INSTABILITY @DRIVEN BOSE-EINSTEIN CONDENSATE (BEC)

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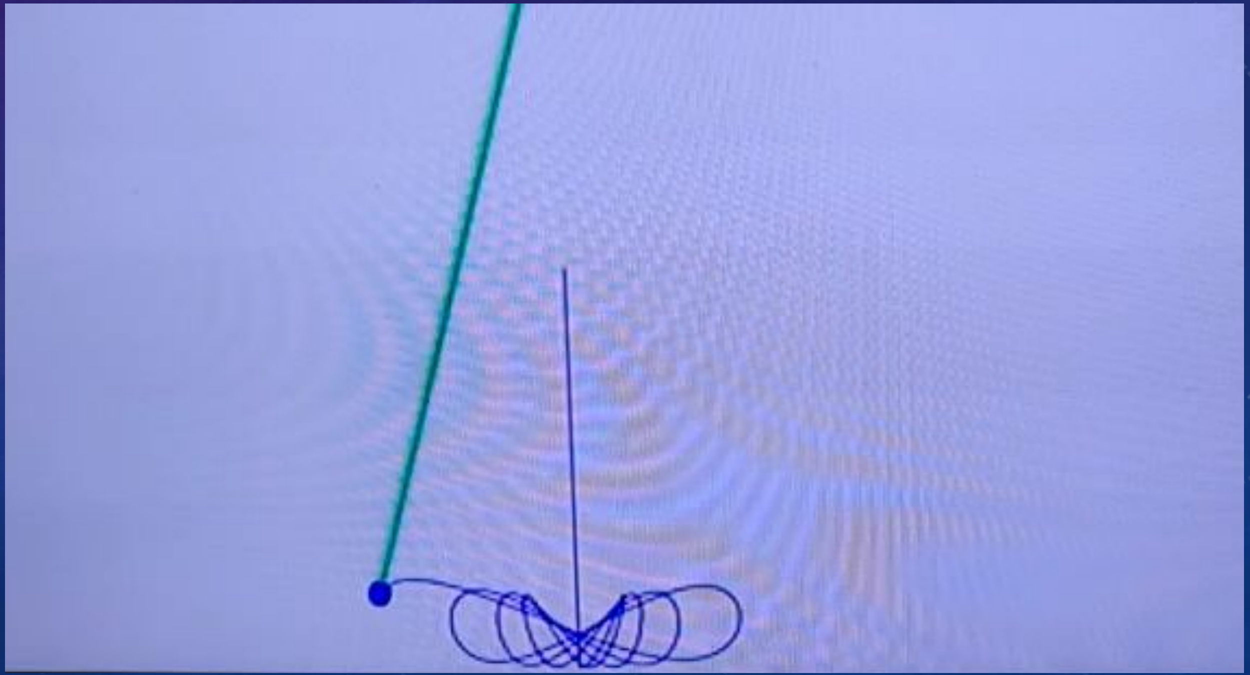
Anirban
Shainee
Arpana
Hari
Topologic -al matter in Opt. Lattice
Ultra-cold Spin system
Instability in Quantum Superfluid
Quan. Algorithm
Group Activities
Subrata
Harshdeep
Foundation in Quantum Mechanics
Supersolid & Droplets
Astrophysics & Neutrino Obs
Soumya
Rohit
Tanima
Swapan

Parametric

- Initial driven oscillation(vertical)
  - ➢ CG is oscillating (g oscillates)
  - ➢ Parameter oscillates
- Small transverse/horizontal motion
- $K.E. = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}mz^2\dot{\theta}^2$
- $P.E. = mgz(1 - \cos\theta)$
- Euler-Lagrange Equation
- $\ddot{\theta} = -\alpha\dot{\theta} - 2\left(\frac{\dot{z}}{z}\right)\dot{\theta} - \left(\frac{g}{z}\right)\sin\theta$



Alexander C: https://www.youtube.com/watch?v=TINqRDFLzV8

C Chambers: https://www.youtube.com/watch?v=Hi_4SsbwaeE

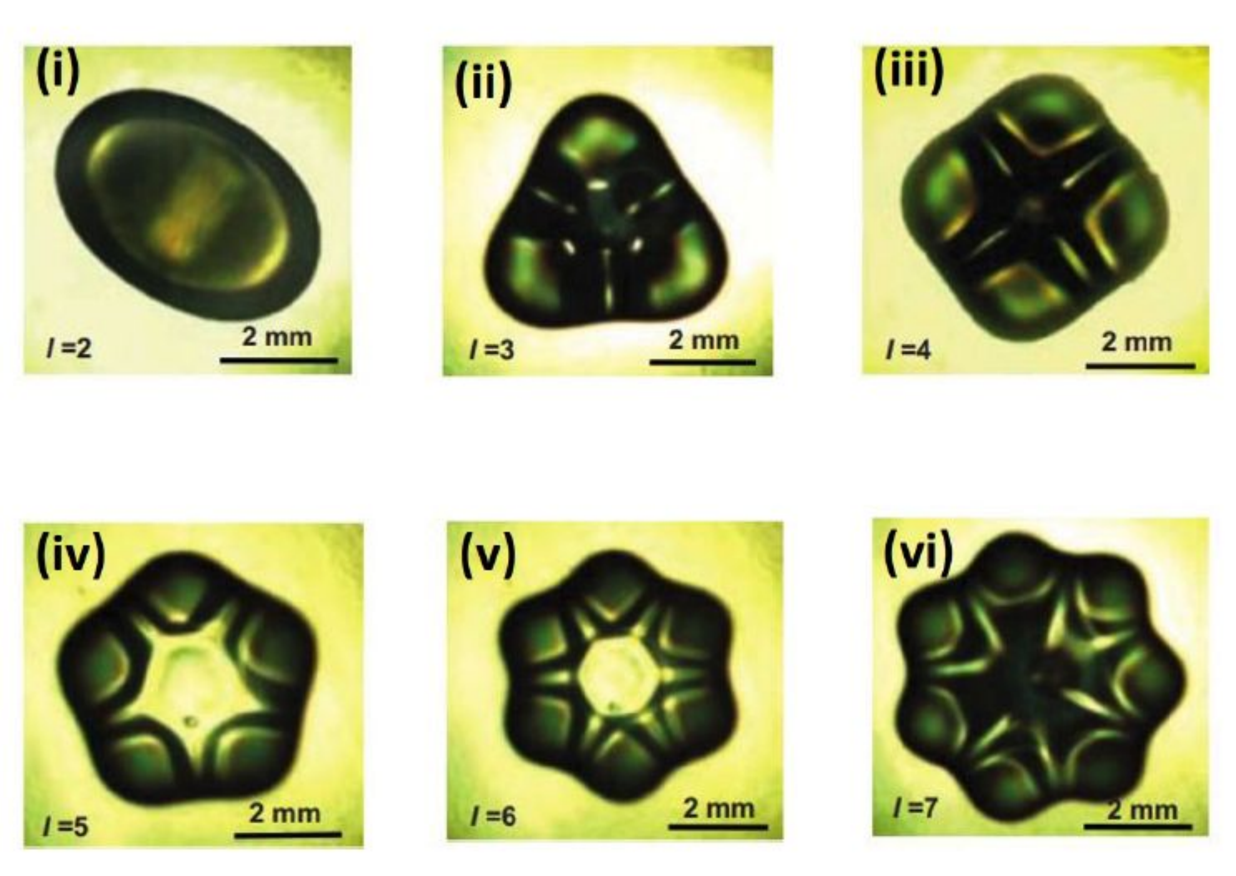
Stability of parametric oscillation

- If the disturbance grows ⎫ with
- If the disturbance decays ⎭ time

- System becomes unstable
- System becomes stable

- Interesting point (fluid/Optics):
- ❖ One can see some surface patterns in the unstable fluid or waveform pattern in optics.
- ❖ Star shaped patterns of liquid drops floating at different forcing frequency

Shen et al., Phys.Rev. E, 81:046305 (2010)



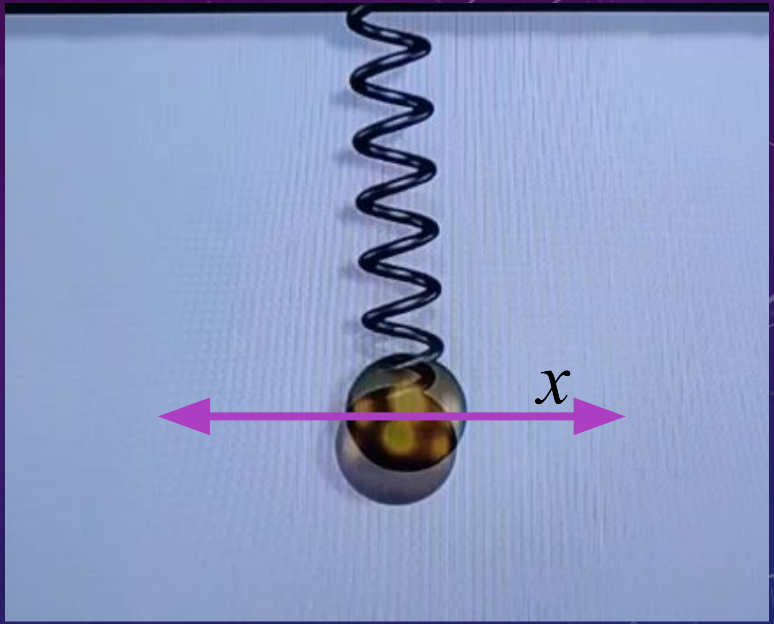
Champneys A. (2009) Dynamics of Parametric Excitation; https://doi.org/10.1007/978-0-387-30440-3_144; Encyclopedia of Complexity and Systems Science.

Parametric Resonance

1D-picture: $\ddot{x} + \frac{g(t)}{l}x = 0;$ $\underbrace{g(t) = g_0 + g_1 \cos \Omega t}$

Time-dep gravitational field

-- Parametric Forcing –Periodic & Sinusoidal



$$\Longrightarrow \underbrace{\ddot{x} + \omega_0^2(1 + h\cos\Omega t)x = 0}; \; \omega_0^2 = \frac{g_0}{l}, h = \frac{g_1}{g_0}$$

Mathieu Equation

- Parametric excitation amplitude= h and its period $= \frac{2\pi}{\Omega}$; $T_{nat} = \frac{2\pi}{\omega_0}$

Question: What frequencies will be important for this driving system?

- Naturally: $\Omega = \frac{2\omega_0}{n}; n = 1,2, \ldots$ (Landau-Lifshitz: Mechanics)

❖ Check $\Omega = 2\omega_0 + \epsilon, \; \epsilon \ll \omega_0 \Longrightarrow$ Trial Soln: $x = a(t)\cos\left(\omega_0 + \frac{\epsilon}{2}\right)t + b(t)\sin\left(\omega_0 + \frac{\epsilon}{2}\right)t$

- Retaining terms linear to $\epsilon \Longrightarrow \dot{a} \sim \epsilon a$ & $\dot{b} \sim \epsilon b$ $\Longrightarrow$ Instability (pumping energy) occurs:

Parametric resonance: $-\frac{1}{2}h\omega_0 < \epsilon < +\frac{1}{2}h\omega_0$

$$a \propto e^{\mu t} \;\&\; b \propto e^{\mu t}; \; \mu^2 = \frac{1}{4}\left[\left(\frac{1}{2}h\omega_0\right)^2 - \epsilon^2\right]$$

Parametric resonance: $-\frac{1}{2}h\omega_0 < \epsilon < +\frac{1}{2}h\omega_0;\ h = \frac{g_1}{g_0}$

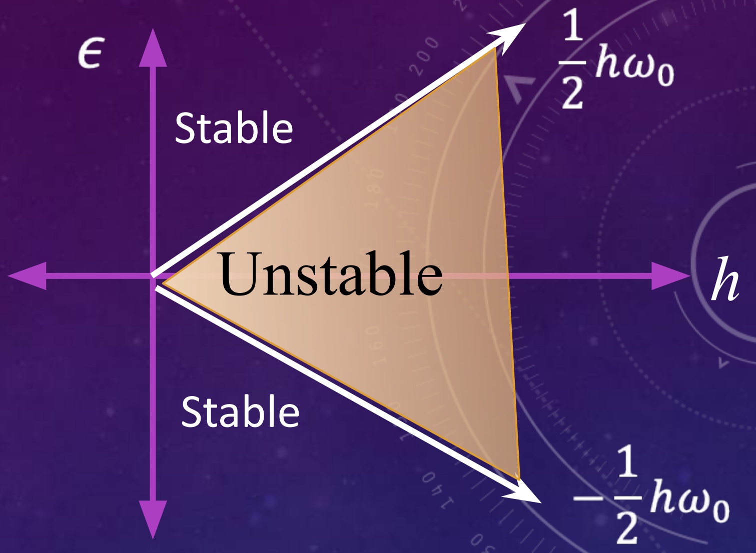
In presence of damping:

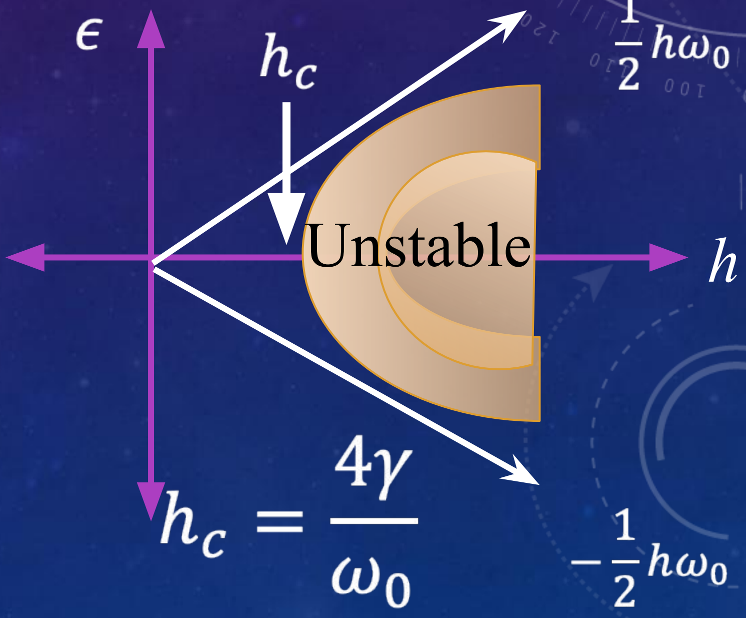
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2(1 + h\cos(2\omega_0 + \epsilon)t)x = 0$$

$$x(t) \sim e^{(\mu-\gamma)t} \times (\text{Oscillation})$$

Instability occurs:

$$-\left[\left(\frac{1}{2}h\omega_0\right)^2 - 4\gamma^2\right]^{1/2} < \epsilon < \left[\left(\frac{1}{2}h\omega_0\right)^2 - 4\gamma^2\right]^{1/2}$$





