DOUBLE SHUFFLE RELATIONS BETWEEN MZVS

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Abstract. We review the family of double shuffle relations between multiple zeta values (MZVs), the construction of the scheme attached to this family of relations, and Racinet's theorem according to which it is a torsor under the action of a certain pro-unipotent group. Time permitting, we will discuss the intepretation of this group and scheme in terms of stabilizers (joint work with H. Furusho).

0.1. **The family of double shuffle relations between multiple zeta values (MZVs).** The dual nature of the MZV numbers, which are both iterated integrals and iterated sums, can be used to exhibit a family of relations called the double shuffle relations, alternative to the associator family of relations ([Rac, IKZ]). Let us give some examples:

Shuffle product:

$$\zeta(a)\zeta(b) = \int_{0<s_1<\cdots<s_a<1} \frac{ds_1}{1-s_1} \wedge \frac{ds_2}{s_2} \wedge \cdots \wedge \frac{ds_a}{s_a} \times \int_{0<t_1<\cdots<t_b<1} \frac{dt_1}{1-t_1} \wedge \frac{dt_2}{t_2} \wedge \cdots \wedge \frac{dt_b}{t_b}$$

$$= \sum \int_0^1 \text{all shuffles} = \sum_{i+j=a+b} \left\{ \binom{i-1}{a-1} + \binom{j-1}{b-1} \right\} \zeta(i,j)$$

Harmonic product:

$$\zeta(a)\zeta(b) = \sum_{0<k} \frac{1}{k^a} \cdot \sum_{0<l} \frac{1}{l^b} = (\sum_{0<k<l} + \sum_{0<k=l} + \sum_{0<l<k}) \frac{1}{k^a l^b}$$

$$= \zeta(a,b) + \zeta(a+b) + \zeta(b,a).$$

All these relations can be condensed in the relations obeyed by the pair $(2\pi\mathrm{i}, \Phi_{\mathrm{KZ}})$, which we explain in the formalism of [Rac].

0.2. **Racinet's formalism.** Let $\mathbf{k}$ be a commutative associative algebra containing $\mathbb{Q}$.

Definition 0.1. Set $\mathcal{V} := \mathbf{k}\langle e_0, e_1\rangle$, free graded algebra over e_0, e_1 of deg=1.
Coproduct $\Delta^{\mathcal{V}} : \mathcal{V} \to \mathcal{V}^{\otimes 2}$, $e_i \mapsto e_i \otimes 1 + 1 \otimes e_i$.

Definition 0.2. Subalgebra $\mathcal{W} := \mathbf{k} \oplus \mathcal{V}e_1 (\hookrightarrow \mathcal{V})$.

Lemma 0.3. *(a) $\mathcal{W}$ is freely generated by $y_1, y_2, \ldots$, where $y_n := -e_0^{n-1}e_1$.*

(b) The harmonic coproduct

$$\Delta^{\mathcal{W}} : \mathcal{W} \to \mathcal{W}^{\otimes 2}, \quad \Delta^{\mathcal{W}}(y_n) = y_n \otimes 1 + 1 \otimes y_n + \sum_{k+l=n} y_k \otimes y_l$$

equips $\mathcal{W}$ with Hopf algebra structure.

Definition 0.4. Define $\mathcal{M} := \mathcal{V}/\mathcal{V}e_0$; this is a left module over $\mathcal{V}$, and therefore $\mathcal{W}$. Define $\mathrm{can} : \mathcal{V} \twoheadrightarrow \mathcal{M}$ to be the canonical projection.

Lemma 0.5. *$\mathcal{M}$ is a free $\mathcal{W}$-module of rank 1, generated by $\mathbf{1} := \mathrm{can}(1)$.*

Definition 0.6. $\Delta^{\mathcal{M}} : \mathcal{M} \to (\mathcal{M})^{\otimes 2}$ is the transport of $\Delta^{\mathcal{W}}$ under the isomorphism $\mathcal{W} \to \mathcal{M}$ induced by action on $\mathbf{1}$.

($\Delta^{\mathcal{W}}$, $\Delta^{\mathcal{M}}$ are both denoted $\Delta_\star$ in [Rac].)

