Introduction to density matrices: Application to semiclassical laser theory

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Plan of lecture

- Pure and mixed states
  - Stern-Gerlach setup
  - Polarization
- Density matrices
  - Diagonal (population) and off-diagonal (coherence) elements
  - Examples
- Light-matter interaction
- Bloch equations and Poincare sphere
- Maxwell-Bloch equations
  - Steady states: dispersion and gain
- Rate equation picture
  - 3-level and 4-level systems
  - Adiabatic elimination and Laser classes
  - Reduction to Statz-deMars equations

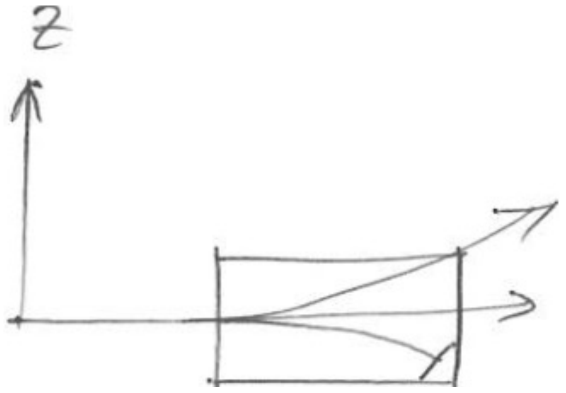
Pure spin states: Stern Gerlach Experiment

Consider a beam of spin $\frac{1}{2}$ particles(hydrogen) passing through a SG setup.

Field gradient along z with respect to fixed coordinate system.

Beam splits vertically into two, each correspond to one of the two possible eigenvalues of the component S_z of the spin operator $\vec{S}$ $(m = \pm\frac{1}{2})$

One of the beams is stopped (eliminated)



$\Rightarrow$ emerging particles are in a state, which corresponds to only one of the eigenvalues. Here it is $+\frac{1}{2}$

State vector

If the state of a given beam is known to be pure, then the joint state of all particles can be represented in terms of one and the same state vector $|\chi\rangle$

			adj.
$m = +\frac{1}{2}$	$\lvert\frac{1}{2}\rangle$	$\lvert+\frac{1}{2}\rangle = \begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}1 & 0\end{pmatrix}$
$m = -\frac{1}{2}$	$\lvert-\frac{1}{2}\rangle$	$\lvert-\frac{1}{2}\rangle = \begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}0 & 1\end{pmatrix}$

If SG magnet is along z' $\quad |\chi\rangle = |+\frac{1}{2}, z'\rangle$

A general spin state $|\chi\rangle$ can always be written as $|\chi\rangle = a_1|+\frac{1}{2}\rangle + a_2|-\frac{1}{2}\rangle$

In another representation, $|\chi\rangle = \begin{pmatrix}a_1\\a_2\end{pmatrix}$, $\qquad \langle\chi| = (a_1^*, a_2^*)$

The state $|\chi\rangle$ is normalized $\Rightarrow |a_1^2| + |a_2^2| = 1 = \langle\chi|\chi\rangle$

Polarization vector

A pure spin state can be characterized either by specifying the polar angles or by (a_1, a_2)

Example: Polarization vector $\vec{P}$

$P_i = \langle\sigma_i\rangle$ expectation value of the Pauli matrices.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\langle\sigma_i\rangle = \langle\chi|\sigma_i|\chi\rangle$$

$$P_x = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$P_y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$P_z = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = +1$$

For a beam of particles in state,

$$|+\frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$P_x^2 + P_y^2 + P_z^2 = 1$$

$|\pm\frac{1}{2}\rangle$ States of opposite polzn.