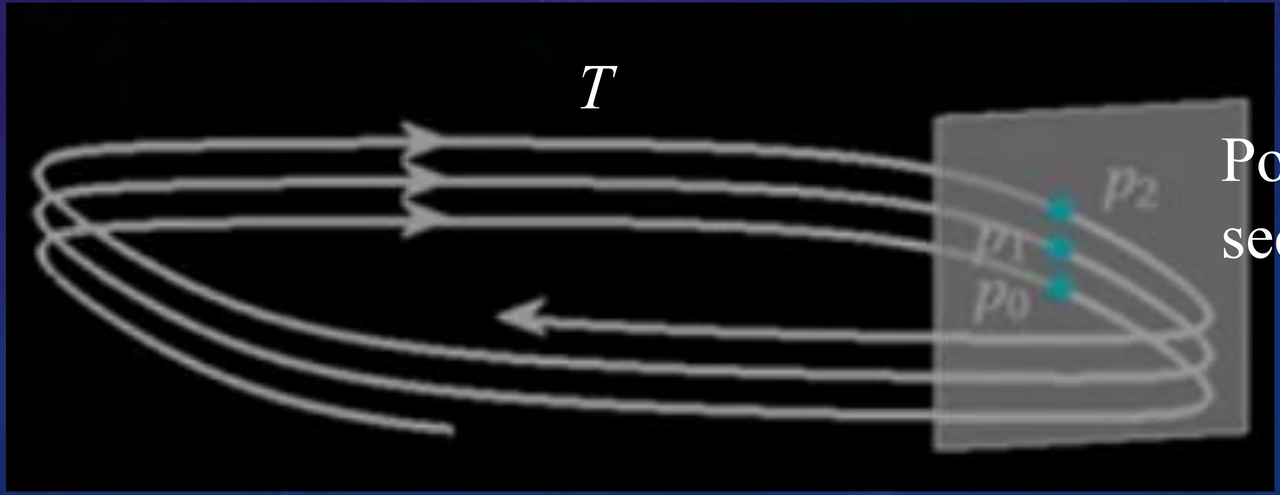
- Instability is strongest at exact resonance: $\Omega = \omega_{exc} = 2\omega_0$
  → Subharmonic instability (T_{exc}=(1/2)T_{nat})
  → Instability occurs for T_{exc}= integer x (1/2)T_{nat}

The parametric spring-mass system, its connection with non-linear optics, and an approach for undergraduate students
Ilario Boscolo, Fabrizio Castelli, Marco Stellato, Stefano Vercellati: arxiv: 1402.5318 (2014)

Floquet theory for 2nd order ODE with Periodic Coefficients

Graphical Representation: Poincare Maps:

- Instead of entire dynamical trajectory → Look into map from one period to another
  → Poincare Map: $x_{n+1} = \varphi(x_n)$
  → Flow map takes x_n to x_{n+1} after time period T
- Use Poincare section: every time the trajectory crosses the Poincare section, marks it
- We will look into dynamics on that section only,
- Stable periodic trajectory: $x_{n+1} = \varphi(x_n)$ where $x_{n+1} \to x_n$
- Unstable periodic trajectory: $x_{n+1} = \varphi(x_n)$ where $x_{n+1} \nrightarrow x_n$; $|x_{n+1} - x_n| > |x_{n-1} - x_n|$



Poincare section

N Kutz: https://www.youtube.com/watch?v=N_zmYDnjACs

Floquet theory for 2nd order ODE with Periodic Coefficients

Solution or point of the trajectory at different time, $x(t) \propto e^{\mu t} \times$ periodic function

Expanded in Fourier Basis

Convert the ODE in eigen-value equation

- Generate matrix in the Fourier basis
- Eigen-value will demonstrate the stability of the system

Faraday Instability
in
Bose-Einstein Condensate

PHYSICAL REVIEW A **102**, 033320 (2020)

Parametrically excited star-shaped patterns at the interface of binary Bose-Einstein condensates

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PHYSICAL REVIEW LETTERS **127,** 113001 (2021)

Spontaneous Formation of Star-Shaped Surface Patterns in a Driven Bose-Einstein Condensate

K. Kwon,[1] ... J. Huh,[1] ... akidis,[3] ... [2] P. G. Kevrekidis,[4]
... ajumder,[2] ... and ...
[1] Depar... orea Advanc... and ... n 34141, Korea
[2] D... s, Indian Ins... Kharag... 1302, India
[3] ... uantum Tec... of P... Hamburg,
... uruper Chau... burg,
[4] Department ... Statistics, U... setts, ... setts 01003-4515, USA
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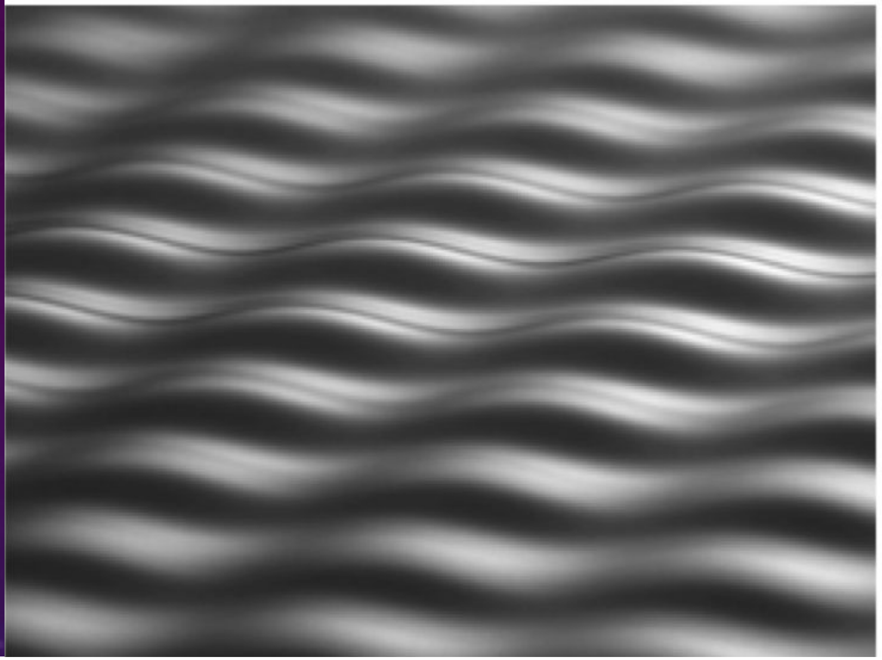
Dr. Koushik Mukherjee
Lund University

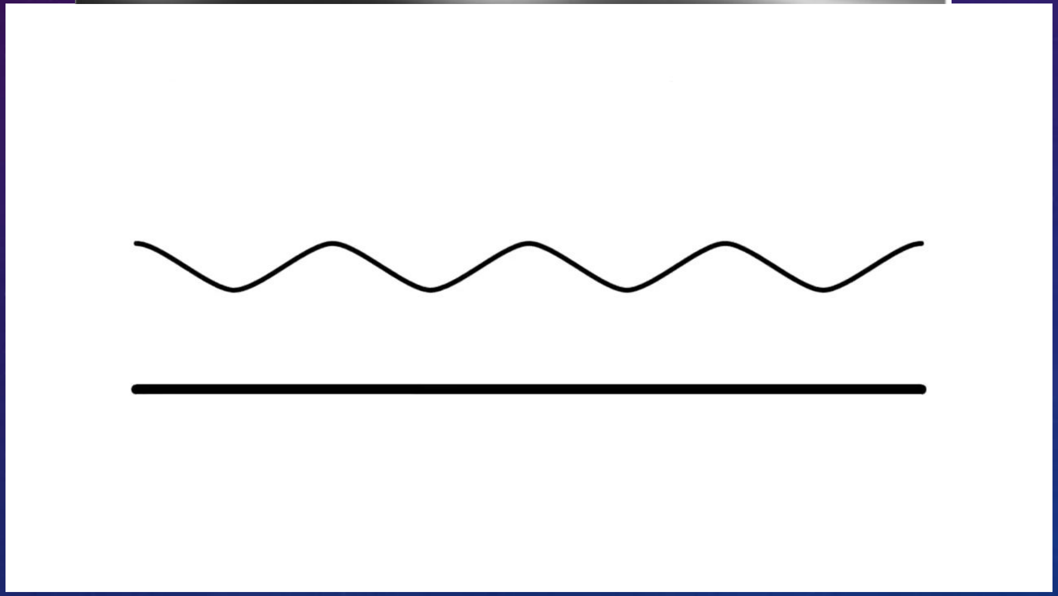
Dr. Dilip Kr Maity
KAUST, Saudi

Subrata Das,
IIT-KGP

Classical Fluid

https://www.youtube.com/watch?v=l5UpWZzWzbA



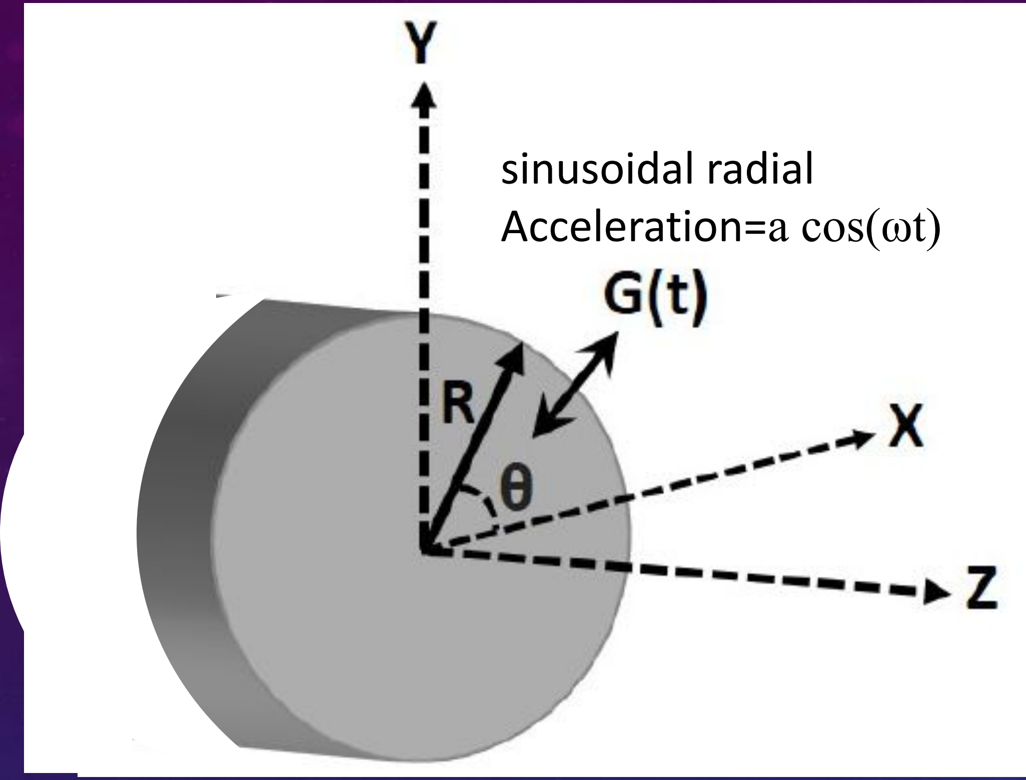


- In 1831, Michael Faraday observed Instability in vertically oscillating fluid,,
  - excited standing waves
- When the vibration frequency exceeds a critical value, the flat hydrostatic surface becomes unstable.
- Instability is subharmonic: fluid oscillates at twice slower than its solid bottom
- Phenomena is well described by
  - Mathieu Equation (Numerical Sol^n)
  - Floquet Theory
- Effective gravitational acceleration:

  $G(t) = g + a\cos(\omega t)$

Fluid dynamic instabilities: theory and application to pattern forming in complex media: F Gallaire & P T Brun, Phil. Trans. A,

Faraday Instability on a (in)viscous cylindrical surface



Navier–Stokes Equations:

$$\rho\left[\frac{\partial U}{\partial t} + (\boldsymbol{U}.\nabla)\boldsymbol{U}\right] = -\nabla \boldsymbol{P} + \mu\nabla^2\boldsymbol{U} + \rho\boldsymbol{G}(t)$$