Denote with hats the degree completions of the graded objects $\mathcal{V}$, $\mathcal{W}$, $\mathcal{M}$, $\Delta^{\mathcal{V}}$, $\Delta^{\mathcal{W}}$, $\Delta^{\mathcal{M}}$, hence $\hat{\mathcal{V}}$, $\hat{\mathcal{W}}$, $\hat{\mathcal{M}}$, $\hat{\Delta}^{\mathcal{V}}$, $\hat{\Delta}^{\mathcal{W}}$, $\hat{\Delta}^{\mathcal{M}}$.

Notation: For $\Phi \in \hat{\mathcal{V}} := \mathbf{k}\langle\langle e_0, e_1\rangle\rangle$, set

$$\Gamma_\Phi(-e_1) := \exp(\sum_{n\geq 1} \frac{1}{n}(\Phi|e_0^{n-1}e_1)e_1^n) \in \hat{\mathcal{V}}, \quad \Phi_\star := \mathrm{can}(\Gamma_\Phi(-e_1)^{-1}\Phi) \in \hat{\mathcal{M}}.$$

0.3. Formulation of double shuffle relations between MZVs.

Definition 0.7. (a) $u : \mathcal{V} \to \mathcal{V}[\alpha, \beta]$ is the algebra morphism given by $e_0 \mapsto e_0 - \alpha$, $e_1 \mapsto e_1 - \beta$.

(b) $v : \mathcal{V}[\alpha, \beta] \to \mathcal{V}$ is the $\mathbf{k}$-module morphism given by $w\alpha^p\beta^q \mapsto e_1^q w e_0^p$.

(c) $\mathrm{Adm} := \sqcup_{m\geq 1}\mathbb{Z}_{\geq 1}^{m-1} \times \mathbb{Z}_{\geq 2}$ is called the set of admissible indices.

(d) $w : \mathrm{Adm} \to \mathcal{V}$ is such that $w(k_1, \ldots, k_m) := v \circ u(e_0^{k_m-1}e_1 \cdots e_0^{k_1-1}e_1)$.

Lemma 0.8. *(see* [LM]*)*

$$\Phi_{\mathrm{KZ}} = 1 + \sum_{(k_1,\ldots,k_m)\in\mathrm{Adm}} (-1)^m \zeta(k_1, \ldots, k_m) \cdot w(k_1, \ldots, k_m) \in 1 + (\hat{\mathcal{V}}_{\mathbb{C}})_{\geq 2}$$

($(-)_{\geq 2}$: sum of components of degree ≥ 2).

Lemma 0.9. *(see* [Rac, IKZ]*) One has:*

(shuffle relation) $\hat{\Delta}(\Phi_{\mathrm{KZ}}) = \Phi_{\mathrm{KZ}} \otimes \Phi_{\mathrm{KZ}}$ *(relation in* $(\hat{\mathcal{V}}_{\mathbb{C}})^{\hat{\otimes} 2}$*);*

(harmonic relation) $\hat{\Delta}^{\mathcal{M}}(\Phi_{\mathrm{KZ},\star}) = \Phi_{\mathrm{KZ},\star} \otimes \Phi_{\mathrm{KZ},\star}$ *(relation in* $(\hat{\mathcal{M}}_{\mathbb{C}})^{\hat{\otimes} 2}$*);*

(normalization) $\zeta(2) = -(2\pi\mathrm{i})^2/24$.

0.4. Double shuffle schemes.

Definition 0.10. The map $\Phi(-) : \mathbf{k}^{\mathrm{Adm}} \to 1 + \hat{\mathcal{V}}^{\mathrm{DR}}_{\geq 2}$ is defined by

$$\Phi(\xi) := 1 + \sum_{(k_1,\ldots,k_m)\in\mathrm{Adm}} (-1)^m \xi(k_1, \ldots, k_m) \cdot w(k_1, \ldots, k_m).$$

Lemma 0.11. *If $\Phi \in 1 + \hat{\mathcal{V}}_{\geq 2}$ is such that $\hat{\Delta}^{\mathcal{V}}(\Phi) = \Phi \otimes \Phi$, then there exists a unique $\xi \in \mathrm{k}^{\mathrm{Adm}}$ such that $\Phi = \Phi(\xi)$.*

