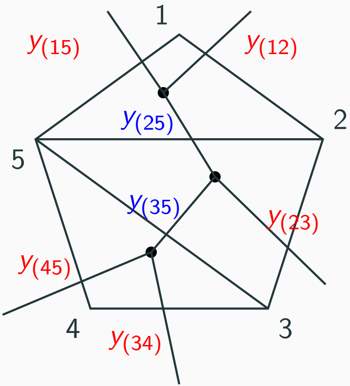
$$(y_{12}y_{34} + y_{14})y_{13}y_{24} = (y_{14} + \tau^4)y_{12}y_{23}y_{34}.$$

Further changing $\tau \to y_{14}^{\frac{1}{4}}\tau$, the above transforms to

$$\mathbf{p_{23}p_{14} + p_{12}p_{34} = (1 + \tau^4)p_{13}p_{24}}.$$

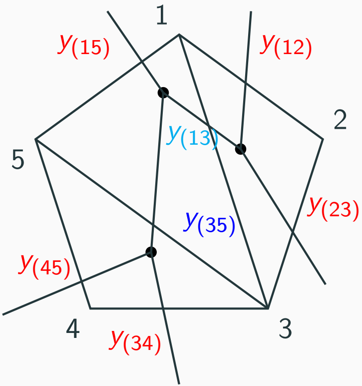
This is just a degeneration of the Plücker relations appearing in $\mathrm{Gr}(2,4)$.

The case $n = 5$



$$y_{(25)}\left(\frac{y_{(15)}}{y_{(12)}}+\frac{y_{(12)}}{y_{(15)}}+\frac{y_{(23)}}{y_{(35)}}+\frac{y_{(35)}}{y_{(23)}}\right)+\frac{1}{y_{(25)}}(y_{(12)}y_{(15)}+y_{(23)}y_{(35)})$$
$$+\,y_{(35)}\left(\frac{y_{(45)}}{y_{(34)}}+\frac{y_{(34)}}{y_{(45)}}\right)+\frac{1}{y_{(35)}}\left(y_{(34)}y_{(45)}\right).$$

The case $n = 5$



$$y_{(13)}\left(\frac{y_{(23)}}{y_{(12)}}+\frac{y_{(12)}}{y_{(23)}}+\frac{y_{(15)}}{y_{(35)}}+\frac{y_{(35)}}{y_{(15)}}\right)+\frac{1}{y_{(13)}}(y_{(12)}y_{(23)}+y_{(15)}y_{(35)})$$
$$+\,y_{(35)}\left(\frac{y_{(45)}}{y_{(34)}}+\frac{y_{(34)}}{y_{(45)}}\right)+\frac{1}{y_{(35)}}\left(y_{(34)}y_{(45)}\right).$$

cont..

Nohara-Ueda uses the following rational change of coordinates that transform the first to the second.

$$y_{(13)}y_{(25)} = \frac{y_{(12)}y_{(23)}y_{(35)}y_{(15)}}{y_{(12)}y_{(35)} + y_{(23)}y_{(15)}}.$$

They are of the form that equates sum of two monomials to a third monomial. In the present paper, the rational change of coordinates for the graph potentials and their deformations are of the following type:

$$y_{(13)}y_{(25)} = \frac{y_{(12)}y_{(23)}y_{(35)}y_{(15)} + \tau^4}{y_{(12)}y_{(35)} + y_{(23)}y_{(15)}}.$$

They equate sum of two monomials to sums of two other monomials.

change of coordinates....

$$y_{(ij)} = \begin{cases} y_{i,i+1}^{1/2}, \; j = i+1 < 6 \\ y_{15} \prod_{k=1}^{4} y_{k,k+1}^{-\frac{1}{2}}, \; (i,j) = (1,5) \\ y_{ij} \prod_{k=i}^{j-1} y_{k,k+1}^{-\frac{1}{2}}, \; |i-j| > 1. \end{cases}$$

Then the Nohara-Ueda mutation takes the form

$$\frac{\mathbf{1}}{\mathbf{y_{13} y_{25}}} = \frac{\mathbf{1}}{\mathbf{y_{15} y_{23}}} + \frac{\mathbf{1}}{\mathbf{y_{12} y_{35} y_{23}}}.$$

In the Plücker coordinates p_{ij}, which related to y_{ij} by:

$$y_{i5} = y_{15} \frac{p_{15}}{p_{i5}}, \text{ and } y_{ij} = \frac{p_{j,j+1}}{p_{ij}} \text{ and for } j \neq 5,$$

$$\boldsymbol{p_{13} p_{25} = p_{12} p_{35} + p_{15} p_{23}}.$$

Similarly for the case of the graph potentials, we get the deformed Plücker relations:

$$\boldsymbol{p_{13} p_{25} = (1 + \tau^4) p_{12} p_{35} + p_{15} p_{23}}.$$

Comparison

Theorem: Belmans-Galkin-M

The graph potential mutuations recover the cluster structures on Gr$(2, n)$ as a limit.

Remark

- As we saw in the beginning, the operator $T_{\vec{z}}$ is nilpotent and hence $\ell >> 0$, conformal blocks equals invariants of tensor product of Lie algebras.
- Geometrically if $\ell >> 0$, the semi-stablity criterion for the moduli of parabolic bundles on $\mathbb{P}^1$ is trivial. Hence the moduli space equals the polygon space.
- This recovers (in the case $n = 2$) the result of Marsh-Rietsch on the mirror of Gr(r, n) constructed by cluster charts

Comparison to Kontsevich-Odesskii

- They consider multiplication kernels $\mathcal{K}(a,b,c) \in L^1(S^1)^{\otimes 3}$ and try to define a product

$$f * g(c) = \int_{S^1 \times S^1} f(a)g(b)\mathcal{K}(a,b,c)$$

  and considers associativity constraints.
- They also come up with our examples of graph potentials i.e. $\mathcal{K}(a,b,c) = \exp W_{\pm}(a,b,c)$ but starting from meromorphic Higgs bundles on $\mathbb{P}^1$ with punctures.
- Meromorphic Higgs bundles appear as Coloumb branches in 3d mirror symmetry.

Question

Is it related to Leung etal's result that relates between 3d and 2d mirror symmetry ?