□Momentum Conservation Equation

$\nabla.U=0$ □Mass conservation Equation

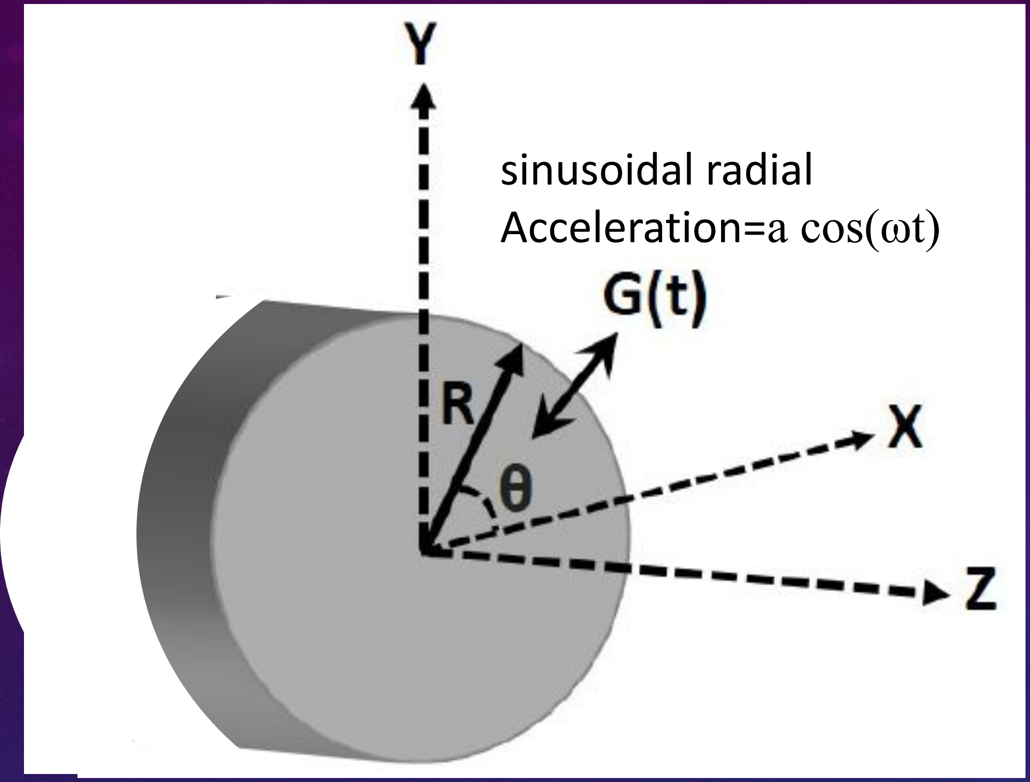
$$\hat{G}(t) = -a\cos(\omega t)\hat{r}$$

Surface is deformed due to the external acceleration $\hat{G}(t)$:
$=> r = R + \eta(\theta, z, t)$

U -Velocity of the fluid element
P -Pressure
μ -Dynamic viscosity
ρ–Density of the fluid

Maity, D.K., *Theor. Comput. Fluid Dyn.* 14th Sep (2020)

Faraday Instability on a viscous cylindrical surface



$$r = R + \eta(\theta, z, t)$$

Ideal case (Inviscid fluid): Mathieu Eq.
(Conserving Pressure Balance at the surface)

$$\ddot{\bar{\eta}} + \bar{\omega}^2 \left(1 + \frac{a}{\bar{a}} \cos \omega t\right) \bar{\eta} = 0$$

Here, $\eta(\theta, z, t) = \bar{\eta}(\mathrm{t}) \exp i(m\theta + kz)$

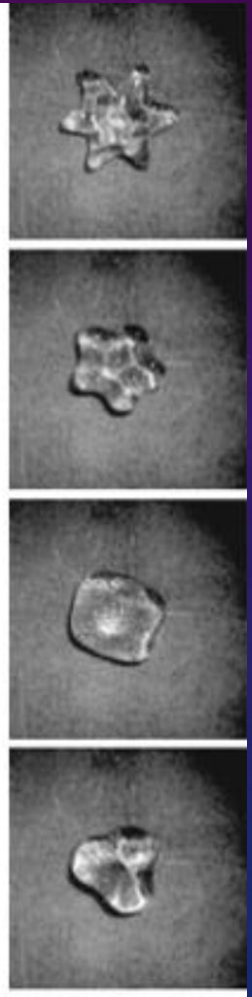
$\bar{\omega}$ & $\bar{a}$ depend on surface tension (σ),
axial wavenumber (k) and
azimuthal wavenumber(m)

U -Velocity of the fluid element
P -Pressure
μ -Dynamic viscosity
ρ–Density of the fluid

Analytic solution: Floquet expansion
Floquet analysis => stability diagram of the fluid

Maity, D.K., *Theor. Comput. Fluid Dyn.* 14th Sep (2020)

Floquet analysis => stability diagram of the fluid



Brunetand Snoeijer, 2011

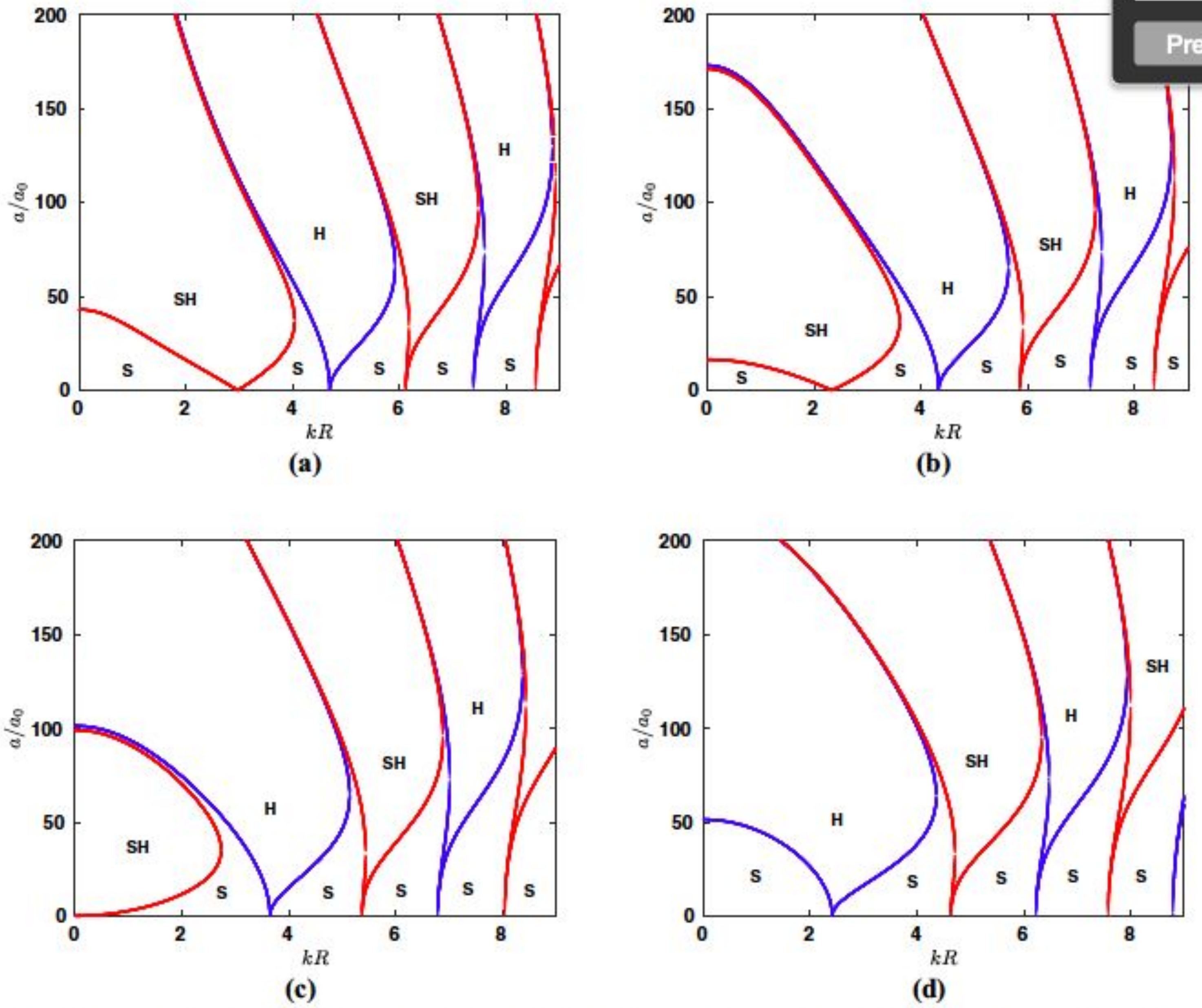
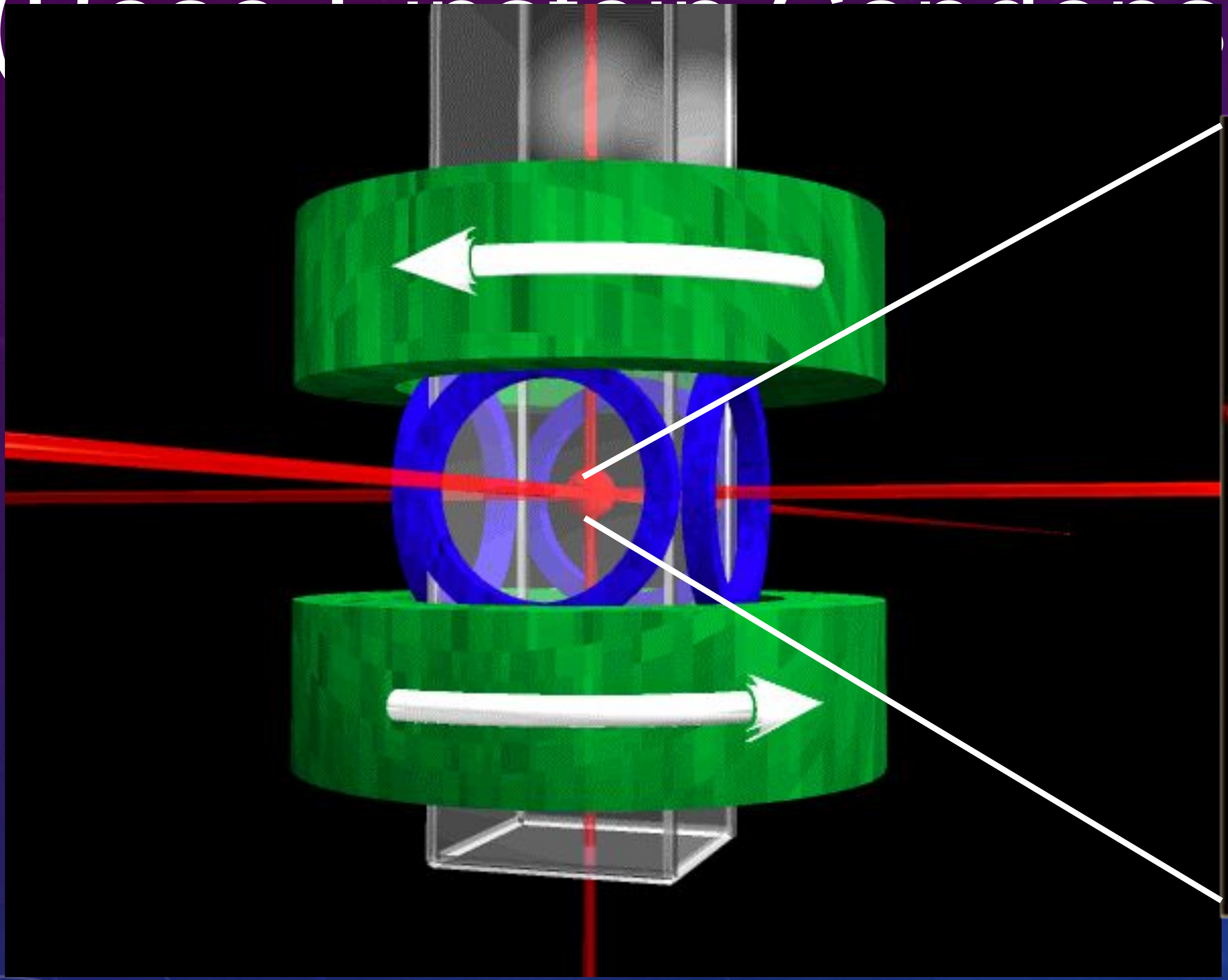


Fig. 2 (Color online) For an ideal fluid, the marginal stability boundaries of the Mathieu equation [Eq. (35)] are plotted for the dimensionless forcing angular frequency $\frac{\omega}{\omega_0} = 9.73$. Red (gray) and blue (black) boundaries represent the subharmonic (SH) and the harmonic (H) case, respectively. S represents the stable region of the system. In the stability curves, the dimensionless forcing amplitudes (a/a_0) are plotted with the dimensionless axial wavenumbers (kR) for **a** $m = 1$, **b** $m = 2$, **c** $m = 3$, **d** $m = 4$

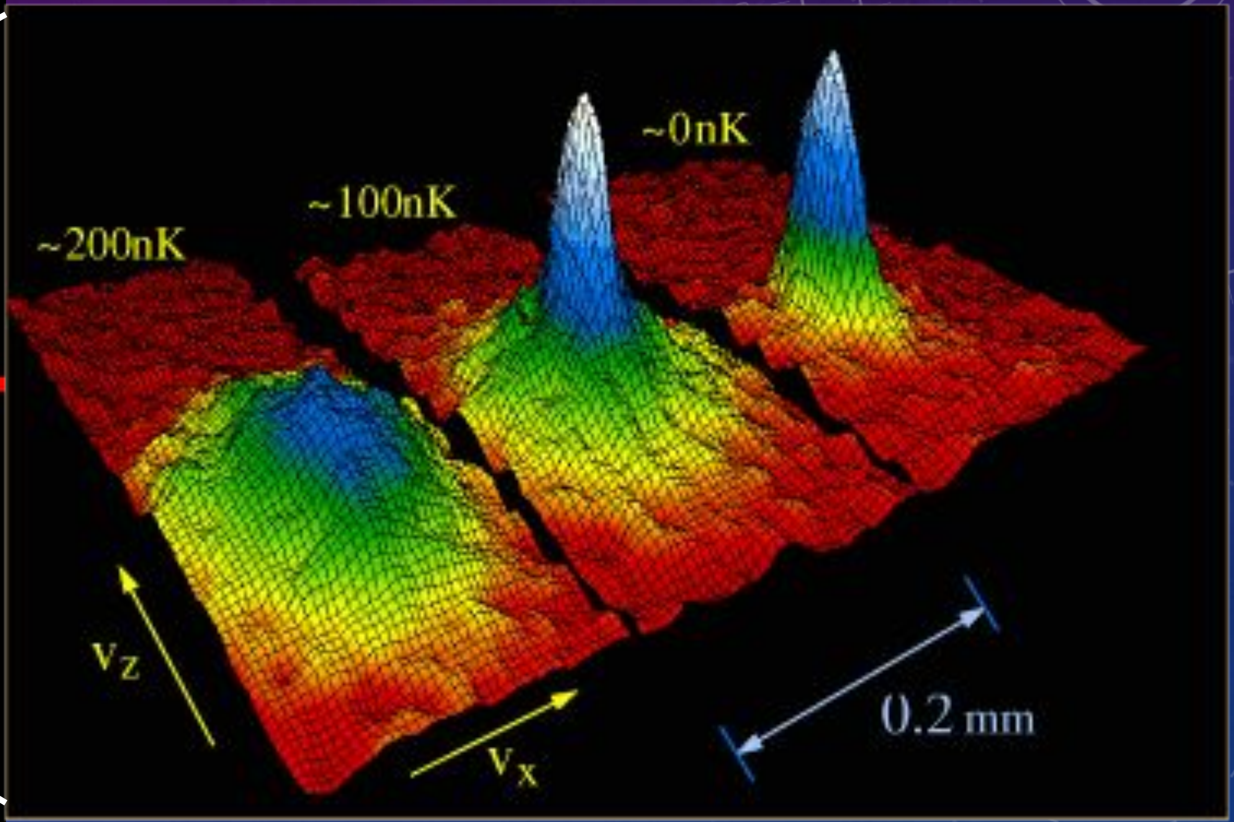
Maity, D.K.,*Theor. Comput. Fluid Dyn.* 14th Sep (2020);