Polar representation

Consider now the general pure state $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$$\text{Let} \begin{cases} a_1 = \cos\frac{\theta}{2} \\ a_2 = \sin\frac{\theta}{2}e^{i\delta}, \end{cases} \quad \delta \text{ is the relative phase}$$

$\longrightarrow$ Completely specified by two real numbers.

$$|\chi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\delta}\sin\frac{\theta}{2} \end{pmatrix} \qquad P_x = \left(\cos\frac{\theta}{2}, e^{-i\delta}\sin\frac{\theta}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\delta}\sin\frac{\theta}{2} \end{pmatrix}$$

$$P_x^2 + P_y^2 + P_z^2 = 1, \quad \begin{cases} P_x = \sin\theta\cos\delta \\ P_y = \sin\theta\sin\delta \\ P_z = \cos\theta \end{cases}, \theta \rightarrow \text{polar angle}, \delta \rightarrow \text{azimuthal angle}$$

In another reference

A second coordinate system x', y', z' can be chosen such that z'- axis is parallel to $\vec{P}$. taking z' as quantization axis

$$P_{x'} = 0, P_{y'} = 0, P_{z'} = 1$$

$\Rightarrow$ all particles have spin up with respect to z'

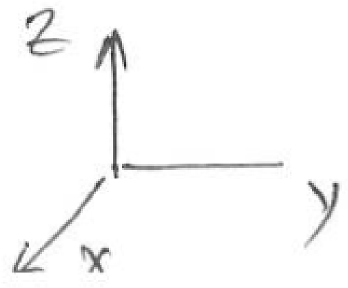
$\Rightarrow$ The direction of the polarization vector is the direction along which all spins are pointing.

SG apparatus pointing along $\vec{P}$ will allow all the particles to pass through.

allows explicit spin functions to be constructed.

Example: pure spin state along x

$|+\frac{1}{2}, x\rangle \quad |\chi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \theta = 90^0, \delta = 0$



$-x'$ direction $\qquad |\chi\rangle = |-\frac{1}{2}, x\rangle$

$$\theta = 90^0, \delta = 180^0, \qquad |\chi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$|\chi\rangle = |\frac{1}{2}, y\rangle \qquad \theta = 90^0, \delta = 90^0$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$|-\frac{1}{2}, y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Note that these four states are constructed using the superposition $|+\frac{1}{2}\rangle$ and $|-\frac{1}{2}\rangle$ states using same magnitudes $|a_1| = |a_2| = \frac{1}{\sqrt{2}}$ but with different relative phases.

Mixed spin states

Most general spin states for an ensemble of particles

Prepare two beams of particles independently one in pure $|+\frac{1}{2}\rangle$ state, other in pure $|-\frac{1}{2}\rangle$ states.

independent: no definite phase relation exist between the two.

Let in the first beam N_1 part

second beam N_2 part

Investigate the polarization state of the combined beam by a SG filter for various orientation.

It is not possible to find any orientation for which the combined beam passes through completely.

$\Rightarrow$ the joint beam is not in a pure state.

Definition: States which are not pure are called mixed states or mixtures.

How to describe a mixed state

(i) It is not possible to describe it by just one state vector $|\chi\rangle \Rightarrow$ since associated with this state there is a direction along which all spins point $\equiv$ direction of the polarization vector.

Whole beam would have passed through a SG apparatus.

(ii) Cannot be represented by a linear superposition of $|+\frac{1}{2}\rangle$ and $|-\frac{1}{2}\rangle$ of the two constituent beams.

For such superposition, we need to know magnitudes and relative phases δ, a_1, a_2

$$\begin{cases} |a_1^2| = W_1 \\ |a_2^2| = W_2 \end{cases}$$

Probabilities of finding the particles in the states $|+\frac{1}{2}\rangle$ or $|-\frac{1}{2}\rangle$ respectively.

Statistical mixture

$$W_1 = \frac{N_1}{N},\ W_2 = \frac{N_2}{N},\ N = N_1 + N_2 \Rightarrow W_1 + W_2 = 1$$

Independently prepared $\Rightarrow$ no definite phase relation

N_1 particles prepared in state $|+\frac{1}{2}\rangle$
N_2 particles prepared in state $|-\frac{1}{2}\rangle$

Mixture to be prepared retaining maximum information.

$\vec{P}$ of the total beam by the statistical average over the separate beams.

$$P_i = W_1|\langle\tfrac{1}{2}|\sigma_i|\tfrac{1}{2}\rangle + W_2|\langle-\tfrac{1}{2}|\sigma_i|-\tfrac{1}{2}\rangle$$

$$P_x = 0,\ P_y = 0,\ P_z = W_1 - W_2 = \frac{N_1 - N_2}{N}$$

$$0 \leq |\vec{P}| \leq 1$$

General treatment

Consider a quantum system denoted by $|\chi\rangle$: complete information: pure state.

