ICTS-RRI Math Circle, Saturday 25th February 2023

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February 21, 2023

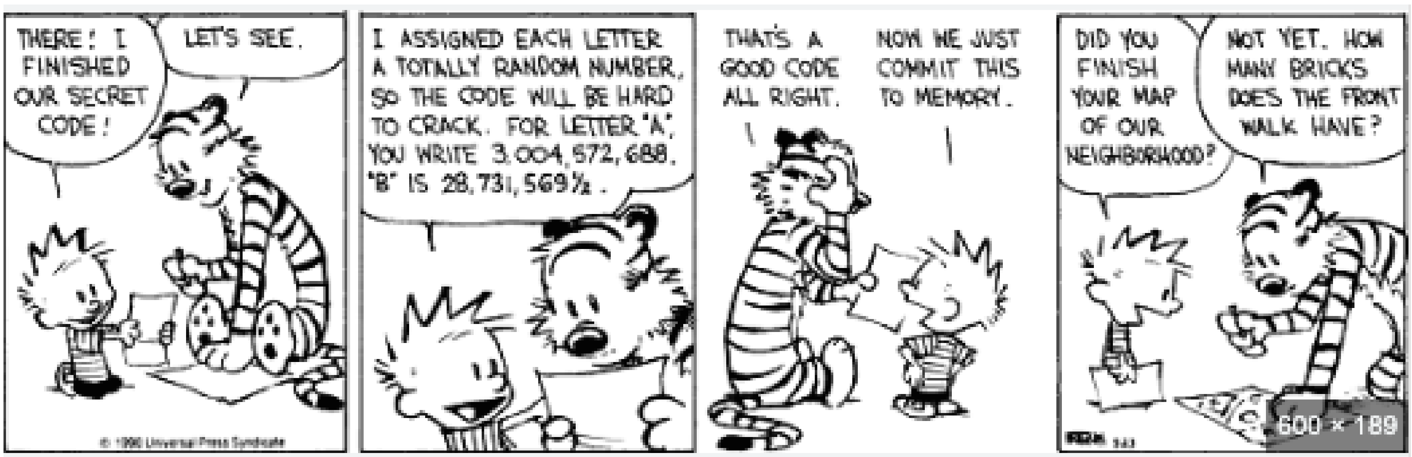

Figure 1:

We are spending a few sessions on cryptography: the art, science and math of secret writing. We were also learning to multiply days of the week! Your parents may have wondered why on Earth you would want to do that. What is such a skill used for? Perhaps you can convince them.

Before you come to the session on Feb 25, try this:
Give your parents some homework: ask them to multiply five sets of two four digit numbers. (Eg 1. 7123 times 6542 2. 1976 times 7676 3. 1184 times 5093 4. 3392 times 4586 5. 9971 times 4425). Time them. Tell them you can check their answers in a fraction of the time they needed to perform the multiplication.

Here's how you do it. Given any number say 148768, you can compute its 'ghost' by adding up all its digits. If the answer has more than one digit repeat till you get a single digit. For example the sum of the digits of 148768 is 34 and the sum of *its* digits is 7. So 7 is the ghost of 148768. You can check a multiplication by checking that the ghosts agree. For example 7123 times 6542= 46598666. Reducing the matter to ghosts, we need only check that ghost(7123) times ghost(6542)= ghost(46598666). Since ghosts are single digit numbers, we can do this very simply: ghost(7123)=4 ghost(6542)=8 and ghost(46598666)=5. And 4 times 8 is 32 whose ghost is 5. Note that multiplication of ghosts include reducing the number to single digits!

Question for you: why does the 'ghost' trick work?

At the end of this note you will find a write-up on how we learn to solve problems. This is a condensation of some ideas due to the mathematician George Polya. Read through this as a preparation for the session on Saturday. Also please bring along some loose sheets of paper, pencils or pens and a ruler.

First Activity: We will have a few puzzles related to modular arithmetic. Why does the ghost trick work? Does it always detect errors? Can we generalise the trick?

Then a few examples of 'discovery' in mathematics. One usually starts by noticing simple examples and finding a pattern. You will be given a set of facts and asked to find a pattern in them. Can you then try to see a bigger picture?

Tea break: 11:15

Second Activity Code breaking (rest of the session):

This game will also be about code making and breaking. Last time we dealt with Caesar shifts. This time we will explore slightly more advanced codes. You will be made into teams A and B. This time we will have vertical grouping so each team has representation across ages.

How to Solve it? by George Polya

This writeup explains in simple language Polya's method for solving problems.

All of us are faced with problems in everyday life and we use a number of tricks to solve them. For instance, you come home from school and find the front door locked and nobody home. What do you do? Think hard. Is there a friendly neighbour? Could there be a window open? Would they have left the key in a hiding place? Can I reach them on the phone? As we get older we get better and better at solving life's problems. We learn from past experience and develop a bag of tricks.

The same is true for problems in mathematics or physics. Many of us use these tricks without being aware that we are using them. For those of you who are learning to solve problems, it is useful to make these rules explicit. This is what George Polya, a Hungarian Mathematician did in a book called "How to solve it?". What follows is a simplified and condensed version of Polya's ideas.

The process is divided into four steps: Understand, Plan, Solve and Review. When you start, you will slowly and consciously go through these steps. As you get more experienced, you will use this method without even thinking about it.

1. *Understand the Problem:* It goes without saying that in order to solve a problem we have to understand it. Here are questions that you ask yourself in order to do this:
   - What form will the answer take? Is it a number, a length? Or is it a logical argument? A proof? An algorithm? A strategy?
   - What is the data that is given to me? Scan the statement of the problem to isolate the data.
   - What are the conditions of the problem? For instance, the answer may have to be a whole number. Or, in a logical proof, the statement may apply only to polygons.
2. *Planning an attack on the Problem:* Ask yourself these questions:
   - Have I seen a related problem before? A problem with a similar unknown?
   - How is the data related to the unknown? Is there too little data? Too much data?
   - Can I simplify the problem by considering a special case? an extreme case?
   - Can I simplify the problem by making it more general?
   - Can I give up a condition? Eg. give up the condition that the answer has to be a whole number and solve it with real numbers.
   - In some problems (a maze for example), it is advantageous to work backwards: start from the end, assuming we have reached our goal. Would this work in our problem?
   - Are there obvious symmetries in the problem?
3. *Solve* In this step, we implement the plan devised in the last section. This may involve calculation or developing the logical steps of a proof. It may be that the first attempt does not succeed. If so we go back to the planning stage and refine it.
4. *Review* This step is important for you to develop problem solving skills for the future. Don't regard a solved problem as dead. You can learn a lot from problems you have already solved and use this knowledge in the future. Ask yourself
   - is there a way to check the solution?
   - Can the solution be generalised?
   - Can I use the solution to devise new problems?

   The next time you are faced with a problem, (even a simple one) try to go through these steps and learn from them. With practice you will find your skills improving. Good luck!