Parametric Oscillation in Quantum Fluid (Bose-Einstein Condensate)



https://plato.stanford.edu/entries/physics-experiment/app3.html

2D velocity Distribution



JILA, University of Colorado, Boulder

Atom Laser: Coherence

Demonstration: https://youtu.be/shdLjIkRaS8

Hamiltonian of BEC

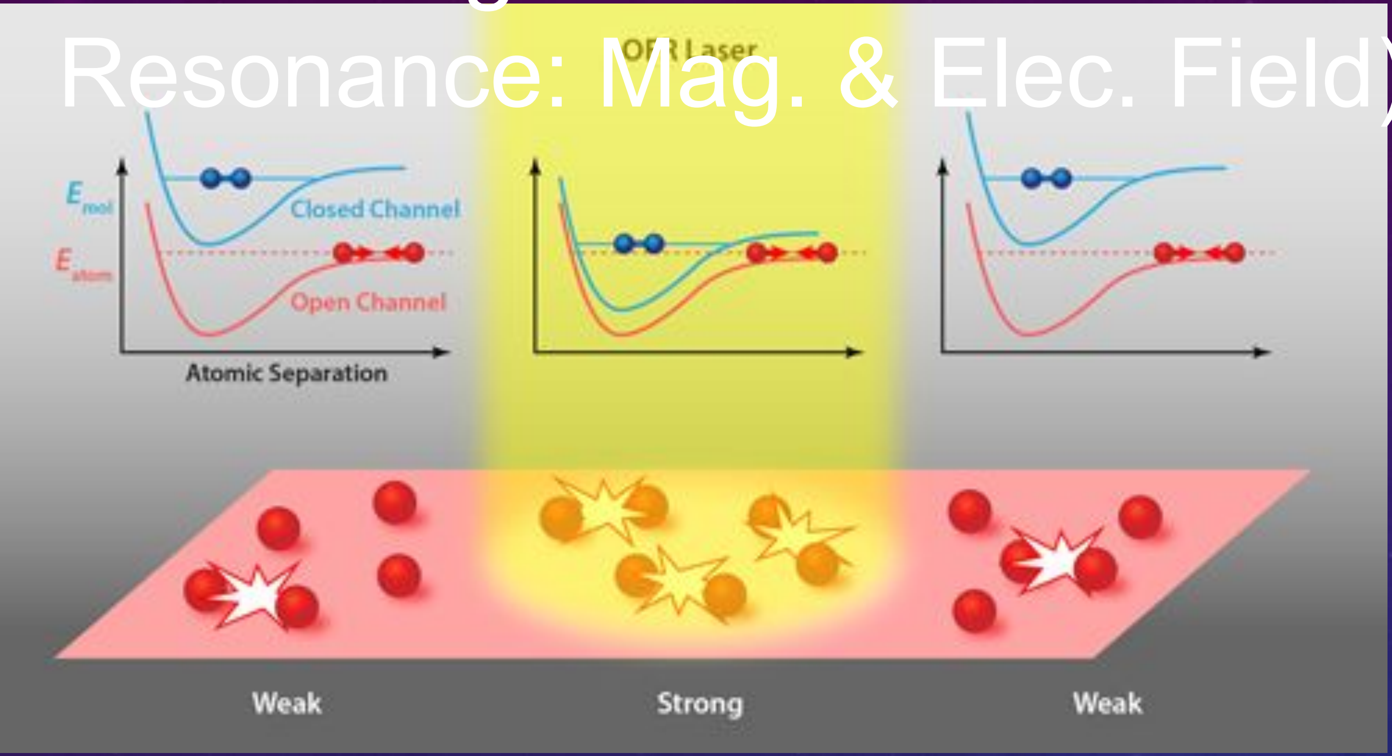
$$\hat{H} = \int dr\, \hat{\Psi}^\dagger(r,t) \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{trap}(r) \right] \hat{\Psi}(r,t)$$

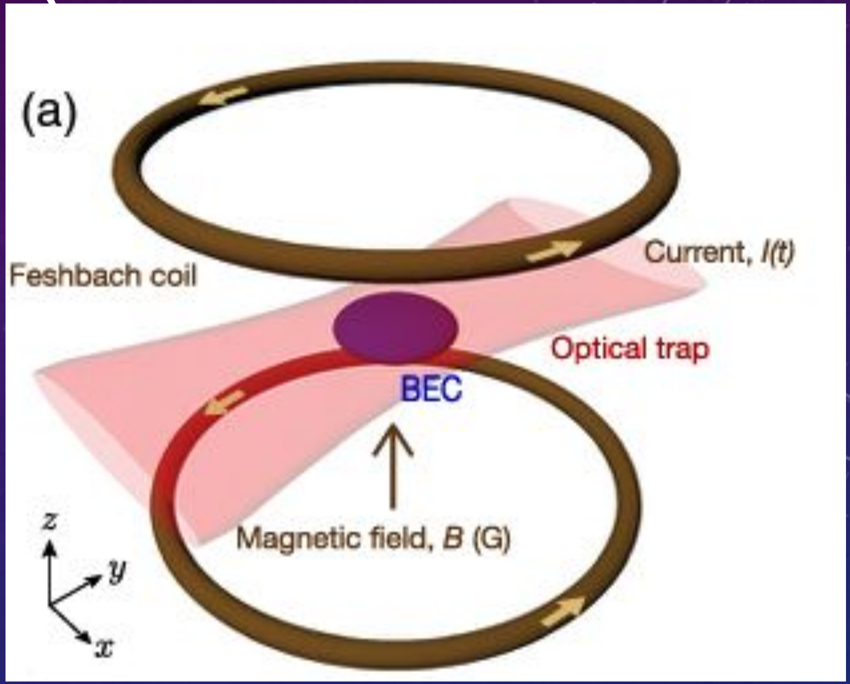
$$+ \frac{1}{2}\int dr \int dr'\, \hat{\Psi}^\dagger(r,t)\, \hat{\Psi}^\dagger(r,t) V(r-r')\, \hat{\Psi}(r',t) \hat{\Psi}(r,t)$$

$V(r-r') = \text{Contact Potential (s-wave approx.)}$

$$= U\delta(r-r') = \frac{4\pi\hbar^2}{m} a_{scat}\, \delta(r-r')$$

Controlling atom-atom interaction (Feshbach Resonance: Mag. & Elec. Field)

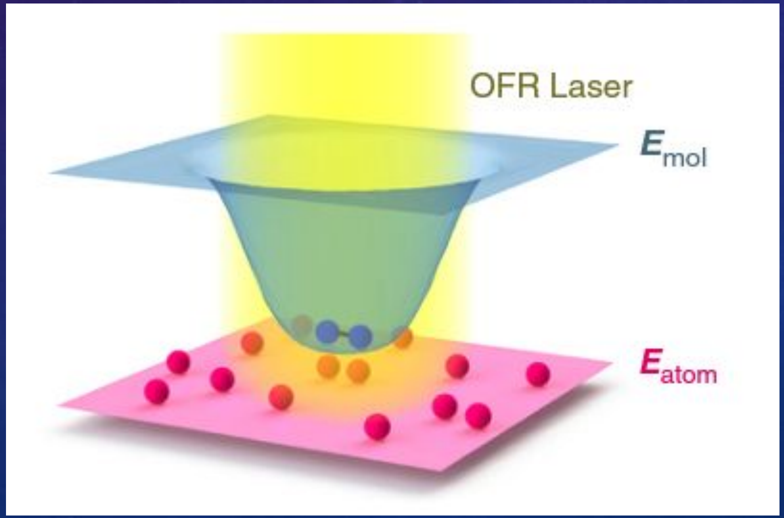


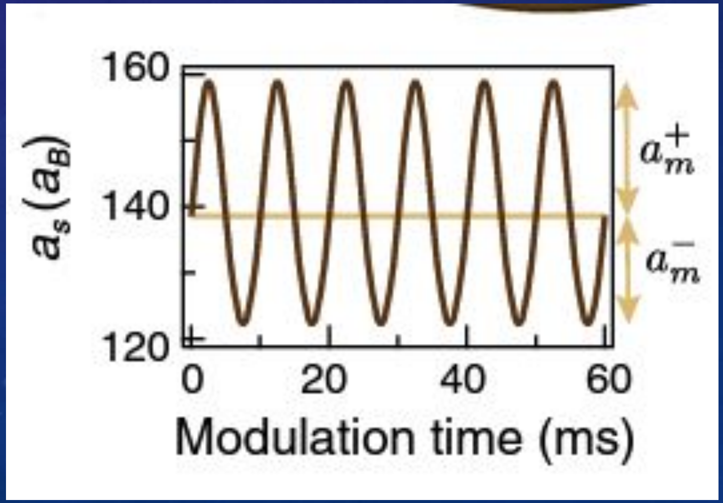


Kwon, Mukherjee, et al., PRL, 127, 113001(2021)

^{7}Li atoms BEC: Pancake-shaped trap

Chris Vale: https://physics.aps.org/articles/v8/95





Red-detuned optical trap for the axial confinement and a magnetic trap

PRL 115, 155301 (2015)

Mean-Field Theory: Gross-Pitaevskii (GP) Equation

$$\hat{H} = \int dr\, \hat{\Psi}^{\dagger}(r,t)\left[-\frac{\hbar^2}{2m}\nabla^2 + V_{trap}(r)\right]\hat{\Psi}(r,t)$$

$$+\frac{1}{2}\int dr \int dr'\, \hat{\Psi}^{\dagger}(r,t)\, \hat{\Psi}^{\dagger}(r,t) V(r-r')\, \hat{\Psi}(r',t)\hat{\Psi}(r,t)$$

$V(r-r') =$ Contact Potential (s−wave approx.)

$$= U\delta(r-r') = \frac{4\pi\hbar^2}{m} a_{scat}\, \delta(r-r') = g_{2D}$$

$$i\hbar\frac{\partial\hat{\Psi}^{\dagger}}{\partial t} = \left[\hat{\Psi}^{\dagger}(r,t), \hat{H}\right]$$

$$\Rightarrow i\hbar\frac{\partial\hat{\Psi}^{\dagger}}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{trap}(r) + U\hat{\Psi}^{\dagger}\hat{\Psi}\right]\hat{\Psi}^{\dagger}(r,t)$$

$$\Rightarrow i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\left(\partial_x^2 + \partial_y^2\right)^2 + \frac{1}{2}m\omega_r^2(x^2+y^2) + g_{2D}\left(1 + \frac{\bar{\bar{a}}_m}{a_{scat}}\cos(\omega_D t)\right)|\psi|^2\right]\psi(r,t)$$

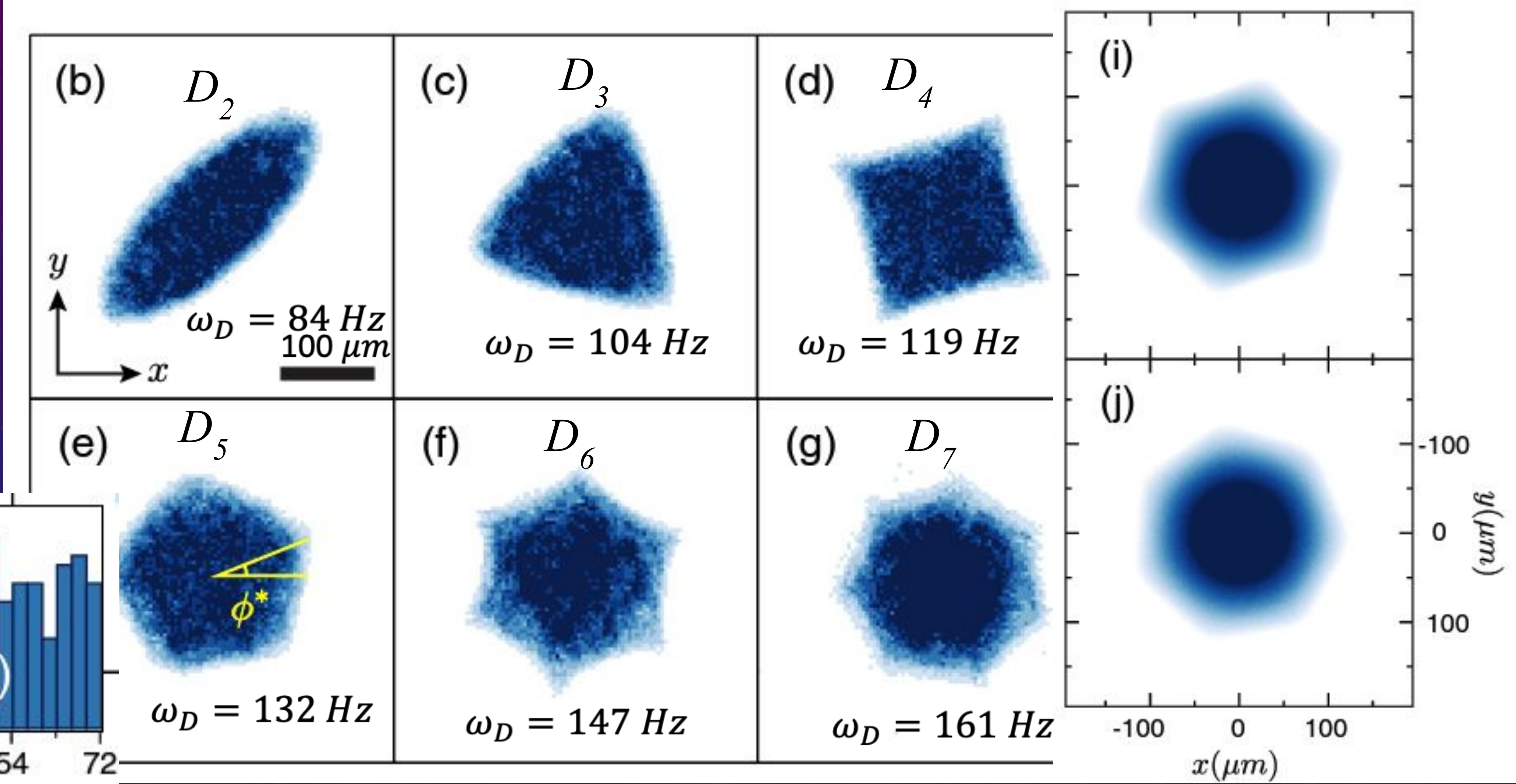
Madelung Transformation: $\psi = \sqrt{n}e^{i\phi}$ & Assume density disturbance $\delta n = \zeta_l r^l e^{il\phi}$

