PRO-UNIPOTENT GROTHENDIECK-TEICHMÜLLER THEORY

BENJAMIN ENRIQUEZ

Abstract. We review the construction of the scheme of associators. We explain its relation with formality isomorphisms for the braid groups of the plane. This leads to the definition of the prounipotent GT group and to the proof of the torsor structure of the scheme of associators. Time permitting, we will discuss the interpretation of the grt Lie algebra in terms of a tower of outer derivation Lie algebras (Ihara)

We refer to [Q] for the notions of complete Hopf algebras (CHA) and Malcev Lie algebras (MLA). We denote by $\mathcal{G}(-)$ (resp. $\mathcal{P}(-)$) the group (resp. Lie algebra) of group-like (resp. primitive) elements of a (topological) Hopf algebra.

0.1. **Computation of group completions.** Let Γ be a finitely generated discrete group. Recall that $\mathrm{gr}(\Gamma)$ is a graded $\mathbb{Z}$-Lie algebra.

Lemma 0.1. *(a)* $(\mathbb{Q}\Gamma)^\wedge := \lim(\mathbb{Q}\Gamma)/(\mathbb{Q}\Gamma)_+^n$ *is a CHA.*

(b) $\mathrm{Lie}(\Gamma) := \mathcal{P}((\mathbb{Q}\Gamma)^\wedge)$ *is a MLA.*

(c) $\mathrm{gr}(\mathbb{Q}\Gamma) := \oplus_{n\geq 0}(\mathbb{Q}\Gamma)_+^n/(\mathbb{Q}\Gamma)_+^{n+1}$ *is a graded Hopf algebra.*

(d) $\mathrm{gr}(\Gamma)_{\mathbb{Q}} = \mathcal{P}(\mathrm{gr}(\mathbb{Q}\Gamma)) = \mathrm{gr}(\mathrm{Lie}(\Gamma))$.

Let $\mathfrak{a}$ be a positively graded $\mathbb{Q}$-Lie algebra, generated in degree 1.

Lemma 0.2. *If there exists:*

(a) a graded Lie algebra morphism $\mathfrak{a} \to \mathrm{gr}(\Gamma)_{\mathbb{Q}}$*, which is an iso in degree 1;*

(b) a group morphism $\Gamma \to \mathcal{G}((U\mathfrak{a})^\wedge_{\mathbb{C}})$*, such that the induced vector space morphism* $\mathrm{gr}(\Gamma)_{\mathbb{C}} \to \mathfrak{a}[1]_{\mathbb{C}}$ *is an iso;*

then:

(c) (a) induces an iso of graded LAs $\mathfrak{a} \to \mathrm{gr}(\Gamma)_{\mathbb{Q}}$*,*

(d) (b) induces a LA morphism $\mathrm{Lie}(\Gamma)_{\mathbb{C}} \to \hat{\mathfrak{a}}_{\mathbb{C}}$ *which is an iso of filtered LAs,*

(e) ... and an iso of CHAs $(\mathbb{C}\Gamma)^\wedge \to (U\mathfrak{a})^\wedge_{\mathbb{C}}$.

Proof. The morphism from (a) is an epi since $\mathrm{gr}(\Gamma)_{\mathbb{Q}}$ is generated in degree 1. The morphism $\Gamma \to \mathcal{G}((U\mathfrak{a})^\wedge)$ from (b) induces a CHA morphism $(\mathbb{C}\Gamma)^\wedge \to (U\mathfrak{a})^\wedge$, which induces a MLA morphism $\mathrm{Lie}(\Gamma)_{\mathbb{C}} \to \hat{\mathfrak{a}}_{\mathbb{C}}$, which induces a graded LA morphism $\mathrm{gr}(\mathrm{Lie}(\Gamma)_{\mathbb{C}}) \to \mathfrak{a}_{\mathbb{C}}$ which is epi as it is epi in degree 1 and $\mathfrak{a}$ is generated in degree 1. The LAs $\mathrm{gr}(\Gamma)_{\mathbb{C}}$ and $\mathfrak{a}_{\mathbb{C}}$ are generated in degree 1 and their degree 1 parts are finite-dimensional, so for each $d \geq 1$, both $\mathrm{gr}(\Gamma)[d]$ and $\mathfrak{a}[d]$ are finite-dimensional. The existence of these two epis imply equality of dimensions, hence both epis are isos. The follows that $\mathrm{Lie}(\Gamma)_{\mathbb{C}} \to \hat{\mathfrak{a}}_{\mathbb{C}}$ is an iso of filtered LAs. □

0.2. **The case of fundamental groups/groupoids.** Let M be a manifold. Let $\mathfrak{a}$ be a positively $\mathbb{Q}$-graded Lie algebra, generated in degree 1.