Often $\Rightarrow$ incomplete information

photon from natural light can have any polarization state with equal probability

system in thermal equilibrium at T has a probability $\sim e^{-\frac{E_n}{kT}}$ of being in state E_n.

in state $|\psi_1\rangle$ with probability p_1
in state $|\psi_2\rangle$ with probability p_2
$\ldots$
in state $|\psi_n\rangle$ with probability p_n

statistical mixture of states $|\psi_1\rangle, |\psi_2\rangle, \ldots$ with probabilities $p_1, p_2, \ldots$

$$p_1 + p_2 + \ldots = \sum_k p_k = 1$$

Superposition vs statistical mixture

single particle in coordinate space in a linear superposition state

$$\psi(r) = \sum_k c_k \psi_k(r)$$

Probability of finding the particle at $r \Rightarrow |\psi(r)|^2 = |\sum_k c_k \psi_k(r)|^2 = P(r)$

$$= \sum_{k,k'} c_k c_{k'}^* \psi_k \psi_{k'}^*$$

interference

In a statistical mixture $P(r) = \sum_k p_k |\psi_k(r)|^2 \quad \Rightarrow$ no interference

Density operator for pure states

Let the state vector of the system be perfectly known
$\Rightarrow$ all probabilities $p_k = 0$ except one

Introduce the operator, $P_{|u_n\rangle} = |u_n\rangle \langle u_n|$

Acting on an arbitrary vector $|V\rangle$, $P_{|u_n\rangle}|V\rangle = |u_n\rangle\langle u_n|V\rangle$

gives a vector aligned along $|u_n\rangle$

Moreover,
$$\begin{aligned} P^2_{|u_n\rangle} &= (|u_n\rangle\langle u_n|)(|u_n\rangle\langle u_n|) \\ &= |u_n\rangle(\langle u_n|u_n\rangle)\langle u_n| \\ &= |u_n\rangle\langle u_n| \\ &= P_{|u_n\rangle} \end{aligned}$$

Thus, $P_{|u_n\rangle}$ is a projection operator onto basis vector $|u_n\rangle$

$$\sum_n P_{|u_n\rangle} = P_{|u_1\rangle} + P_{|u_2\rangle} + \ldots = \sum_n |n\rangle\langle n| = 1$$

Density operator for pure states contd.

Description by a density matrix

$$\langle A\rangle = \langle\psi|A|\psi\rangle = \sum_{nm} c_n^* c_m A_{nm}$$

matrix elements of the operator $|\psi\rangle\langle\psi| \Leftrightarrow$ Projection onto ket $|\psi\rangle$

$$|\psi\rangle = \sum_n c_n|u_n\rangle$$

$$\langle\psi| = \sum_n \langle u_n|c_n^*$$

$$c_n = \langle u_n|\psi\rangle$$

$$c_n^* = \langle\psi|u_n\rangle$$

$$c_n^* c_m = \langle\psi|u_n\rangle\langle u_m|\psi\rangle$$

$$= \langle u_m|\psi\rangle\langle\psi|u_n\rangle$$

Natural to introduce the density operator $\rho = |\psi\rangle\langle\psi|$

Mean value of an observable

$$\begin{aligned}\langle A\rangle &= \langle\psi|A|\psi\rangle \\ &= \sum_{nm} c_n^* c_m A_{nm} = \sum_{mn} \rho_{mn} A_{nm} = Tr(\rho A) \\ &= Tr(A\rho)\end{aligned}$$

Mathematically $|\psi\rangle\langle\psi|$ is projection operator.

$P_{|\psi\rangle} = |\psi\rangle\langle\psi|$, projection onto ket $|\psi\rangle$

$P_{|\psi\rangle}|V\rangle = |\psi\rangle(\langle\psi|V\rangle)$

$$\begin{aligned}P^2_{|\psi\rangle} &= |\psi\rangle\langle\psi|\psi\rangle\langle\psi| \\ &= |\psi\rangle\langle\psi| = P_{|\psi\rangle}\end{aligned}$$

$\Rightarrow P^2_{|\psi\rangle} = P_{|\psi\rangle}$, Projection optr.