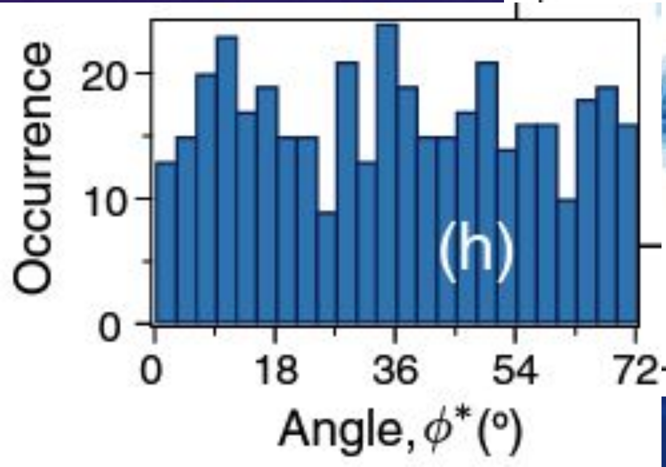
Mathieu equation: $\ddot{\zeta}_l(t) + \omega_l^2[1 + (\bar{\bar{a}}_m/a_{scat})\cos(\omega_m t)]\zeta_l(t) = 0$

Experimental observation of star-shaped condensates

Atoms: ^7Li; $|F = 1, m_F = 1\rangle$; $[\omega_r, \omega_z] = 2\pi\times[29.4(2), 725(5)]Hz$; $a_A = 138(6)a_B$

After 1 s modulation, BEC boundary is strongly deformed, displaying 2D regular polygon patterns D_l symmetry.

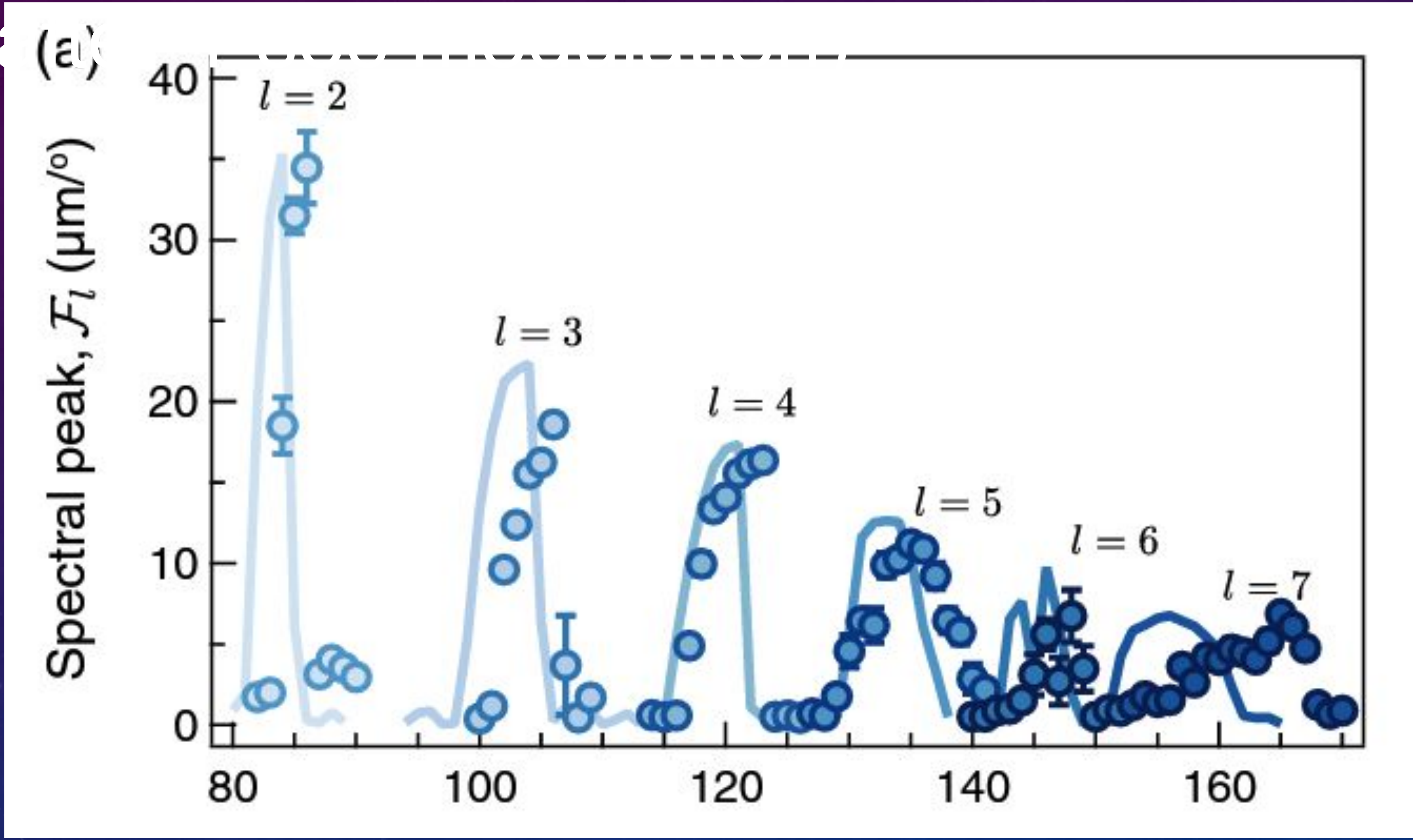




Kwon, Mukherjee, et al., PRL, 127, 113001 (2021)

Spectral peak of various l-fold star patterns

(a)

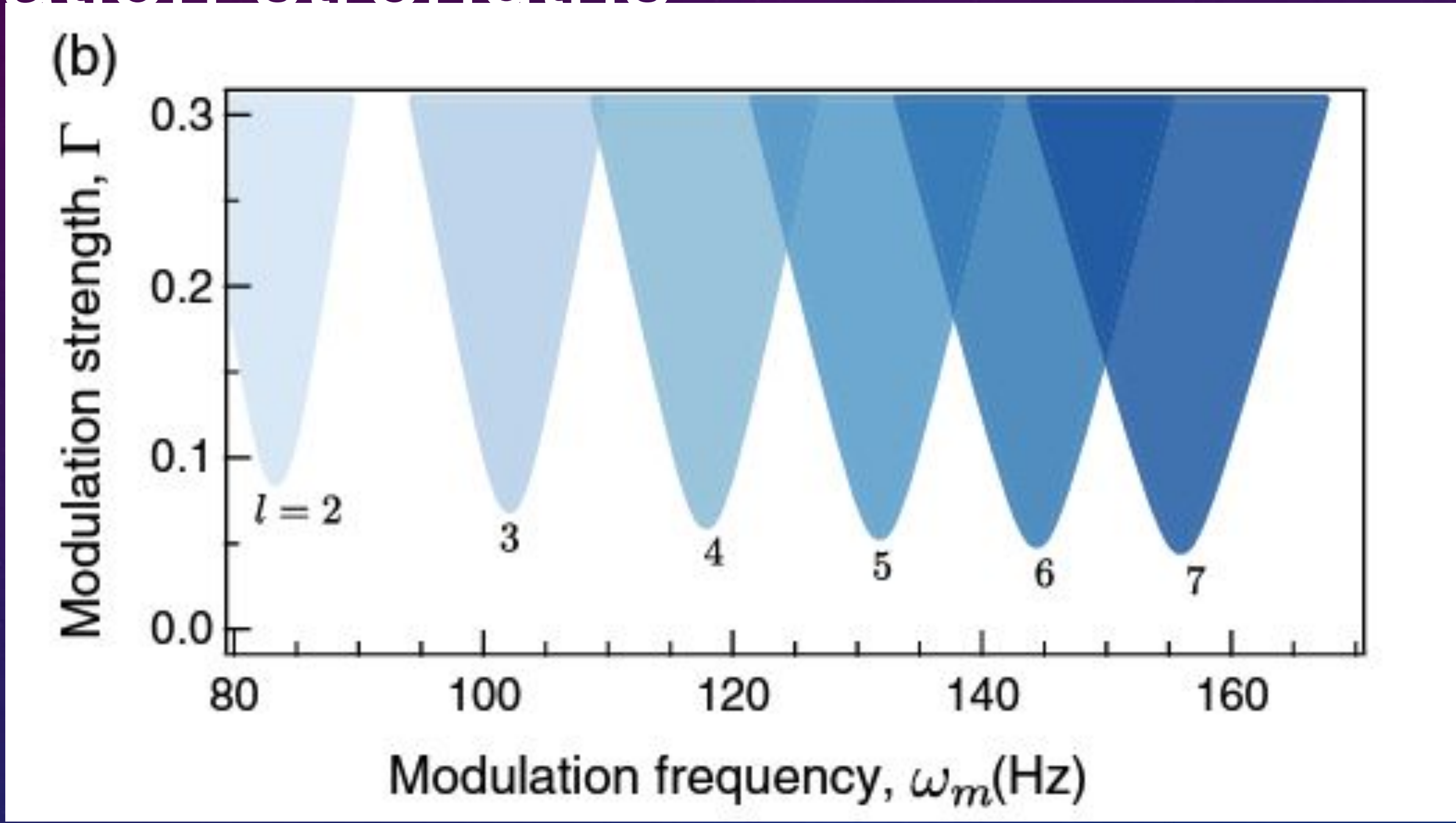


Modulation frequency, (Hz)

Kwon, Mukherjee, et al., PRL, 127, 113001 (2021)

Floquet stability tongues for different modulation strengths

Kwon, Mukherjee, et al., PRL, 127, 113001 (2021)



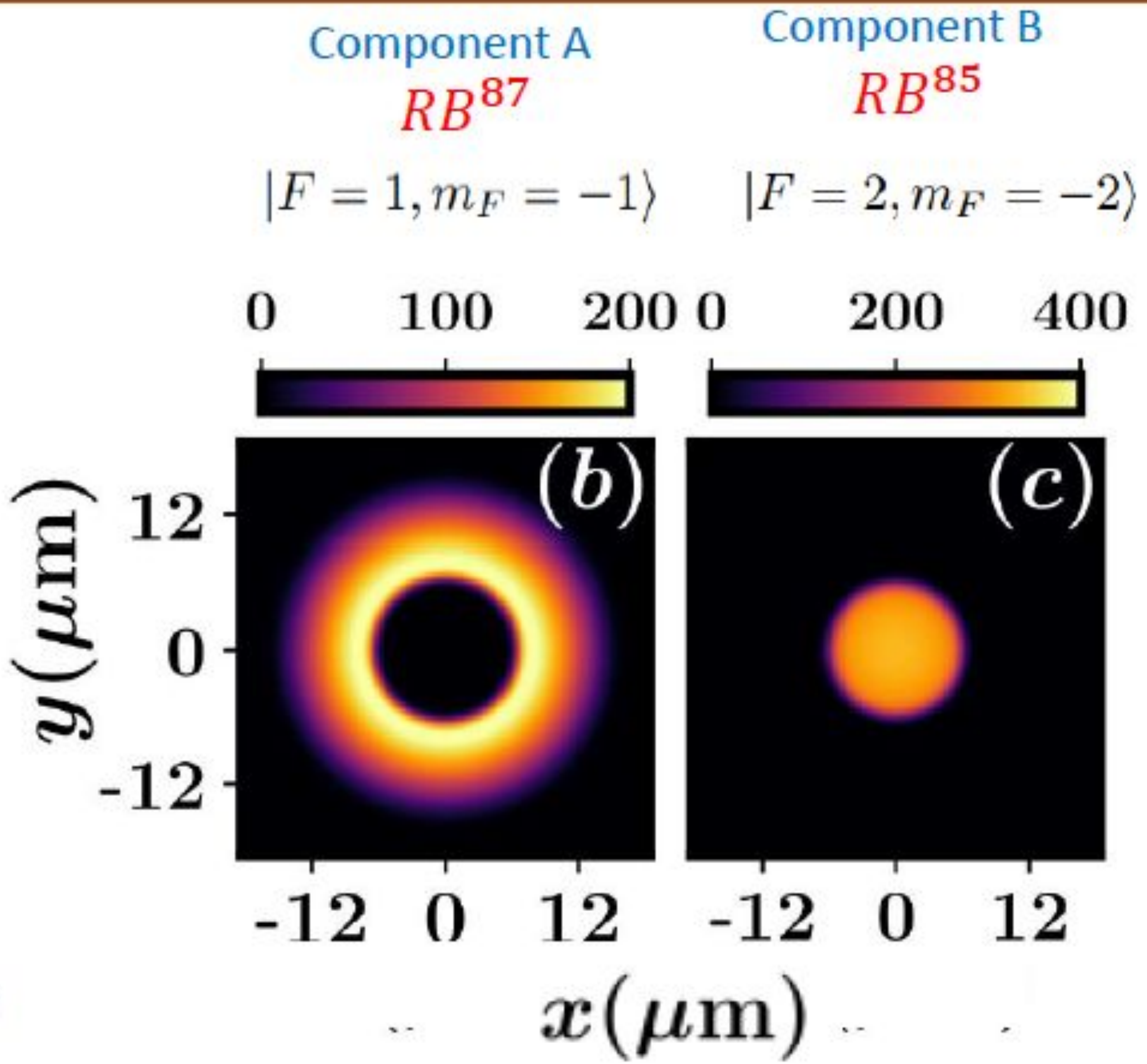
- Precisely measure the dispersion relation of the collective excitations
- Identify the resonant frequencies of the patterns by comparing experimental/ theoretical (effective Mathieu equation) patterns vs Floquet analysis

Phase Separated Binary Bose Einstein Condensates

- A **Bose-Einstein condensate (BEC)** is a state of matter which typically formed when a gas of bosons at low densities is cooled to temperatures very close to absolute zero.
- BEC is a superfluid state. Many of the classical fluid instabilities have been studied in BEC.
- To study interfacial instability, immiscible binary BEC is required.
- Condition for Immiscible 2-comp BEC:

$$a^2_{AB}/(a_{AA}a_{BB}) \geq 1$$

- RB^{87}-RB^{85} are well-studied Two-comp. BEC due to their near degenerate ground state energies.