Proof. By assumption, $\varphi := \log\Phi \in \hat{\mathbb{L}}(e_0, e_1)_{\geq 2}$. The restriction of u on $\hat{\mathbb{L}}(e_0, e_1)_{\geq 2}$ is the identity, which since u is an algebra morphism implies $u(\Phi) = \Phi$. Therefore $u(\Phi) \in \hat{\mathcal{V}}$, which since the restriction of v to $\hat{\mathcal{V}} \subset \hat{\mathcal{V}}[[\alpha,\beta]]$ is the identity implies $v(u(\Phi)) = u(\Phi)$, so that finally $\Phi = v \circ u(\Phi)$, and

$$\Phi - 1 = v \circ u(\Phi - 1). \tag{0.4.1}$$

One has $\mathcal{V}_{\geq 1} = e_0\mathcal{V}e_1 \oplus (e_1\mathcal{V} + \mathcal{V}e_1)$. The restriction of $v \circ u$ to $e_1\mathcal{V} + \mathcal{V}e_1$ is zero, which implies that if one denotes by $\pi : \mathcal{V}_{\geq 1} \to e_0\mathcal{V}e_1$ the projection corresponding to the above decomposition, one has the identity $v \circ u = v \circ u \circ \pi$, in particular

$$v \circ u(\Phi - 1) = v \circ u \circ \pi(\Phi - 1). \tag{0.4.2}$$

Then $\pi(\Phi - 1)$ belongs to $e_0\mathcal{V}e_1$. The map $\mathrm{Adm} \to e_0\mathcal{V}e_1$ given by $(k_1, \dots, k_m) \mapsto e_0^{k_m - 1}e_1 \cdots e_0^{k_0 - 1}e_1$ is a basis of $e_0\mathcal{V}e_1$, therefore there exists $\xi \in \mathbf{k}^{\mathrm{Adm}}$ such that

$$\pi(\Phi - 1) = \sum_{(k_1,\dots,k_m)\in \mathrm{Adm}} (-1)^m \xi(k_1, \dots, k_m) e_0^{k_m - 1}e_1 \cdots e_0^{k_0 - 1}e_1. \tag{0.4.3}$$

Then

$$\Phi = 1 + v \circ u(\Phi - 1) = 1 + v \circ u \circ \pi(\Phi - 1) = 1 + v \circ u\Big(\sum_{(k_1,\dots,k_m)\in \mathrm{Adm}} (-1)^m \xi(k_1, \dots, k_m) e_0^{k_m - 1}e_1 \cdots e_0^{k_0 - 1}e_1\Big)$$

$$= 1 + \sum_{(k_1,\dots,k_m)\in \mathrm{Adm}} (-1)^m \xi(k_1, \dots, k_m) w(k_1, \dots, k_m),$$

where the first equality follows from (0.4.1), the second equality from (0.4.2), the third equality from (0.4.3), and the last equality from the definition of $w(k_1, \dots, k_m)$. □

Definition 0.12. For $\mu \in \mathbf{k}$, set

$$\mathsf{DMR}_\mu(\mathbf{k}) := \{\Phi \in 1 + \hat{\mathcal{V}}_{\geq 2} | \hat{\Delta}^{\mathcal{V}}(\Phi) = \Phi \otimes \Phi, \hat{\Delta}^{\mathcal{M}}(\Phi_\star) = \Phi_\star \otimes \Phi_\star, (\Phi|e_0e_1) = \mu^2/24\}.$$

Corollary 0.13. *For $\lambda \in \mathbf{k}$, the map*

$$\{\xi \in \mathbf{k}^{\mathrm{Adm}} | \Phi(\xi) \in \mathsf{DMR}_\mu(\mathbf{k})\} \to \mathsf{DMR}_\mu(\mathbf{k})$$

given by $\xi \mapsto \Phi(\xi)$ is a bijection.

Then the double shuffle relations on MZVs are expressed as the statement $\Phi_{\mathrm{KZ}} \in \mathsf{DMR}_{2\pi\mathrm{i}}(\mathbb{C})$.

0.5. **The torsor of double shuffle relations family of double shuffle relations.**

Definition 0.14. $\mathcal{G} := \{g \in 1 + \hat{\mathcal{V}}_{\geq 2} | \hat{\Delta}^{\mathcal{V}}(g) = g \otimes g\}$.

$\mathcal{G}$ may be equipped with a group product $*$ (twisted Magnus group structure), such that the map $\mathcal{G} \to \mathrm{Aut}(\hat{\mathcal{V}})$, $g \mapsto \mathrm{aut}_g^{\mathcal{V},(1)} = [e_1 \mapsto e_1, e_0 \mapsto g e_0 g^{-1}]$ is a group morphism.

