Maths Circle: What is Euclidean Construction?

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One sentence answer to this question is ***"Euclidean Construction is whatever construction we learn in school."***

Two more questions:

1) What do we mean by the above sentence with quotes?
2) Why should we care?

Let's first try to answer Question 2) above.

- I personally care about them for aesthetic reasons. Euclidean geometry is the main reason why I loved mathematics in middle school. Constructions played a very important role in this attraction towards my most favourite subject.

- There is another reason why Euclidean constructions were important. *Compass* and *ruler* (i.e., a scale without any mark for measuring length) were way easier to design historically than the protractor (the "D" shaped object in our geometry box). Even in those days, constructing 90 degrees was important in architecture and more generally, science. Who other than Euclid could have helped?

Now let's try to answer Question 1) above.

According to Wikipedia, Euclidean Construction, "also known as ruler-and-compass construction or classical construction, is the construction of lengths, angles, and other geometric figures using only an idealized ruler and compass".

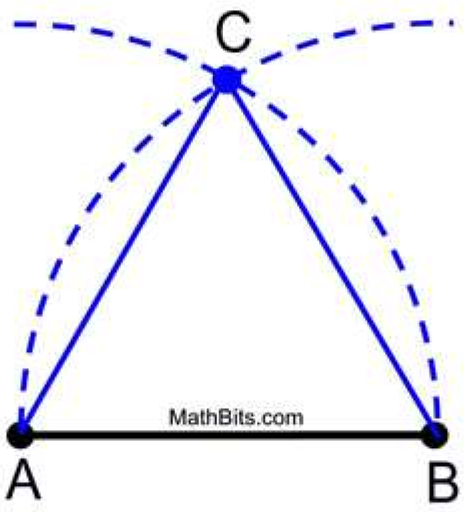
An idealized ruler is a ruler ("scale" in our geometry box) except that someone has erased all the marks on it. In particular, you cannot use it to measure length or draw a line segment of a specified length.

Euclidean constructions consist of repeated application of five basic constructions using the points, lines and circles that have already been constructed. These are:

- Creating the line through two existing points.
- Creating the circle through one point with centre another point.
- Creating the point which is the intersection of two existing, non-parallel lines.
- Creating the one or two points in the intersection of a line and a circle (if they intersect).
- Creating the one or two points in the intersection of two circles (if they intersect).

I forgot to mention one more thing. We actually start with two given points (that can be joined to form a line segment, whose length, will be assumed to be unity). In other words, these two points give us a unit of length in our mathematics copy, where Euclid resurrects and repeats his tricks.

Just to give an example, starting with just two distinct points (A and B in the picture below), we can create a line AB and two circles (in turn, using each point as centre and passing through the other point). If we draw both circles, one new point (C in the picture) is created at their intersection. Drawing the lines AC and BC completes the construction of an equilateral triangle ABC. (source: https://en.wikipedia.org/wiki/Straightedge_and_compass_construction).



(Picture is borrowed from MathBits.com)

Other most-used Euclidean constructions include:

- Constructing the perpendicular bisector from a segment.
- Finding the midpoint of a segment.
- Drawing a perpendicular line from a point to a line.
- Bisecting an angle.
- Mirroring a point in a line.
- Constructing a line through a point tangent to a circle.
- Constructing a circle through 3 non-collinear points.
- Drawing a line through a given point parallel to a given line.

Anything that's bound by rules and regulations is expected to have some limitations. *Are there such in Euclidean construction?* The answer is YES. For example, ***not all angles can be trisected by Euclidean means***.

What is an example of such an angle (that cannot be trisected)?

Before answering this question, let us first think of an example of an angle that can actually be trisected. Such an angle in 90 degrees. This is because we can construct 60 degrees (recall the equilateral triangle) and 30 degrees (by bisecting 60 degrees). And these two do the job!

Giving an example of an angle that cannot be trisected using Euclidean means is very easy – 60 degrees. This means, in particular, that neither 20 degrees nor 40 degrees can be drawn using Euclid's help.

The bigger question is, ***how to prove the above statements?***

Even though this is a question from geometry, the answer lies in algebra – more precisely a beautiful and elegant theory, initiated by Évariste Galois in his rather short span of life (25 October 1811 – 31 May 1832), now known as Galois theory. Roughly speaking, 20 degrees cannot be constructed because cosine of 20 degrees has a "cubic minimal polynomial". However, the detailed explanation needs deeper venture into the mathematics which will be done in a series of Maths Circles.