Gross–Pitaevskii equations (GP Equations)

- Using the mean-field approximation at $T = 0$

GP equations

Component A $$i\hbar\frac{\partial\psi_A}{\partial t} = \left(-\frac{\hbar^2}{2m_A}\nabla^2 + V_A + g_{AA}|\psi_A|^2 + g_{AB}|\psi_B|^2\right)\psi_A$$ $$\int|\psi_A|^2 d\boldsymbol{r} = N_A$$

Component B $$i\hbar\frac{\partial\psi_B}{\partial t} = \left(-\frac{\hbar^2}{2m_B}\nabla^2 + V_B + g_{BB}|\psi_B|^2 + g_{BA}|\psi_A|^2\right)\psi_B$$ $$\int|\psi_B|^2 d\boldsymbol{r} = N_B$$

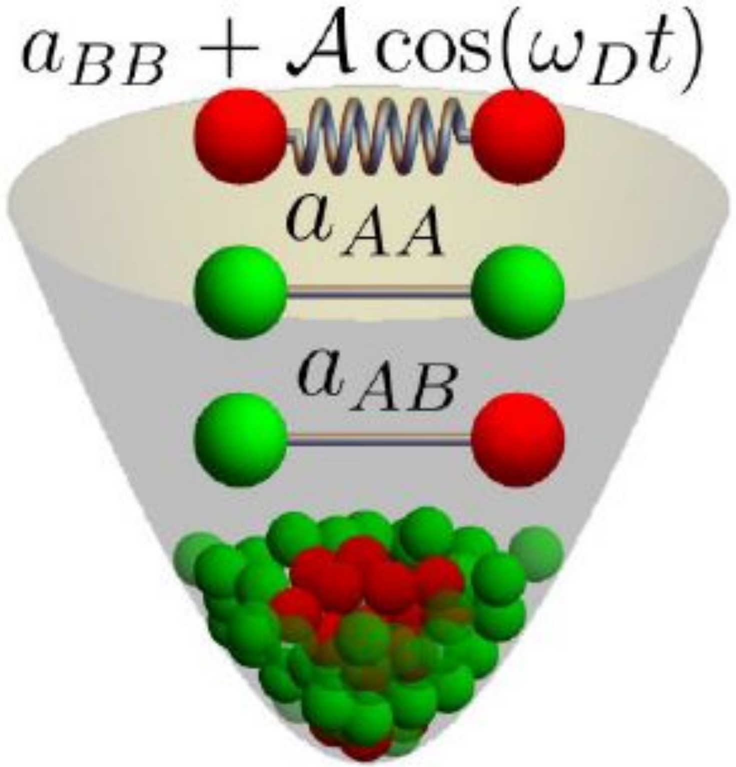
$$g_{ij} = 2\pi\hbar^2 a_{ij}({m_i}^{-1} + {m_j}^{-1})$$

$(i, j = A, B)$

$$V_i = \frac{m_i}{2}\omega^2\left[\alpha^2 r^2 + \lambda^2 z^2\right]$$

$\alpha = 1, \quad \lambda = 40$ ➡ Condensate is highly oblate disk shaped

Scattering length modulation



Maity, et al., PRA, 102, 033320 (2020)

$$a_{AA} = 99\, a_0$$

$$a_{AB} = 213\, a_0$$

a_0 - Bohr radius

Using Feshbach resonance (Papp et al. 2008)

a_{BB} can be modulated $(50 - 900)\, a_0$

$$a_{BB} = 75\, a_0 + \mathcal{A}\cos(\omega_D t)$$

Amplitude of the modulation

Angular frequency of the modulation

Simulation of Faraday Instability in BEC

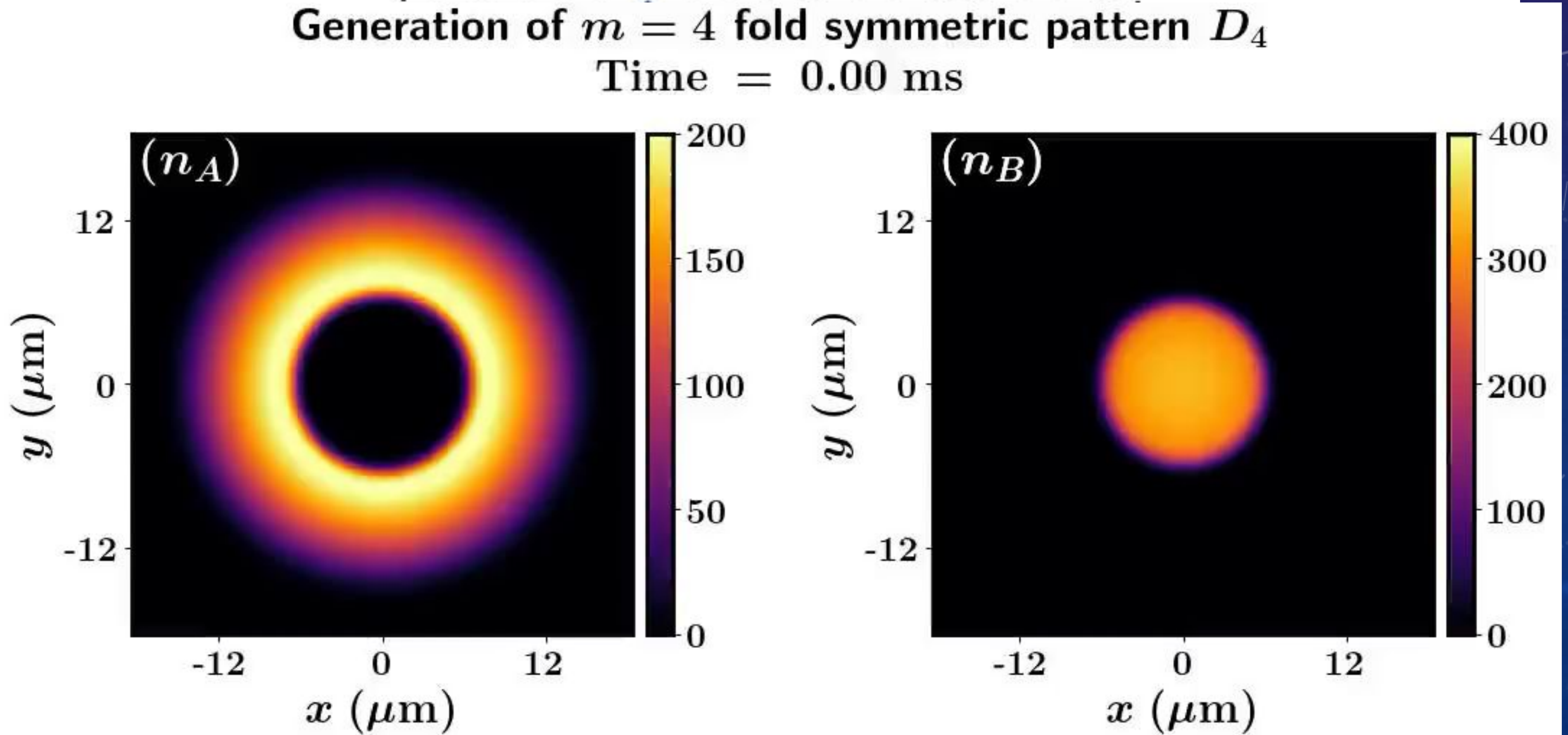
Simulations

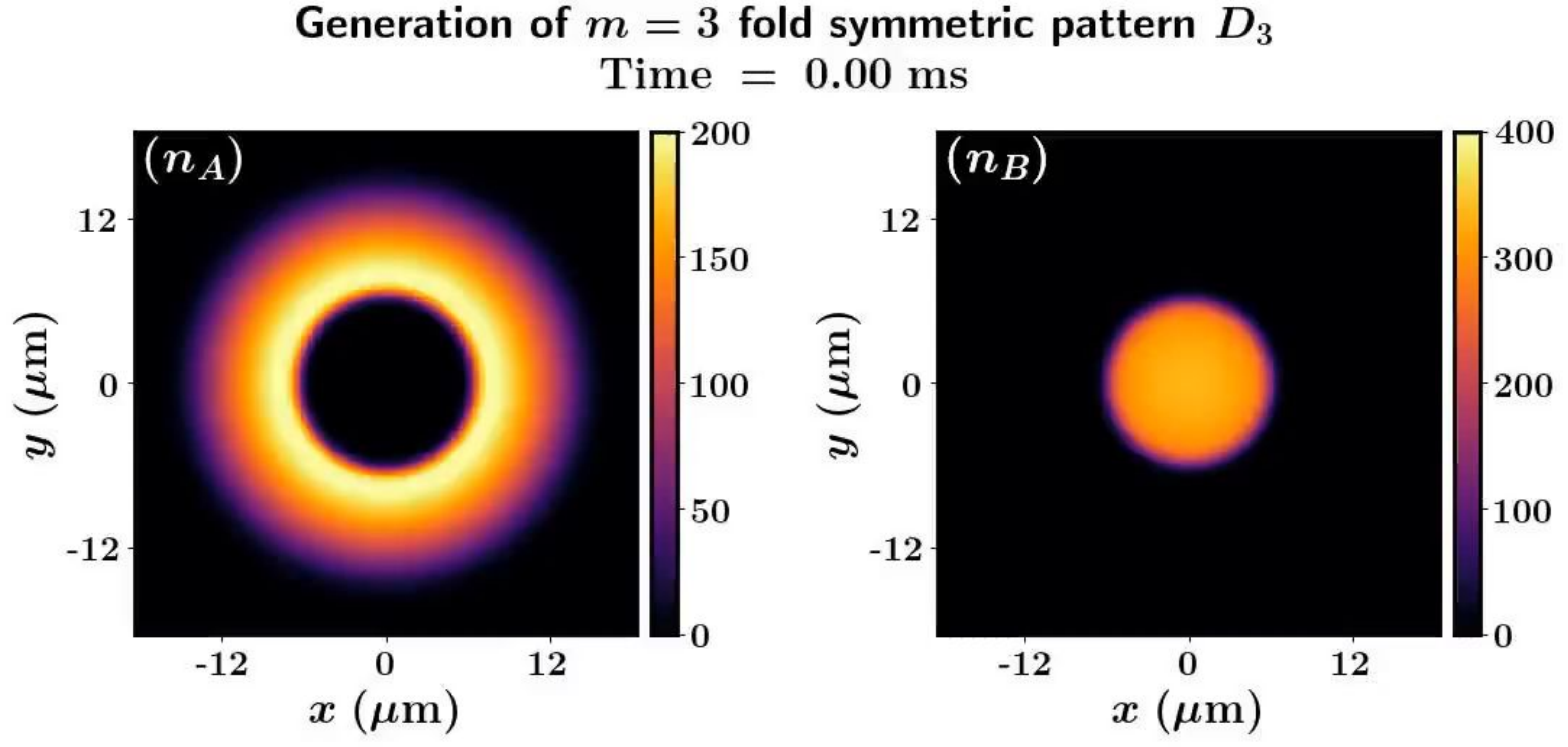
$$i\hbar\frac{\partial\psi_B}{\partial t} = \left(-\frac{\hbar^2}{2m_B}\nabla^2 + V_B + g_{BB}\left[1+\frac{\mathcal{A}}{a_{BB}}\cos(\omega_D t)\right]|\psi_B|^2 + g_{BA}|\psi_A|^2\right)\psi_B$$