Theorem 0.15. *(see* [Rac]*) (a)* $\mathsf{DMR}_0(\mathbf{k})$ *is a subgroup of* $(\mathcal{G}, *)$.

(b) The left action of $\mathcal{G}$ on itself induces a torsor structure on $\mathsf{DMR}_\lambda(\mathbf{k})$ *over* $\mathsf{DMR}_0(\mathbf{k})$.

0.6. **Double shuffle-associator relations.**

Definition 0.16. For $\mu \in \mathbf{k}^\times$, set $\mathsf{M}_\mu(\mathbf{k}) := \{\Phi \in \mathsf{M}(\mathbf{k}) | (\Phi|e_0e_1) = \mu^2/24\}$.

Theorem 0.17. *(a) Group inclusion* $\mathsf{GRT}_1(\mathbf{k})^{op} \subset \mathsf{DMR}_0(\mathbf{k})$.

(b) For $\mu \in \mathbf{k}^\times$*, inclusion* $\mathsf{M}_\mu(\mathbf{k}) \subset \mathsf{DMR}_\mu(\mathbf{k})$ *compatible with group actions.*

Proof. Beside the tentative proof of [DT], there are two (at the moment unrelated) proofs available:

• the proof from [Fu] uses explicit formulas describing both sides: will be reviewed in lecture by L. Schneps.

• the proof from [EF2] is based on the stabilizer interpretation of $\mathsf{DMR}_0(\mathbf{k})$ ([EF0]) and on relations DMR-braids ([DT, EF1]). The first topic, as well as the stabilizer interpretation of $\mathsf{DMR}_0(\mathbf{k})$ (itself a consequence of the DMR-braids relations), will be reviewed in the rest of the lecture. □

0.7. **Interpretation of the group** $\mathsf{DMR}_0(\mathbf{k})$**.**

Definition 0.18. An $\mathbf{k}$-algebra-module is a pair (A, M) of an associative $\mathbf{k}$-algebra A and a A-module M. A morphism $(A, M) \to (A', M')$ of algebra-modules is a pair of an algebra morphism $A \to A'$ and a $\mathbf{k}$-module morphism $M \to M'$ which are compatible with the actions.

$\mathbf{k}$-algebra-modules forms a category, hence a group $\mathrm{Aut}_{\mathbf{k}\text{-alg-mod}}(A, M)$ for (A, M) an algebra-module.

Lemma 0.19. *(a)* $(\hat{\mathcal{V}}, \hat{\mathcal{V}})$ *is an algebra-module, where* $\hat{\mathcal{V}}$ *is viewed as a left* $\hat{\mathcal{V}}$*-module.*

(b) there is a diagram $(\hat{\mathcal{V}}, \hat{\mathcal{V}}) \to (\hat{\mathcal{V}}, \hat{\mathcal{M}}) \leftarrow (\hat{\mathcal{W}}, \hat{\mathcal{M}})$ *of algebra-modules.*

(c) There is a group morphism $(\mathcal{G}, *) \to \mathrm{Aut}_{\mathbf{k}\text{-alg-mod}}(\hat{\mathcal{V}}, \hat{\mathcal{V}})$ *given by* $g \mapsto (\mathrm{aut}_g^{\mathcal{V},(1)}, \mathrm{aut}_g^{\mathcal{V},(10)})$*, where* $\mathrm{aut}_g^{\mathcal{V},(10)}(v) := \mathrm{aut}_g^{\mathcal{V},(1)}(v) \cdot g$.

(d) ... and group morphisms from $(\mathcal{G}, *)$ *to the automorphism groups of the objects of the diagram from (b) and compatible with this diagram, resulting in a group morphism*

$$(\mathcal{G}, *) \to \mathrm{Aut}_{\mathbf{k}\text{-alg-mod}}(\hat{\mathcal{W}}, \hat{\mathcal{M}})$$

denoted $g \mapsto (\mathrm{aut}_g^{\mathcal{W},(1)}, \mathrm{aut}_g^{\mathcal{M},(10)})$ *...*

