Divisibility

Sachchidanand

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Suppose that we want to check whether the number 327841989 is divisible by $2, 3, 5, 7, 11, \ldots$. We can use elementary methods to check the divisibility. What about, if asked is this number is divisible by $41, 43$ or 79. Do we have a divisibility test for these numbers or do we have to divide and check?

We recall the divisibility test for the number 7. A positive integer n is divisible by 7 if and only if the difference between twice the unit digit of n and the remaining digits of n, excluding the unit digit, is divisible by 7. For example, consider the number $n = 9874$. Applying the divisibility test for 7, we have:

$$987 - 2 \times 4 = 979.$$

Now the above number is divisible by 7 if and only if 9874 is divisible by 7 (why?). So we use the test again

$$97 - 18 = 69,$$

which is not divisible by 7 and hence 979 is not divisible by 7 and therefore, 9784 will also be not divisible by 7. Formally, if we write a number $N = 10a + b$, where b is the unit digit of N, then we have

$$(10a + b) - 3(a - 2b) = 7(a + b) = 0 \bmod 7.$$

Therefore,

$$\begin{aligned} 7 \mid (10 + b) &\text{ if and only if } 7 \mid 3(a - 2b) \\ &\text{ if and only if } 7 \mid (a - 2b). \end{aligned}$$

Note that we can also write

$$(10 + b) - 3(a - 9b) = 7(a + 4b) = 0 \bmod 7.$$

So $7 \mid (10a+b)$ if and only if $7 \mid (a-9b)$. Thus, this could also be a divisibility test for 7.

First Activity: Take some numbers and try to see if the number is divisible by $11, 13, 17$ and 19. Try to find a similar divisibility test for these numbers. Also, try to see if the same divisibility test (as for 7) works for the numbers $17, 37$ and 47.

Finally, we will try to find a general pattern for the divisibility test for any prime number.