$A = 15\ a_0$

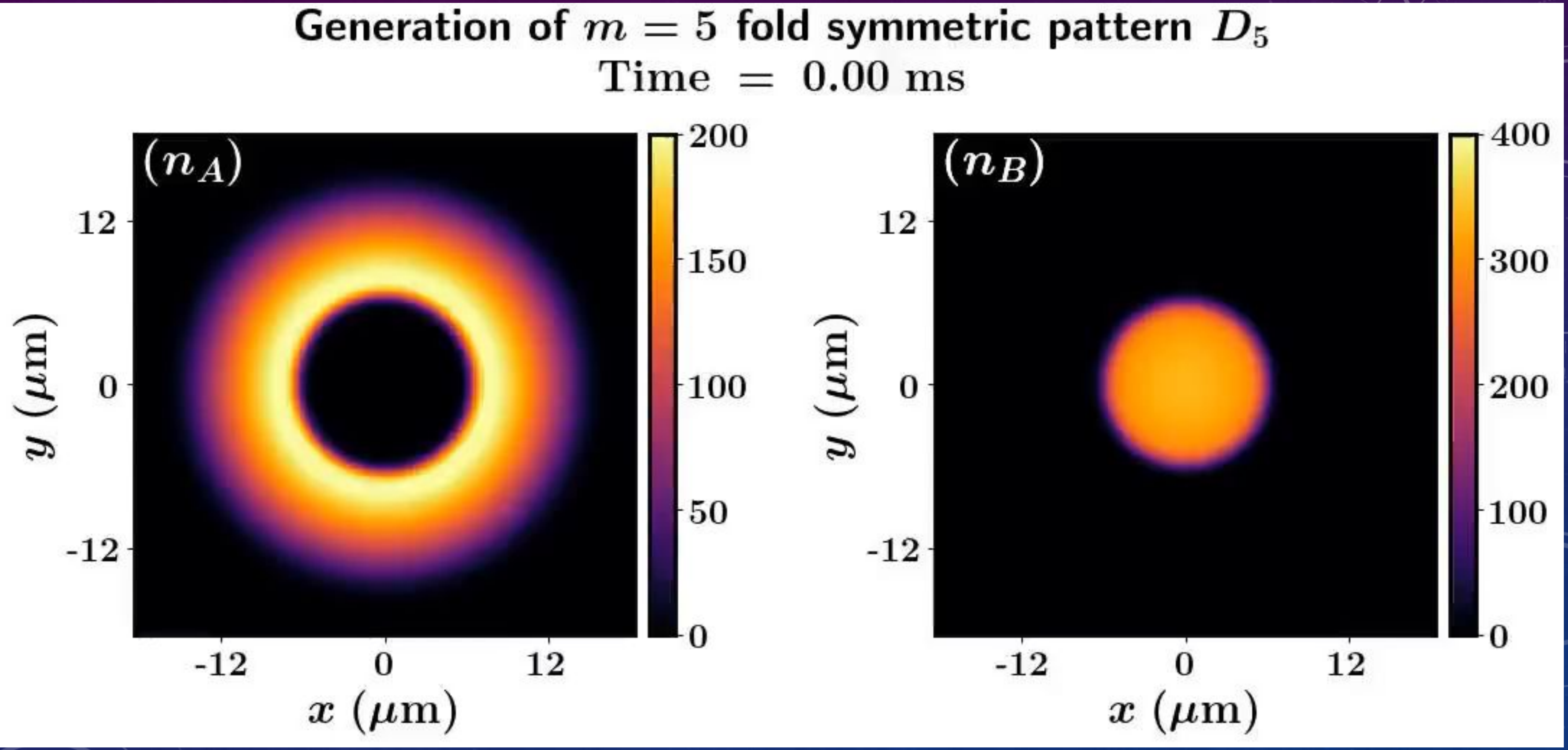
$\frac{\omega_D}{2\pi} = 69\ Hz$

Maity, et al., PRA, 102, 033320 (2020)

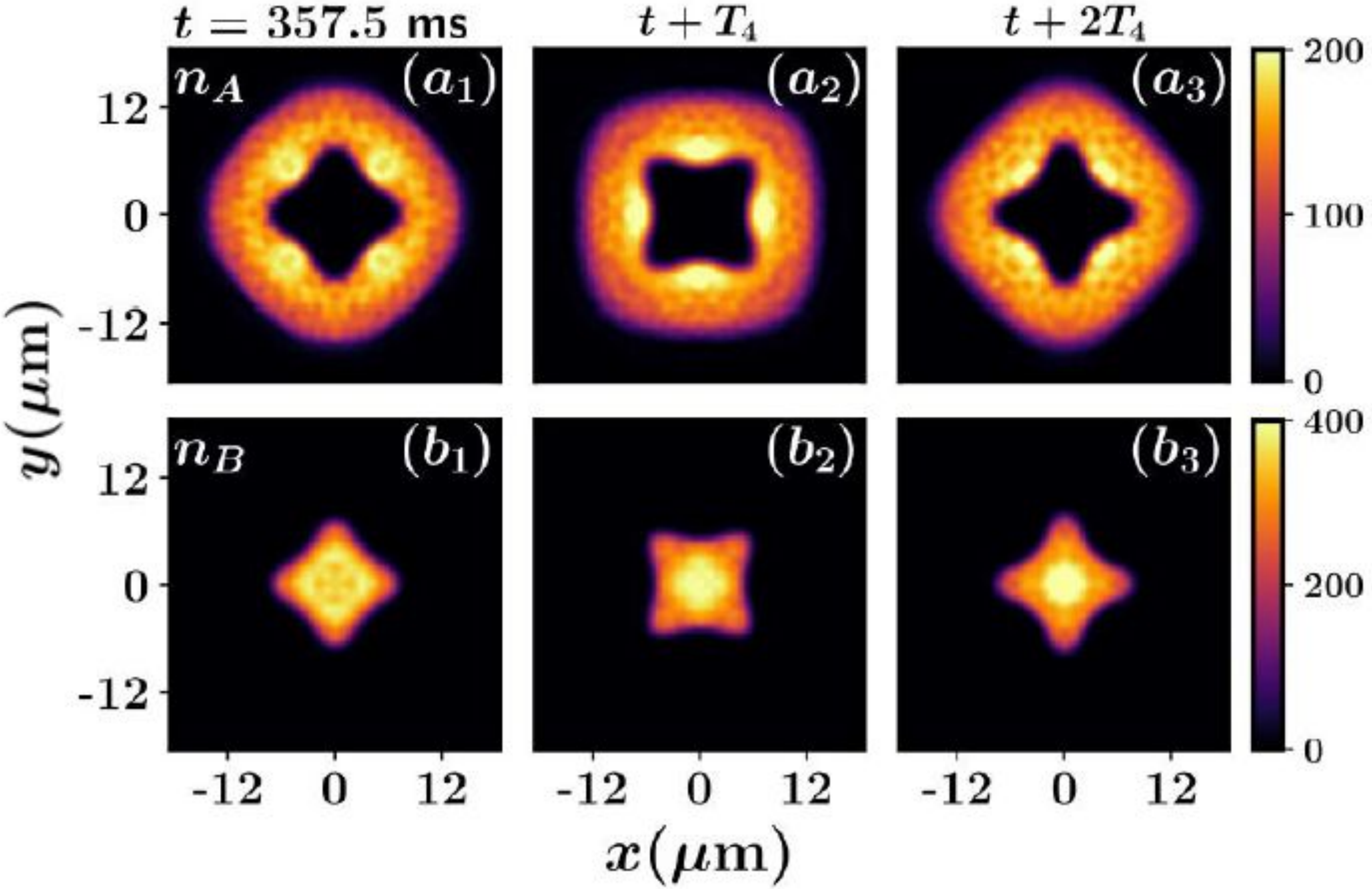




Generation of $m = 3$ fold symmetric pattern D_3
Time = 0.00 ms
(n_A)
(n_B)
y (μm)
x (μm)
12
0
-12
200
150
100
50
0
400
300
200
100
0


Generation of $m = 5$ fold symmetric pattern D_5
Time = 0.00 ms
(n_A)
(n_B)
y (μm)
x (μm)
12
0
-12
0
50
100
150
200
300
400

Subharmonically Excited Pattern



$\mathcal{A} = 15\, a_0$

$\frac{\omega_D}{2\pi} = 69$ Hz

Floquet Analysis: An Analytical Approach

Simulations

$$i\hbar\frac{\partial\psi_B}{\partial t} = \left(-\frac{\hbar^2}{2m_B}\nabla^2 + V_B + g_{BB}[1+\frac{\mathcal{A}}{a_{BB}}\cos(\omega_D t)]|\psi_B|^2 + g_{BA}|\psi_A|^2\right)\psi_B$$

$$i\hbar\frac{\partial\psi_B}{\partial t} = \left(-\frac{\hbar^2}{2m_B}\nabla^2 + V_B[1+b\cos(\omega_D t)] + g_{BB}|\psi_B|^2 + g_{BA}|\psi_A|^2\right)\psi_B$$

Maity, et al., PRA, 102, 033320 (2020)

Here we need to specify
Surface tension coefficient

Floquet Theory

Floquet Analysis

Mathieu Equation : $$\ddot{\zeta}_m + \omega_m^2 \left[1 - (b/b_{0m}) \cos(\omega_D t)\right] \zeta_m = 0$$

$$\omega_m^2 = \frac{\sigma}{R^3} \frac{m(m^2-1)}{(m_B n_B - m_A n_A)}$$

$$b_{0m} = \frac{\sigma(m^2-1)}{m_B \omega^2 n_B R^3}$$

$$\lim_{k\to 0} \frac{K_m(kR)}{kK'_m(kR)} = \lim_{k\to 0} \frac{I_m(kR)}{kI'_m(kR)} = \frac{R}{m}$$

Floquet Expansion : $$\zeta_m(t) = e^{(s+i\alpha\omega_D)t} \sum_{p=-\infty}^{\infty} \zeta_m^{(p)} e^{ip\omega_D t}$$

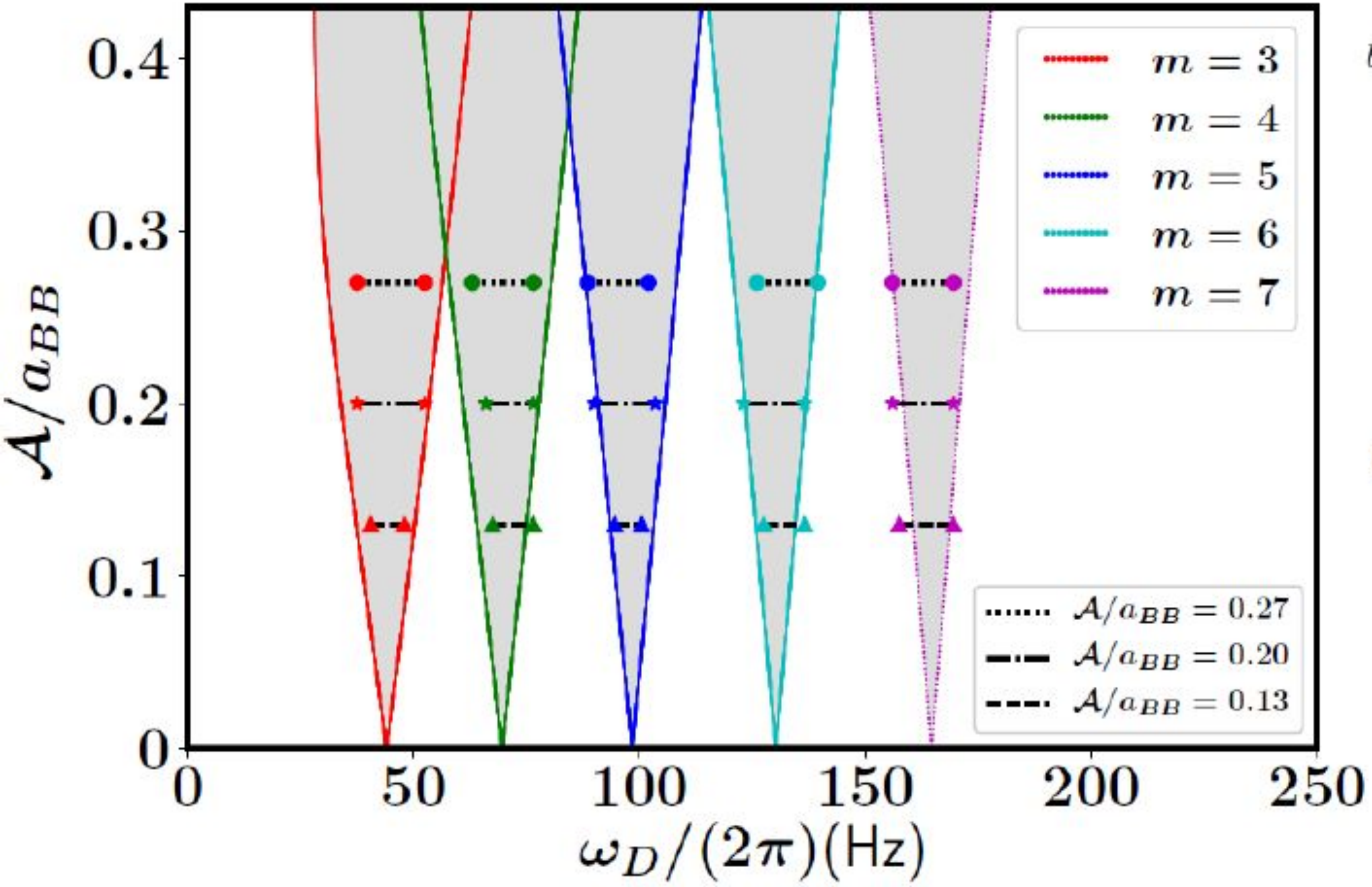
s - growth rate

$(s+i\alpha\omega_D)$ - Floquet exponent

Linear Difference Equation : $$A_m^{(p)} \zeta_m^{(p)} = b\left(\zeta_m^{(p-1)} + \zeta_m^{(p+1)}\right)$$

$$A_m^{(p)} = \left[-2(p+\alpha)^2 \omega_D^2 \frac{(m_B n_B - m_A n_A)}{m m_B \omega^2 n_B} + \frac{2\sigma(m^2-1)}{R^3 n_B m_B \omega^2}\right]$$

Marginal Stability Diagram



$b = 8\pi\hbar^2\mathcal{A}|\Psi_B(r=R,\theta,z)|^2/(m_B^2\omega^2R^2)$

σ = 1.1*10^{-18} ± 5% N/m

Conclusions

- Observed spontaneously formed star-shaped surface patterns in single component BEC : Experiment

- Patterns are controlled externally by changing amplitude and frequency of driving field--- parametrically excited by modulating the scattering length near the Feshbach resonance

- Parametric oscillations simulated at the mean-field level: Numerically solving Gross-Pitaveskii equation (Reduced Mathieu Equation)

- Another interpretation: Known oscillating patterns help to characterize unknown scattering lengths of ultra-cold atoms.