Time evolution

$$\frac{d}{dt}\langle\psi| = -\frac{1}{i\hbar}\langle\psi| H^\dagger = -\frac{1}{i\hbar}\langle\psi| H$$

$$\frac{d}{dt}|\psi\rangle = \frac{1}{i\hbar}H|\psi\rangle$$

$$\begin{aligned}
\frac{d}{dt}\rho &= \frac{d}{dt}|\psi\rangle\langle\psi| \\
&= \frac{d|\psi\rangle}{dt}\langle\psi| + |\psi\rangle\frac{d\langle\psi|}{dt} \\
&= \frac{1}{i\hbar}(H|\psi\rangle\langle\psi| - |\psi\rangle\langle\psi| H) = \frac{1}{i\hbar}(H\rho - \rho H)
\end{aligned}$$

A compendium for pure states

$$\frac{d}{dt}\rho = \frac{1}{i\hbar}[H, \rho]$$

Generalized Schrödinger equation

Properties of the density operator in case of pure state

$$\rho^\dagger = \rho$$

$$\text{Tr } \rho = 1$$

$$\langle A \rangle = \text{Tr } (\rho A) = \text{Tr } (A\rho)$$

$$i\hbar \frac{d}{dt}\rho = [H, \rho]$$

These properties are general and hold also for mixed case

In case of pure states: two specific properties

$$\rho^2 = \rho$$

$$\text{Tr } \rho^2 = 1$$

These can be used to find out if a state is pure or not

Mixed case density operator

$$\text{definition } \rho = \sum_k p_k \rho_k = \sum_k p_k |\psi_k\rangle\langle\psi_k|$$

For mixed states: $\rho^2 \neq \rho, \quad \rho$ is not a projection operator.

$$\text{Hence, Tr } (\rho^2) \neq \text{Tr } (\rho) = 1$$

$$\text{For mixed states Tr } \rho^2 < 1$$

Diagonal elements : interpretation

Express $|\psi_k\rangle$ in basis $|u_n\rangle$ as

$$|\psi_k\rangle = \sum_n c_n^{(k)} |u_n\rangle, \quad c_n^{(k)} = \langle u_n|\psi_k\rangle$$

$$\rho_{nn} = \sum_k p_k \left| c_n^{(k)} \right|^2 \qquad \Rightarrow +\text{ve real number}$$

$\left| c_n^{(k)} \right|^2$ - probability of $|u_n\rangle$ in pure state $|\psi_k\rangle$

$\Rightarrow \rho_{nn}$ - probability of $|u_n\rangle$ in state ρ

Diagonal matrix elements are called population of the state $|u_n\rangle$

Physically if N times the same experiment is carried out with the same initial conditions, (N is large) then $\Rightarrow N\rho_{nn}$ systems will be found in the state $|u_n\rangle$

Off-diagonal elements : interpretation

$$\begin{aligned}\rho_{nm} &= \langle u_n|\rho|u_m\rangle \\ &= \sum_k p_k \langle u_n|\psi_k\rangle\langle\psi_k|u_m\rangle \\ &= \sum_k p_k c_n^{(k)} c_m^{(k)*}\end{aligned}$$

$c_n^{(k)} c_m^{(k)*}$ is a cross term expressing interference between $|u_n\rangle$ and $|u_m\rangle$. These appear when $|\psi_k\rangle$ is a coherent linear superposition of these states.

ρ_{nm}- weighted average of these terms taken over all possible states of the mixture.

If $\rho_{nm} = 0 \Rightarrow$ the statistical average has cancelled out any interference effects between $|u_n\rangle$ and $|u_m\rangle$

If it is non zero $\Rightarrow$ certain coherence persists between $|u_n\rangle$ and $|u_m\rangle$

$\Rightarrow$ off diagonal terms $\equiv$ called coherence.

Basis dependence

'population' and 'coherence' depends on the choice of basis $\{|u_n\rangle\}$

Since ρ is Hermitian: always possible to find an orthonormal basis $\{|\chi_n\rangle\}$ in which ρ is diagonal.

$\rho = \sum_l \pi_l |\chi_l\rangle \langle\chi_l| \Rightarrow \rho$ can thus be thought of as a statistical mixture of orthonormal states $|\chi_n\rangle$ with probability π_n.

$\Rightarrow$ no coherence between states $|\chi_n\rangle$

$\text{Tr } \rho^2 = \sum_l \pi_l^2 \leq \sum_l \pi_l = 1$ When one π_l equals 1, all others must be zero.