(e) ... which can be modified into a group morphism denoted $g \mapsto ({}^\Gamma\mathrm{aut}_g^{\mathcal{W},(1)}, {}^\Gamma\mathrm{aut}_g^{\mathcal{M},(10)})$*, where*

$${}^\Gamma\mathrm{aut}_g^{\mathcal{W},(1)}(w) := \Gamma_g(-e_1)^{-1} \cdot \mathrm{aut}_g^{\mathcal{W},(1)}(w) \cdot \Gamma_g(-e_1), \quad {}^\Gamma\mathrm{aut}_g^{\mathcal{M},(10)}(m) := \Gamma_g(-e_1)^{-1} \cdot \mathrm{aut}_g^{\mathcal{M},(10)}(m).$$

(f) the group morphism $(\mathcal{G}, *) \to \mathrm{Aut}_{\mathbf{k}\text{-mod}}(\hat{\mathcal{M}})$ *resulting from (e) induces a group morphism* $(\mathcal{G}, *) \to \mathrm{Aut}_{\mathbf{k}\text{-mod}}(\mathrm{Hom}_{\mathbf{k}\text{-mod}}(\hat{\mathcal{M}}, \hat{\mathcal{M}}^{\hat{\otimes}2}))$ *given by*

$$g \bullet \Delta := ({}^\Gamma\mathrm{aut}_g^{\mathcal{M},(10)})^{\otimes 2} \circ \Delta \circ ({}^\Gamma\mathrm{aut}_g^{\mathcal{M},(10)})^{-1}.$$

Theorem 0.20. *(*[EF0]*) $\mathsf{DMR}_0(\mathbf{k}) = \mathsf{Stab}_{\mathcal{G}}(\hat\Delta^{\mathcal{M}})$ where the stabilizer is relative to the action of (f), and we recall $\hat\Delta^{\mathcal{M}} \in \mathrm{Hom}_{\mathbf{k}\text{-mod}}(\hat{\mathcal{M}}, \hat{\mathcal{M}}^{\hat\otimes 2})$.*

Proof. Based on [Rac]. □

0.8. **Interpretation of the torsor structure of $\mathsf{DMR}_\mu(\mathbf{k})$.** The objects $\mathcal{V}, \mathcal{W}, \mathcal{M}, \Delta^{\mathcal{V}}, \Delta^{\mathcal{W}}, \Delta^{\mathcal{M}}$ and their completed versions should be viewed as associated to the de Rham geometry of $\mathfrak{M}_{0,4}$. They will henceforth be denoted with a DR superscript, hence $\mathcal{V}^{\mathrm{DR}}, \mathcal{W}^{\mathrm{DR}}, \mathcal{M}^{\mathrm{DR}}, \Delta^{\mathcal{V},\mathrm{DR}}, \Delta^{\mathcal{W},\mathrm{DR}}, \Delta^{\mathcal{M},\mathrm{DR}}$. We now introce the Betti counterparts, denoted $\mathcal{V}^{\mathrm{B}}, \mathcal{W}^{\mathrm{B}}, \mathcal{M}^{\mathrm{B}}, \Delta^{\mathcal{V},\mathrm{B}}, \Delta^{\mathcal{W},\mathrm{B}}, \Delta^{\mathcal{M},\mathrm{B}}$ and their completions.

Definition 0.21. (a) F_2 is the free group with two generators X_0, X_1.

(b) $\mathcal{V}^{\mathrm{B}} := \mathbf{k}F_2$ is the group algebra of F_2.

(c) $\mathcal{W}^{\mathrm{B}} := \mathbf{k} \oplus \mathcal{V}^{\mathrm{B}}(X_1 - 1)$ ($\hookrightarrow \mathcal{V}^{\mathrm{B}}$) is defined as a subalgebra of $\mathcal{V}^{\mathrm{B}}$.

(d) $\mathcal{M}^{\mathrm{B}} := \mathcal{V}^{\mathrm{B}}/\mathcal{V}^{\mathrm{B}}(X_0 - 1)$ is a left $\mathcal{V}^{\mathrm{B}}$-module, therefore also $\mathcal{W}^{\mathrm{B}}$-module.

The coproduct on $\mathcal{V}^{\mathrm{B}}$ arising from the group algebra structure is denoted $\Delta^{\mathcal{V},\mathrm{B}}$ (not to be used here).