- Modes of experimentally observed excited subharmonic star shaped patterns are identified theoretically: Floquet analysis

- Precisely measure the dispersion relation of the collective excitations

- Identify the resonant frequencies of the patterns by comparing experimental/ theoretical (effective Mathieu equation) patterns vs Floquet analysis

- Floquet analysis is carried out to estimate interfacial tension of the binary phase separated BEC in experimentally possible regime.

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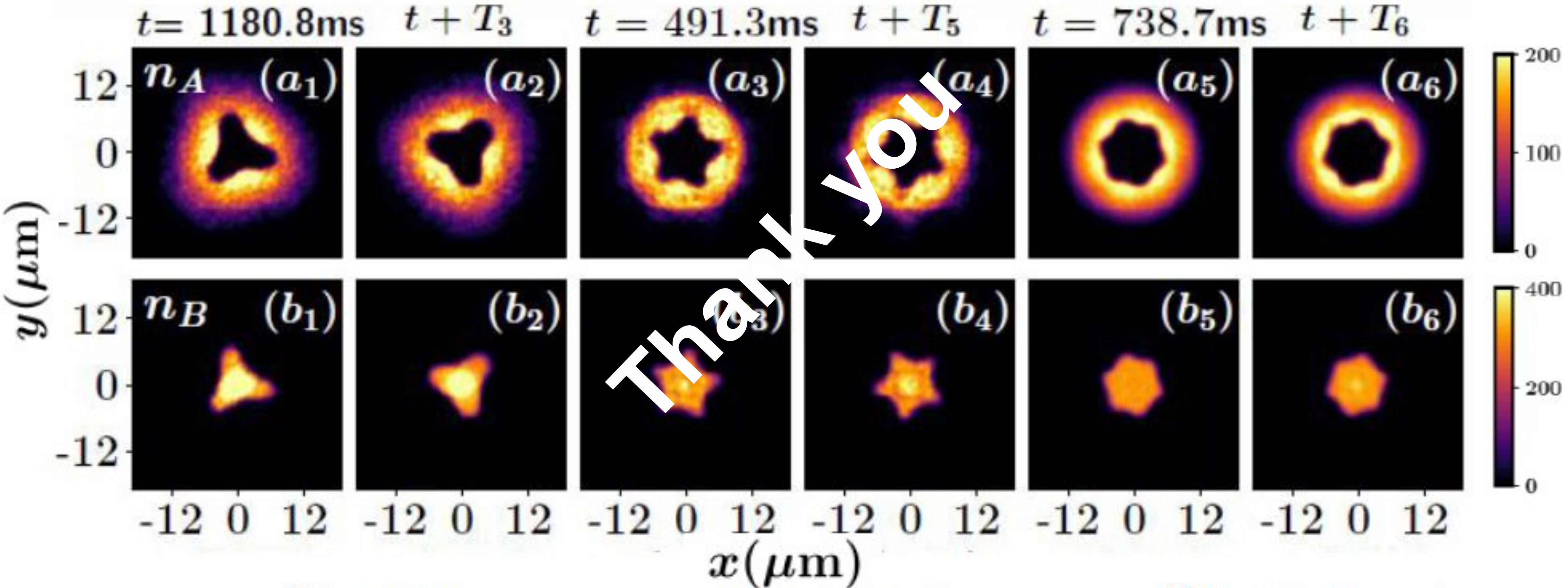
Subharmonically Excited Patterns

m=3 **m=5** **m=6**



$\frac{\omega_D}{2\pi}$ **= 48 Hz**

$\frac{\omega_D}{2\pi}$ **= 95.1 Hz**

$\frac{\omega_D}{2\pi}$ **= 132 Hz**

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Pulse- and continuously driven many-body quantum dynamics of bosonic impurities in a Bose-Einstein condensate

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Periodically driven harmonic oscillator potential of the impurities

$$V(x,t) = \frac{1}{2}M\omega^2\left(x - A\sin\left(\omega_D t\right)\right)^2$$

Acousto-optical modulators

⇒ Induce extra interaction potential among the bosonic impurities apart from s-wave repulsion

Floquet theory for 2nd order ODE with Periodic Coefficients

□ Stability Analysis

$\ddot{x} + f(t)x = 0; \quad f(t+T) = f(t)$

→ Linear equation: $x_1(t)$ & $x_2(t)$ two linearly ind. sol^{n}s => $x(t) = c_1x_1(t) + c_2x_2(t)$ *also sol*^{n}

Shift the ODE $@t = t + T$;

$$\ddot{x}(t+T) + f(t+T)x(t+T) = 0; \implies \ddot{x}(t+T) + f(t)x(t+T) = 0$$

$\Rightarrow x(t)$ solution means $x(t+T)$ is also solution.

$\Rightarrow$ Define $x_1(t+T) = \alpha x_1(t) + \beta x_2(t)$ & $x_2(t+T) = \gamma x_1(t) + \gamma x_2(t)$

$$\Rightarrow x(t+T) = c_1[\alpha x_1(t) + \beta x_2(t)] + c_2[\gamma x_1(t) + \delta x_2(t)] = \underbrace{(c_1\alpha + c_2\beta)}_{A}x_1 + \underbrace{(c_1\gamma + c_2\delta)}_{B}x_2$$

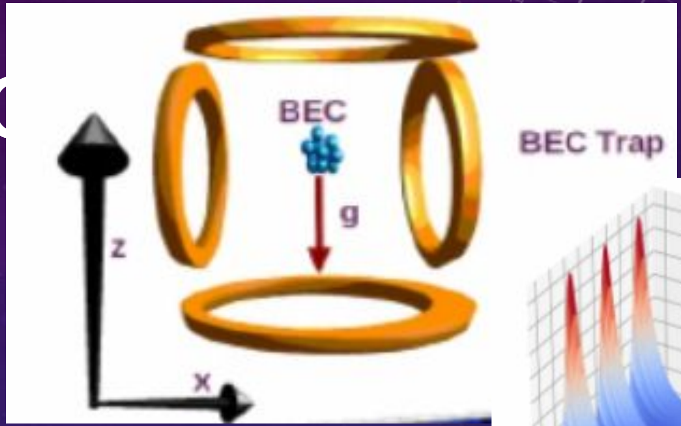
$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$=M $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$; Choose $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is eigenvector of M associated with eigenvalue λ

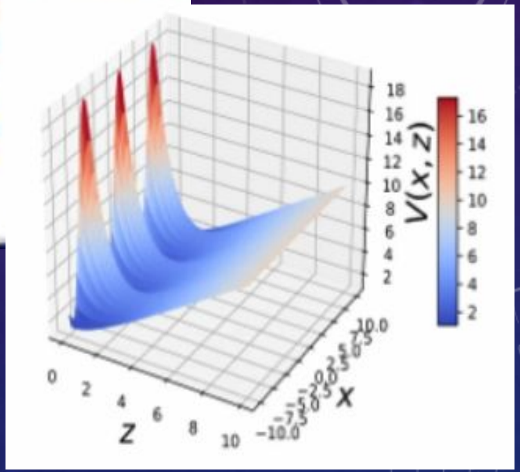
$\implies x(t+T) = \lambda x(t) \implies x(t)$ is periodic within a scale factor $\lambda = e^{\mu T}$, say.

- Define, periodic func^n $P(t)\ s.t.\ x(t) = e^{\mu t}P(t)$ & $P(t+T) = P(t) \Leftrightarrow x(t+T) = e^{\mu T}x(t)$
- Sign of μ decides the stability => Either exponential growth or decay x periodic function

Mean-Field Theory: Gross-Pitaevskii (GP) Equatio

$$\hat{H} = \int dr\, \hat{\Psi}^\dagger(r,t)\left[-\frac{\hbar^2}{2m}\nabla^2 + V_{trap}(r)\right]\hat{\Psi}(r,t)$$





$$+\frac{1}{2}\int dr \int dr'\, \hat{\Psi}^\dagger(r,t)\, \hat{\Psi}^\dagger(r,t) V(r-r')\, \hat{\Psi}(r',t)\hat{\Psi}(r,t)$$

$$V(r-r') = \text{Contact Potential (s−wave approx.)}$$

$$= U\delta(r-r') = \frac{4\pi\hbar^2}{m} a_{scat}\, \delta(r-r')$$

$$i\hbar\frac{\partial\hat{\Psi}^\dagger}{\partial t} = \left[\hat{\Psi}^\dagger(r,t), \hat{H}\right]$$

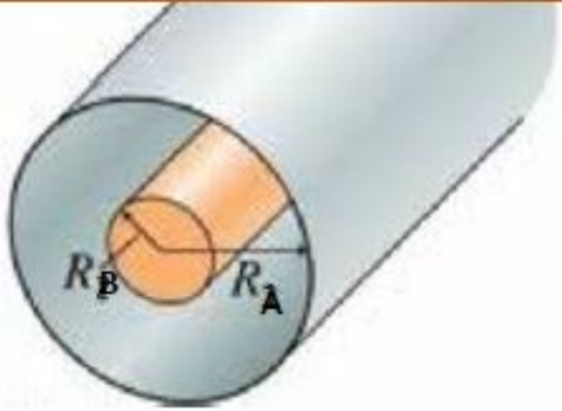
$$\Longrightarrow i\hbar\frac{\partial\hat{\Psi}^\dagger}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{trap}(r) + U\hat{\Psi}^\dagger\hat{\Psi}\right]\hat{\Psi}^\dagger(r,t)$$

$$\Longrightarrow i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\left(\partial_x^2 + \partial_y^2\right)^2 + \frac{1}{2}m\omega_r^2(x^2+y^2) + g_{2D}\left(1 + \frac{\bar{\bar{a}}_m}{a_{scat}}\cos(\omega_D t)\right)|\psi|^2\right]\psi(r,t)$$

Madelung Transformation: $\psi = \sqrt{n}e^{i\phi}$ & Assume density disturbance $\delta n = \zeta_l r^l e^{il\phi}$

Mathieu equation: $\ddot{\zeta}_l(t) + \omega_l^2[1 + (\bar{\bar{a}}_m/a_{scat})\cos(\omega_m t)]\zeta_l(t) = 0$

Floquet Analysis

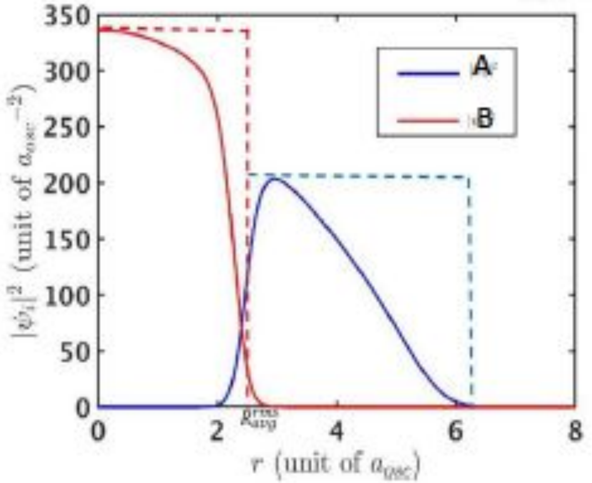


Wavefunctions: $\Psi_j(r,\theta,z) = \sqrt{n_j(r,\theta,z)}e^{i\phi_j} \qquad (i,j = A, B)$

Velocity: $\boldsymbol{v}_j = \frac{\hbar}{m_j}\nabla\phi_j$

Hydrodynamical form: $-m_j\dfrac{\partial \boldsymbol{v}_j}{\partial t} = \dfrac{\nabla P_j}{n_j}, \qquad \nabla^2\phi_j = 0$

$$n_A = 0 \text{ for } r < R \text{ and } n_B = 0 \text{ for } r > R$$



Effective Pressure:

$$P_A = \frac{1}{2}(m_A n_A v_A^2) + \frac{\hbar^2\sqrt{n_A}}{2m_A}\nabla^2\sqrt{n_A} + g_{AA}n_A^2 + \frac{1}{2}m_A n_A\omega^2(r^2 + \lambda_A^2 z^2)$$

$$P_B = \frac{1}{2}(m_B n_B v_B^2) + \frac{\hbar^2\sqrt{n_B}}{2m_B}\nabla^2\sqrt{n_B} + g_{BB}n_B^2 + \frac{1}{2}m_B n_B\omega^2(r^2 + \lambda_B^2 z^2) + \frac{1}{2}m_B n_B\omega^2 r^2 b\cos(\omega_D t))$$

Floquet Analysis

Normal stress jump condition :

$$\left[P_B - P_A\right]_{r=R+\zeta} = \sigma\left[\frac{1}{R_1} + \frac{1}{R_2}\right]$$

(Young–Laplace equation)

$$\left(\hbar n_A \frac{\partial \phi_A}{\partial t} - \hbar n_B \frac{\partial \phi_B}{\partial t}\right)_{|r=R} = -R m_B \omega^2 n_B b \cos(\omega_D t)\zeta - \sigma\left(\frac{1}{R^2} + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)\zeta$$

Expansions :

$$\zeta(\theta, z, t) = \sum_{m=1}^{\infty} \zeta_m(t) e^{i(m\theta+kz)}$$

$$\phi_A(r, \theta, z, t) = \sum_{m=1}^{\infty} \frac{d\zeta_m(t)}{dt} \frac{m_A K_m(kr)}{\hbar k K'_m(kR)} e^{i(m\theta+kz)}$$

$$\phi_B(r, \theta, z, t) = \sum_{m=1}^{\infty} \frac{d\zeta_m(t)}{dt} \frac{m_B I_m(kr)}{\hbar k I'_m(kR)} e^{i(m\theta+kz)}$$