Definition 0.3. $\mathrm{MC}(M,\mathfrak{a}_{\mathbb{C}}) := \{J \in \mathcal{A}^1(M)\hat{\otimes}\hat{\mathfrak{a}}_{\mathbb{C}} | dJ = (1/2)[J,J]\}$.

For $J \in \mathrm{MC}(M,\mathfrak{a}_{\mathbb{C}})$, the projection of $J[1]$ of J in $\mathcal{A}^1(M)\otimes\mathfrak{a}[1]_{\mathbb{C}}$ belongs to $\mathrm{Ker}(d : \mathcal{A}^1(M) \to \mathcal{A}^2(M)) \otimes \mathfrak{a}[1]_{\mathbb{C}}$.

Lemma 0.4. *Let $b \in M$ and $J \in \mathrm{MC}(M,\mathfrak{a})$. Assume given a graded Lie algebra morphism $\mathfrak{a} \to \mathrm{gr}(\pi_1(M,b))_{\mathbb{Q}}$, inducing an iso $\mathfrak{a}[1] \to H_1(M,\mathbb{Q})$, such that the image of $J[1] \in \mathrm{Ker}(d : \mathcal{A}^1(M) \to \mathcal{A}^2(M)) \otimes \mathfrak{a}[1]_{\mathbb{C}} \to H^1_{dR}(M,\mathbb{C}) \otimes H_1(M,\mathbb{C})$ is non-degenerate.*

Then Lem. 0.2 may be applied, using in (b) the group morphism induced by integration of the flat connection $d - J$, and its conclusions hold with $\Gamma = \pi_1(M,b)$.

To a set $B \subset M$, we associate the groupoid $\pi_1(M,B)$ with base B, and its completion $\pi_1(M,B)^\wedge$. We also associate the trivial groupoid $\mathcal{G}((U\mathfrak{a})^\wedge)_B$ with base B.

Definition 0.5. If $\mathcal{C}$ is a tensor category, a $\mathcal{C}$-enriched category with base B is a category with set of objects B, where the sets of morphisms are equipped with structures which make them objects in $\mathcal{C}$, and where the compositions are morphisms in $\mathcal{C}$.

When $\mathcal{C}$ is the category of coalgebras, the notion of $\mathcal{C}$-enriched category coincides with that of Hopf groupoid ([Fr]). We will take for $\mathcal{C}$ the category of filtered coalgebras.

Lemma 0.6. *Let $B \subset M$ be a subset and $J \in \mathrm{MC}(M,\mathfrak{a}_{\mathbb{C}})$ satisfying (a) and (b) of Lem. 0.4. Then integration of the flat connection $d - J$:*

(d') induces a filtered groupoid isomorphism $\pi_1(M,B)^\wedge \to \mathcal{G}((U\mathfrak{a})^\wedge_{\mathbb{C}})_B$,

(e') ... and an iso of categories enriched in filtered coalgebras $(\mathbb{C}\pi_1(M,B))^\wedge \to ((U\mathfrak{a})^\wedge_{\mathbb{C}})_B$.

Assume further that Σ is a discrete group, acting freely on M, preserving B, as well as on $\mathfrak{a}$, and that J is Σ-equivariant.

Lemma 0.7. Σ *acts on the source and target of (e'), and the iso (e') is equivariant.*

In this situation, orbifold fundamental groups may be constructed as follows: for $b \in B$, $\pi_1(\Sigma \setminus M, \Sigma \cdot b) = \sqcup_{\sigma\in\Sigma}\pi_1(M; b, \sigma \cdot b)$ with appropriate group law. Then there is an exact sequence $1 \to \pi_1(M,b) \to \pi_1(\Sigma \setminus M, \Sigma \cdot b) \to \Sigma \to 1$ and an iso

$$\begin{array}{ccc} \pi_1(\Sigma \setminus M, \Sigma \cdot b)^\wedge & \longrightarrow & \mathcal{G}((U\mathfrak{a})^\wedge) \rtimes \Sigma \\ & \searrow \quad \swarrow & \\ & \Sigma & \end{array}$$

where $\pi_1(\Sigma \setminus M, \Sigma \cdot b)^\wedge$ is the relative completion of the morphism $\pi_1(\Sigma \setminus M, \Sigma \cdot b) \to \Sigma$.