In that case ρ is a pure state, $\text{Tr } \rho^2 = 1$; for mixed states $\text{Tr } \rho^2 < 1$

An important inequality

$$\rho_{nn}\rho_{mm} \geq |\rho_{mn}|^2$$

Proof: LHS $\Rightarrow \left(\sum_k p_k \left|c_n^{(k)}\right|^2\right)\left(\sum_k p_k \left|c_m^{(k)}\right|^2\right)$

$$\geq \left(\sum_k p_k \left|c_n^{(k)} c_m^{(k)}\right|\right)^2 \geq \left|\sum_k p_k c_n^{(k)} c_m^{(k)}\right|^2 = |\rho_{nm}|^2$$

Consequence $\Rightarrow$ ρ can have coherence only between states whose populations are not zero.

$$\text{Tr } \rho^2 = \sum_{mn} \rho_{mn}\rho_{nm} = \sum_{mn} |\rho_{mn}|^2 \leq \sum_{nm} \rho_{nn}\rho_{mm}$$

$$= \sum_n \rho_{nn} \sum_m \rho_{mm} = 1$$

Example: thermal equilibrium

$$\rho = Z^{-1} e^{-\frac{H}{kT}}, \qquad Z = \text{Tr}\left\{ e^{-\frac{H}{kT}} \right\}$$

Use basis vectors $|u_n\rangle$ of H

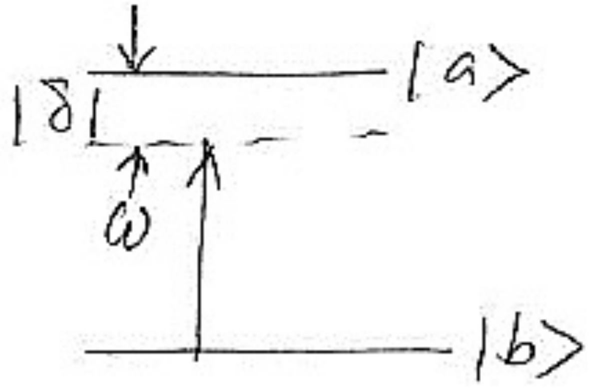
$$\rho_{nn} = \frac{\langle n | e^{-\frac{H}{kT}} | n \rangle}{Z} = Z^{-1} e^{-\frac{E_n}{kT}}$$

$$\rho_{nm} = Z^{-1} \langle u_n | e^{-\frac{H}{kT}} | u_m \rangle = 0 \quad \text{for } n \neq m$$

$\Rightarrow$ At thermal equilibrium population of the stationary states are exponentially decreasing functions of energy. Coherence between stationary states =0

Two-level system interacting with monochromatic field

Total Hamiltonian $H = H_0 + H_I$



$$H_0 = \hbar\omega_a |a\rangle \langle a| + \hbar\omega_b |b\rangle \langle b|$$

$$\begin{aligned} H_I &= -\vec{d}\,.\,\vec{E} = -exE \\ &= -e\left(|a\rangle \langle a|\, x_{aa} + |b\rangle \langle b|\, x_{bb} + |a\rangle\langle b|\, x_{ab} + |b\rangle\langle a|\, x_{ba}\right) E \\ &= -e\left(|a\rangle\langle b|\, x_{ab} + |b\rangle\langle a|\, x_{ba}\right) E \\ &= -\left(|a\rangle\langle b| + |b\rangle\langle a|\right) d_x E \end{aligned}$$

Let $E = E_0 \cos\omega t = \frac{1}{2}\left(e^{-i\omega t} + e^{i\omega t}\right) E_0$

$$
\begin{aligned}
\Rightarrow H_I &= -\frac{\hbar}{2}\frac{d_x E_0}{\hbar}(e^{-i\omega t} + e^{i\omega t})(|a\rangle\langle b| + |b\rangle\langle a|) \\
&= -\frac{\hbar}{2}\Omega(e^{-i\omega t} + e^{i\omega t})(|a\rangle\langle b| + |b\rangle\langle a|) \\
&= -\frac{\hbar}{2}\Omega(e^{-i\omega t} + e^{i\omega t})(\sigma_+ + \sigma_-)
\end{aligned}
$$

$$\sigma_+ = \sigma_{ab} = |a\rangle\langle b| \qquad \sigma_+ |a\rangle = 0$$

$$\sigma_- = \sigma_{ba} = |b\rangle\langle a| \qquad \sigma_+ |b\rangle = |a\rangle$$

