Maths Circle: What is Euclidean Construction? Part II

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As mentioned in the previous exploration sheet, Euclidean construction starts with two points at unit distance. Without loss of generality, we shall assume these two points to be (0, 0) and (1, 0) in the Cartesian coordinate system. Euclidean construction proceeds with the repeated applications of five basic constructions. These are:

1. Creating the line through two existing points.
2. Creating the circle through one point with centre another point.
3. Creating the point which is the intersection of two existing, non-parallel lines.
4. Creating the one or two points in the intersection of a line and a circle (if they intersect).
5. Creating the one or two points in the intersection of two circles (if they intersect).

In other words, we keep on using the points, lines and circles that have already been constructed and get to construct new ones using the five basic constructions (1 – 5) above.

Definition: *We say that a point in the plane is constructible if it is either (0, 0) or (1, 0) or it can be obtained as a new point of intersection in 3 – 5 above after repeating any of 1 – 5 finitely many times in any order.*

Definition: *We say that a real number r is constructible if the point (r, 0) is constructible as defined above.*

In the last MCI session, we discussed the proof of the following results:

- Any integer is constructible.
- If a and b are constructible numbers, then so are a+b, a–b and ab.
- If a and b are constructible numbers and b is nonzero, then a/b is also constructible.

We also gave the following homework assignment.

Exercise: *If $a>0$ is constructible, then show that $\sqrt{a}$ is also constructible.*

Once we solve the above exercise, the following result will be established.

Proposition 1: *A real number a is constructible if a can be written using integers and the mathematical operations +, –, ×, ÷, √ (each used finitely many times in any order).*

Our next goal is to prove the converse of Proposition 1 and thus proving the following theorem.

Theorem 2: *A real number a is constructible if and only if a can be written using integers and the mathematical operations +, –, ×, ÷, √ (each used finitely many times in any order).*

The proof of the converse of Proposition 1 will need some machineries and we shall develope them in the upcoming MCI sessions. Finally, we shall use Theorem 2 to verify that 60 degrees cannot be trisected using Euclidean constructions. This will be a very lucid introduction to a beautiful and elegant mathematical theory, initiated by Évariste Galois in his rather short span of life (25 October 1811 – 31 May 1832), now known as Galois theory.