0.3. **Tangential base points.** Recall that a *filter basis* $\mathcal{F}$ on a set X is a nonempty subset of $\mathfrak{P}(X)$, not containing $\emptyset$, and stable under intersections. A function $f : X \to \mathbb{C}$ is then such that $\lim_{\mathcal{F}} f = 1$ iff for any $\epsilon > 0$, $f^{-1}(B(1,\epsilon))$ contains an element of $\mathcal{F}$. Example : the collection over $\epsilon > 0$ of preimages of $B(0,\epsilon)$ by $C_3(\mathbb{R}) \to \mathbb{R}$, $(z_1, z_2, z_3) \mapsto |z_{13}/z_{23}|$.

The data of a subset $B \subset M$ can be generalized to an assignment $\mathbf{B} \ni \mathbf{b} \mapsto (M_{\mathbf{b}}, \mathcal{F}_{\mathbf{b}}, F_{\mathbf{b}})$, where $M_{\mathbf{b}}$ is a connected and 1-connected submanifold, $\mathcal{F}_{\mathbf{b}}$ is a filter basis on $M_{\mathbf{b}}$, and $F_{\mathbf{b}}$ is a solution of the restriction of $dF = JF$ to $M_{\mathbf{b}}$ (the situation $B \subset M$ is recovered with $\mathbf{B} = B$, $\mathbf{b} \mapsto M_{\mathbf{b}}$ is the identity map, the datum of $\mathcal{F}_{\mathbf{b}}$ is empty, and $F_{\mathbf{b}} = 1$).

For $\mathbf{b}, \mathbf{b}' \in \mathbf{B}$, one defines the set $\pi_1(M; \mathbf{b}, \mathbf{b}')$ and the groupoid $\pi_1(M; \mathbf{B})$. Under the assumptions of Lem. 0.6, the collection of functions $F_{\mathbf{b}}$ gives rise to a groupoid morphism $\pi_1(M; \mathbf{B})^\wedge \to \mathcal{G}((U\mathfrak{a})^\wedge_{\mathbb{C}})_{\mathbf{B}}$. Lemmas 0.6 and 0.7 can then be generalized to the setup to triples $(M_{\mathbf{b}}, \mathbf{b}, F_{\mathbf{b}})$.

0.4. **A particular case:** $M = C_n(\mathbb{C})$**.** When $M = C_n(\mathbb{C})$, $\mathfrak{a} = \mathfrak{t}_n$ (the Kohno-Drinfeld Lie algebra), $J = \sum_{i<j} t_{ij} d\ln(z_i - z_j)$, $\mathbf{B}_n = S_n \times \mathbf{Pa}_n$, and for $(\sigma, P) \in \mathbf{B}_n$, $M_{\mathbf{b}} := \sigma \cdot \{(z_1, \ldots, z_n) \in C_n(\mathbb{R}) | z_1 < \ldots < z_n\}$, $\mathcal{F}_{\mathbf{b}}$ is given in the case $\sigma = 1$, $P = ((**)*)*$ by the collection of $\pi^{-1}(B(0,\epsilon)^2)$ avec $\pi : (z_1, \ldots, z_4) \mapsto (z_{12}/z_{13}, z_{13}/z_{14})$, $F_{\mathbf{b}}$ la solution de comportement asymptotique $(z_1, \ldots, z_4) \mapsto z_{12}^{t_{12}} z_{13}^{t_{12,3}} z_{14}^{t_{123,4}}$.

0.5. **Operadic aspects.** The collection of groupoids $n \mapsto \pi_1(C_n(\mathbb{C}), \mathbf{B}_n)^\wedge$ is equipped with an operadic structure; the same is true of the collection $n \mapsto \mathcal{G}((U\mathfrak{t}_n)^\wedge_{\mathbb{C}})_{\mathbf{B}_n}$. They correspond to the operads in Hopf groupoids $n \mapsto \mathbb{C}\pi_1(C_n(\mathbb{C}), \mathbf{B}_n)^\wedge$ and $n \mapsto ((U\mathfrak{t}_n)^\wedge_{\mathbb{C}})_{\mathbf{B}_n}$.