Lemma 0.22. *(a) $\mathcal{M}^{\mathrm{B}}$ is a free $\mathcal{W}^{\mathrm{B}}$-module of rank 1, generated by $\mathbf{1}^{\mathrm{B}}$:=projection of 1.*

(b) $\mathcal{W}^{\mathrm{B}}$ is presented by the generators $Y_n^{\pm}$ ($n > 0$) X_1 and X_1^{-1} and the relations $X_1 \cdot X_1^{-1} = X_1^{-1} \cdot X_1 = 1$, where $Y_n^+ = (X_0 - 1)^{n-1}X_0(1 - X_1)$ and $Y_n^- = (X_0^{-1} - 1)^{n-1}X_0^{-1}(1 - X_1^{-1})$.

(c) There exists an algebra morphism $\Delta^{\mathcal{W},\mathrm{B}} : \mathcal{W}^{\mathrm{B}} \to (\mathcal{W}^{\mathrm{B}})^{\otimes 2}$, uniquely determined by

$$\Delta^{\mathcal{W},\mathrm{B}} : X_1^{\pm 1} \to X_1^{\pm 1} \otimes X_1^{\pm 1}, \quad Y_n^{\pm} \mapsto Y_n^{\pm} \otimes 1 + 1 \otimes Y_n^{\pm} + \sum_{n',n''>0, n'+n''=n} Y_{n'}^{\pm} \otimes Y_{n''}^{\pm}.$$

(d) There exists a $\mathbf{k}$-module map $\Delta^{\mathcal{M},\mathrm{B}} : \mathcal{M}^{\mathrm{B}} \to (\mathcal{M}^{\mathrm{B}})^{\otimes 2}$, uniquely determined by $\Delta^{\mathcal{M},\mathrm{B}}(w \cdot \mathbf{1}^{\mathrm{B}}) = \Delta^{\mathcal{W},\mathrm{B}}(w) \cdot (\mathbf{1}^{\mathrm{B}})^{\otimes 2}$ for any $w \in \mathcal{W}^{\mathrm{B}}$.

The algebra $\mathcal{V}^{\mathrm{B}}$ is completed according to the powers of the augmentation ideal. This induces completions of $\mathcal{W}^{\mathrm{B}}, \mathcal{M}^{\mathrm{B}}, \Delta^{\mathcal{V},\mathrm{B}}, \Delta^{\mathcal{W},\mathrm{B}}, \Delta^{\mathcal{M},\mathrm{B}}$, denoted with hats.

Lemma 0.23. *Let $\mu \in \mathbf{k}^\times$.*

(a) There is an topological algebras isomorphism $\mathrm{iso}_\mu^{\mathcal{V}} : \hat{\mathcal{V}}^{\mathrm{B}} \to \hat{\mathcal{V}}^{\mathrm{DR}}$ induced by $X_0 \mapsto \exp(\mu e_0)$, $X_1 \mapsto \exp(\mu e_1)$.

(b) It induces an isomorphism of topological $\mathbf{k}$-modules $\mathrm{iso}_\mu^{\mathcal{M}} : \hat{\mathcal{M}}^{\mathrm{B}} \to \hat{\mathcal{M}}^{\mathrm{DR}}$.

Theorem 0.24. *(*[EF1]*) Let $\mu \in \mathbf{k}^\times$. Then $\mathsf{DMR}_\mu(\mathbf{k}) = \{g \in \mathcal{G} | $the diagram below commutes$\}$, the diagram being*

$$\begin{tikzcd} \hat{\mathcal{M}}^{\mathrm{B}} \arrow[r, "\hat\Delta^{\mathcal{M},\mathrm{B}}"] \arrow[d, "{}^\Gamma\mathrm{aut}_g^{\mathcal{M}} \circ \mathrm{iso}_\mu^{\mathcal{M}}"'] & (\hat{\mathcal{M}}^{\mathrm{B}})^{\hat\otimes 2} \arrow[d, "({}^\Gamma\mathrm{aut}_g^{\mathcal{M}} \circ \mathrm{iso}_\mu^{\mathcal{M}})^{\otimes 2}"] \\ \hat{\mathcal{M}}^{\mathrm{DR}} \arrow[r, "\hat\Delta^{\mathcal{M},\mathrm{DR}}"'] & (\hat{\mathcal{M}}^{\mathrm{DR}})^{\hat\otimes 2} \end{tikzcd}$$

Proof. Based on the interpretation of $\Delta^{\mathcal{M},\mathrm{B}/\mathrm{DR}}$ in term of the Betti/de Rham geometry of $\mathfrak{M}_{0,4}$ and $\mathfrak{M}_{0,5}$ (see [DT], also [EF1]). □

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