In Heisenberg picture $\sigma_+ = |a\rangle\langle b|$, $\sigma_- = |b\rangle\langle a|$ oscillates as $e^{i\omega_{ab}t}$ and $e^{-i\omega_{ab}t}$ respectively for a free atom

$$\langle\sigma_-\rangle = \mathrm{Tr}\ \rho\,|b\rangle\langle a| = \langle a|\rho|b\rangle = \rho_{ab} = \rho_{ab}(0)e^{-i\omega_{ab}t}$$

$|a\rangle\langle b|\, e^{i\omega t}$ and $|b\rangle\langle a|\, e^{-i\omega t}$ vary quickly as $e^{\pm i(\omega_{ab}+\omega)t}$

In contrast, $|a\rangle\langle b|\, e^{-i\omega t}$ and $|b\rangle\langle a|\, e^{i\omega t}$ vary slowly as $e^{\pm i(\omega_{ab}-\omega)t}$

$$|a\rangle\langle b|\, e^{-i\omega t} \rightarrow e^{i(\omega_{ab}-\omega)t}$$
$$|b\rangle\langle a|\, e^{i\omega t} \rightarrow e^{-i(\omega_{ab}-\omega)t}$$

Coming back to the interaction Hamiltonian,

$$\begin{aligned} H_I &= -\frac{\hbar\Omega}{2}(\sigma_+ e^{-i\omega t} + \sigma_- e^{+i\omega t}) \\ &= -\frac{\hbar\Omega}{2}(|a\rangle\langle b|\, e^{-i\omega t} + |b\rangle\langle a|\, e^{+i\omega t}) \end{aligned}$$

Evolution equation without decay

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H,\rho]$$

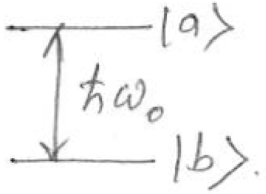
$$\text{Let } O = \frac{H\rho}{\hbar} \qquad \Rightarrow \frac{d\rho}{dt} = -i(O - O^{\dagger})$$

$$O = \frac{H\rho}{\hbar} = \left(\omega_a |a\rangle\langle a| + \omega_b |b\rangle\langle b|\right)\rho - \frac{\Omega}{2}\left(|a\rangle\langle b| e^{-i\omega t}\rho + |b\rangle\langle a| e^{i\omega t}\rho\right)$$

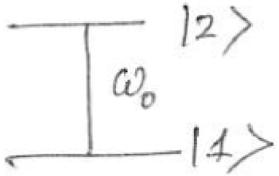
$$\begin{aligned}
\Rightarrow \frac{d}{dt}\rho_{aa} &= \frac{i\Omega}{2}\left(e^{-i\omega t}\rho_{ba} - e^{+i\omega t}\rho_{ab}\right) \\
\frac{d}{dt}\rho_{bb} &= -\frac{i\Omega}{2}\left(e^{-i\omega t}\rho_{ba} - e^{+i\omega t}\rho_{ab}\right) \\
\frac{d}{dt}\rho_{ab} &= -i\omega_{ab}\,\rho_{ab} - \frac{i\Omega}{2}e^{-i\omega t}\left(\rho_{aa} - \rho_{bb}\right)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \frac{d}{dt}\tilde{\rho}_{aa} &= \frac{i\Omega}{2}(\tilde{\rho}_{ba} - \tilde{\rho}_{ab}) \\
\frac{d}{dt}\tilde{\rho}_{bb} &= -\frac{i\Omega}{2}(\tilde{\rho}_{ba} - \tilde{\rho}_{ab}) \\
\frac{d}{dt}\tilde{\rho}_{ab} &= i\delta\tilde{\rho}_{ab} - \frac{i\Omega}{2}(\tilde{\rho}_{aa} - \tilde{\rho}_{bb})
\end{aligned}$$