Monodromy of the KZ connection gives rise to an element of

$$\mathrm{Iso}_{\text{operads in Hopf groupoids}}(n \mapsto \mathbb{C}\pi_1(C_n(\mathbb{C}), \mathbf{B}_n)^\wedge, n \mapsto ((U\mathfrak{t}_n)^\wedge_{\mathbb{C}})_{\mathbf{B}_n}).$$

0.6. **Associators and operads.** For any commutative algebra $\mathbf{k}$ containing $\mathbb{Q}$, there is an injection

$$\mathrm{Iso}_{\text{operads in Hopf groupoids}}(n \mapsto \mathbf{k}\pi_1(C_n(\mathbb{C}), \mathbf{B}_n)^\wedge, n \mapsto ((U\mathfrak{t}_n)^\wedge_{\mathbf{k}})_{\mathbf{B}_n}) \hookrightarrow \mathbf{k}^\times \times \mathcal{G}(\hat{T}(e_0, e_1)_{\mathbf{k}})$$

induced by the images of generators for $n = 2, 3$. The relations satisfied by the pair $(2\pi \mathrm{i}, \Phi_{\mathrm{KZ}}) \in \mathbb{C}^\times \times \mathcal{G}(\hat{T}(e_0, e_1)_{\mathbb{C}})$ can be used to define a $\mathbb{Q}$ scheme $\mathbf{k} \mapsto \mathsf{M}(\mathbf{k}) \subset \mathbf{k}^\times \times \mathcal{G}(\hat{T}(e_0, e_1)_{\mathbf{k}})$ ([Dr]). Explicitly, $\mathsf{M}(\mathbf{k}) \subset \mathbf{k}^\times \times \mathcal{G}(\hat{T}(e_0, e_1)_{\mathbf{k}})$ is defined as the set of pairs (μ, ϕ), such that

$$\phi(B, A) = \phi(A, B)^{-1}, \quad \phi(B, C) e^{\mu B/2} \phi(A, B) e^{\mu A/2} \phi(C, A) e^{\mu C/2} = 1 \tag{0.6.1}$$

$(A := e_0,\ B := e_1,\ C := -A - B)$,

$$\phi(t_{12}, t_{13} + t_{14}) \phi(t_{13} + t_{23}, t_{34}) = \varphi(t_{23}, t_{34}) \varphi(t_{12} + t_{13}, t_{24} + t_{34}) \varphi(t_{12}, t_{12}) \tag{0.6.2}$$

(in $\mathcal{G}(U(\mathfrak{t}_4)^\wedge_{\mathbf{k}})$).

Moreover ([BN, Fr]), there is an isomorphism of schemes

$$\mathrm{Iso}_{\text{operads in Hopf groupoids}}(n \mapsto \mathbf{k}\pi_1(C_n(\mathbb{C}), \mathbf{B}_n)^\wedge, n \mapsto ((U\mathfrak{t}_n)^\wedge_{\mathbf{k}})_{\mathbf{B}_n}) = \mathsf{M}(\mathbf{k}).$$

0.7. **The group schemes $\mathsf{GT}(-)$ and $\mathsf{GRT}(-)$.**

$$\mathrm{Iso}_{\text{operads in Hopf groupoids}}(n \mapsto \mathbf{k}\pi_1(C_n(\mathbb{C}), \mathbf{B}_n)^\wedge, n \mapsto ((U\mathfrak{t}_n)^\wedge_{\mathbf{k}})_{\mathbf{B}_n})$$

is naturally a bitorsor over

$$\mathsf{GRT}(\mathbf{k}) := \mathrm{Aut}_{\text{operads in Hopf groupoids}}(n \mapsto ((U\mathfrak{t}_n)^\wedge_{\mathbf{k}})_{\mathbf{B}_n})$$

and

$$\mathsf{GT}(\mathbf{k}) := \mathrm{Aut}_{\text{operads in Hopf groupoids}}(n \mapsto \mathbf{k}\pi_1(C_n(\mathbb{C}), \mathbf{B}_n)^\wedge).$$

By the generation results of these operads, one has scheme inclusions

$$\mathsf{GRT}(\mathbf{k}) \hookrightarrow \mathbf{k}^\times \times \mathcal{G}(\hat{T}(e_0, e_1)_{\mathbf{k}}), \quad \mathsf{GT}(\mathbf{k}) \hookrightarrow \mathbf{k}^\times \times \mathcal{G}((\mathbf{k}F_2)^\wedge),$$

where the precise description of the left-hand sides as subschemes, as well as their group structures, are made explicit in [Dr]; namely, $\mathsf{GRT}(\mathbf{k}) = \mathsf{GRT}_1(\mathbf{k}) \rtimes \mathbf{k}^\times$, and $\mathsf{GRT}_1(\mathbf{k}) \subset \mathcal{G}(\hat{T}(e_0, e_1)_{\mathbf{k}})$ is defined by (0.6.2), (0.6.1) with $\mu = 0$, and