NMR Bloch equations without decay



$$a \to 2, \qquad b \to 1, \text{ drop tilde}$$



$$\begin{aligned} u &= \rho_{21} + \rho_{12} \\ v &= -i(\rho_{12} - \rho_{21}) \\ w &= \rho_{22} - \rho_{11} \end{aligned}$$

$$\boxed{\begin{aligned} \dot{u} &= \delta v \\ \dot{v} &= -\delta u + \Omega w \\ \dot{w} &= -\Omega v \end{aligned}}$$

Let $\vec{R} = (u, v, w)$

$$\boxed{\vec{M} = (-\Omega, 0, -\delta)}$$

$$\frac{d\vec{R}}{dt} = \vec{M} \times \vec{R}$$

Interpretation

$$\frac{d\vec{R}^2}{dt} = 2\vec{R}\,.\,\frac{d\vec{R}}{dt} = 2\vec{R}\,.\,[\vec{M}\times\vec{R}] = 0 \qquad \Rightarrow R^2 = \text{constant}$$

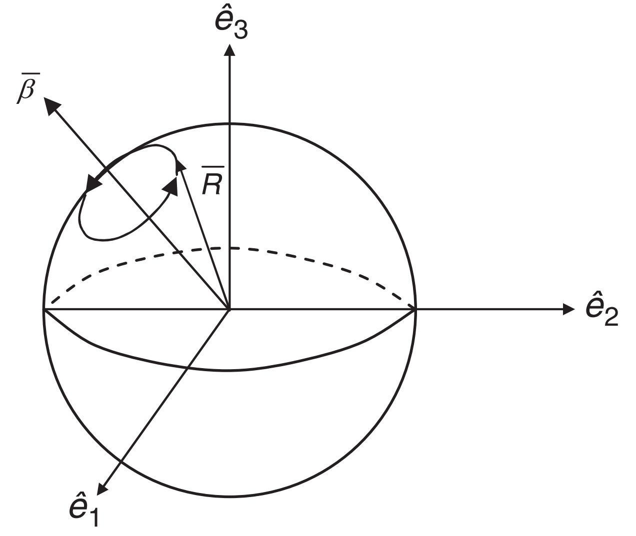
$$\begin{aligned} R^2 &= u^2 + v^2 + w^2 = (\rho_{21}+\rho_{12})^2 - (\rho_{21}-\rho_{12})^2 + (\rho_{22}-\rho_{11})^2 \\ &= (c_2c_2^* + c_1c_1^*)^2 = 1 \end{aligned}$$

North pole: $w = \rho_{22} - \rho_{11} = 1$

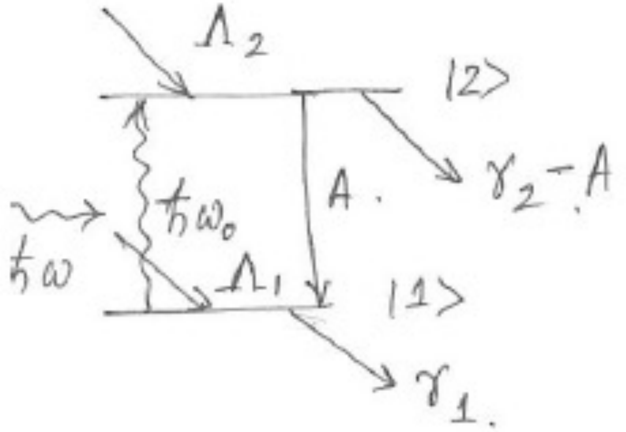
$\Rightarrow \rho_{22} = 1, \quad \rho_{11} = 0$

South pole: $w = -1,\ u = v = 0$

$\rho_{22} = 0, \quad \rho_{11} = 1$



Inclusion of relaxation and pumping



Introduce relaxation rates,

$$\gamma_1 = \frac{1}{\tau_1},\ \gamma_2 = \frac{1}{\tau_2} \text{ for } \rho_{11} \text{ and } \rho_{22}.$$

Part of population decaying from $2 \rightarrow 1$, by spontaneous emission with rate A