$$A + g(A,B)^{-1}Bg(A,B) + g(A,C)^{-1}Cg(A,C) = 0.$$

and $(g \circ h)(A,B) := g(h(A,B)Ah(A,B)^{-1}, B)h(A,B)$ and $\mathsf{GT}(\mathbf{k})$ is the set of pairs (λ, f), such that

$$f(Y,X) = f(X,Y)^{-1}, \quad f(X_3, X_1)X_3^m f(X_2, X_3)X_2^m f(X_1, X_2)X_1^m = 1$$

if $m = (\lambda - 1)/2$, $(X_1, X_2, X_3) := (X, Y, (XY)^{-1})$, and

$$\partial_3(f)\partial_1(f) = \partial_0(f)\partial_2(f)\partial_4(f)$$

where $\partial_i : F_2 \to \mathrm{PB}_4$ are suitable morphisms.

The definition of $\mathsf{GT}(\mathbf{k})$ is formally similar to the of the group $\widehat{\mathrm{GT}}$. This can be made precise as follows: fpr any prime ℓ, one can construct a pro-ℓ analogue GT_ℓ of $\widehat{\mathrm{GT}}$ and a sequence of group morphisms

$$\widehat{\mathrm{GT}} \to \mathrm{GT}_\ell \hookrightarrow \mathsf{GT}(\mathbb{Q}_l).$$

0.8. **The Ihara approach.** For any $n \geq 0$, there is a diagram

$$\begin{array}{ccccc}
\mathrm{Aut}_{\text{Hopf groupoids}}(((U\mathfrak{t}_n)^\wedge_{\mathbf{k}})_{\mathbf{B}_n}) & \longrightarrow & \mathrm{Out}_{\text{Hopf alg.}}(((U\mathfrak{t}_n)^\wedge_{\mathbf{k}}))^{S_n} & \xrightarrow{=} & \mathrm{Out}_{\text{Lie alg.}}((\hat{\mathfrak{t}}_n)_{\mathbf{k}})^{S_n} \\
\uparrow & & & & \uparrow \subset \\
\mathrm{Aut}_{\text{operads in Hopf groupoids}}(n \mapsto ((U\mathfrak{t}_n)^\wedge_{\mathbf{k}})_{\mathbf{B}_n}) & & \longrightarrow & & \mathrm{Out}^*_{\text{Lie alg.}}((\hat{\mathfrak{t}}_n)_{\mathbf{k}})^{S_n}
\end{array}$$

where Out^* means the classes of automorphisms taking t_{ij} to a conjugate for any (i,j). In contrast the the family of groups $\mathrm{Out}_{\text{Lie alg.}}(\hat{\mathfrak{t}}_n)^{S_n}$, the groups $\mathrm{Out}^*_{\text{Lie alg.}}((\hat{\mathfrak{t}}_n))^{S_n}$ are related to morphisms as n varies, and there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{Out}^*_{\text{Lie alg.}}((\hat{\mathfrak{t}}_n)_{\mathbf{k}})^{S_n} & \xrightarrow{p_n} & \mathrm{Out}^*_{\text{Lie alg.}}((\hat{\mathfrak{t}}_{n-1})_{\mathbf{k}})^{S_{n-1}} \\
{}_{i_n}\nwarrow & & \nearrow{}_{i_{n-1}} \\
& \mathsf{GRT}(\mathbf{k}) &
\end{array}$$

(see [Ih]). The map p_n is injective when $n = 4$, and an iso for $n \geq 5$; i_n is an iso for $n \geq 5$.

References

[BN] D. Bar-Natan, On associators and the Grothendieck-Teichmüller group I. Selecta Math. (N.S.) 4:2 (1998), 183–212.

[Dr] V. Drinfeld, On quasitriangular quasi-Hopf algebras and on a group that is closely connected with Gal($\bar{\mathbb{Q}}/\mathbb{Q}$). Leningrad Math. J. 2:4 (1990), 829–860.

[Fr] B. Fresse, Homotopy of Operads and Grothendieck-Teichmüller Groups, Mathematical Surveys and Monographs, Vol. 217, AMS.

[Ih] Ya. Ihara, On the stable derivation algebra associated with some braid groups. Israel J. Math. 80, 135–153 (1992).

[Q] D. Quillen, Rational homotopy theory, Ann. of Math. (2) 90 (1969), 205–295.

IRMA (UMR 7501) et Département de Mathématiques, Université de Strasbourg, 7 rue René-Descartes, 67084 Strasbourg (France)

Email address: b.enriquez@math.unistra.fr