Let Γ be the decay rate of coherence ρ_{12}. One has $\Gamma \geq \dfrac{\gamma_1 + \gamma_2}{2}$

$$\begin{aligned}
\frac{d\rho_{22}}{dt} &= \Lambda_2 - \gamma_2\rho_{22} - i\frac{\Omega}{2}(\sigma_{21} - \sigma_{12}) \\
\frac{d\rho_{11}}{dt} &= \Lambda_1 - \gamma_1\rho_{11} + i\frac{\Omega}{2}(\sigma_{21} - \sigma_{12}) + A\rho_{22} \\
\frac{d\sigma_{21}}{dt} &= -(\Gamma - i\delta)\sigma_{21} - i\frac{\Omega}{2}(\rho_{22} - \rho_{11})
\end{aligned}$$

Steady state without the laser field

$$\Omega = 0$$

Label the solution with subscript 0.

$$\sigma_{21}^{(0)} = \sigma_{12}^{(0)} = 0$$

$$\rho_{22}^{(0)} - \rho_{11}^{(0)} = \frac{\Lambda_2}{\gamma_2} - \frac{\Lambda_1}{\gamma_1} - A\frac{\Lambda_2}{\gamma_1\gamma_2} = \frac{\Lambda_2(\gamma_1 - A) - \Lambda_1\gamma_2}{\gamma_1\gamma_2}$$

For population inversion, $\rho_{22} > \rho_{11}$, one must have $\gamma_1 > A$.

$$\Lambda_2\gamma_1 > \Lambda_1\gamma_2$$

Upper level is pumped more efficiently than the lower level.

With laser field

$$\Omega \neq 0$$

$$\rho_{22} - \rho_{11} = \left(\rho_{22}^{(0)} - \rho_{11}^{(0)}\right) - \frac{i\Omega}{2\gamma_2\gamma_1}\left(\gamma_1 + \gamma_2 - A\right)\left(\sigma_{21} - \sigma_{12}\right)$$

$$\sigma_{21} - \sigma_{12} = \frac{\Omega}{2}(\rho_{22} - \rho_{11})\left(-\frac{2i\Gamma}{\delta^2 + \Gamma^2}\right) = \left(-\frac{i\Omega\Gamma}{\delta^2 + \Gamma^2}\right)(\rho_{22} - \rho_{11})$$

$$\rho_{22} - \rho_{11} = \frac{\rho_{22}^{(0)} - \rho_{11}^{(0)}}{\left(1 + \frac{\gamma_1+\gamma_2-A}{\gamma_2\gamma_1}\frac{\Omega^2}{2}\frac{\Gamma}{\delta^2+\Gamma^2}\right)}$$

Saturation

$$\vec{E} = \hat{e}\,\frac{E_0}{2}e^{-i\omega t} + c.c$$

$$u = \frac{1}{2}\left(\vec{E}\,.\,\vec{D} + \vec{B}\,.\,\vec{H}\right)$$

$$\vec{S} = \vec{E}\times\vec{H}$$

$$I = 2\epsilon_0 n_0 c_0 \frac{|E_0|^2}{4}$$

$$\Rightarrow \Omega^2 = \frac{d^2 E_0^2}{\hbar^2} = \frac{4d^2 I}{2\hbar^2\epsilon_0 n_0 c_0}$$

Define I_{sat}

$$I_{sat} = \frac{\epsilon_0 c_0 n_0 \hbar^2}{d^2\Gamma}\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2-A}\left(\Gamma^2+\delta^2\right)$$

$$\rho_{22}-\rho_{11} = \frac{\rho_{22}^{(0)}-\rho_{11}^{(0)}}{1+\frac{I}{I_{sat}}}$$

$I >> I_{sat}$, system becomes transparent and no longer responds to incident wave.

Polarization

The incident wave creates a polarization in the atomic medium

$$P_{at,x} = n\langle d_x \rangle = n\text{Tr}\ \{\rho \hat{d}_x\} = 2nd\ \text{Re}(\rho_{21})$$

$$\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} 0 & d_{12} \\ d_{21} & 0 \end{pmatrix} = \rho_{12}d_{21} + \rho_{21}d_{12} = 2d\ \text{Re}(\rho_{21})$$

atoms embedded in the matrix

$$P_{mat,x} = \epsilon_0 \chi_{mat} \frac{E_0}{2} e^{-i\omega t} + c.c$$

Susceptibility

$$\chi_{at} = \chi' + i\chi'' = \frac{2nd\,\sigma_{21}}{E_0\epsilon_0}$$

$$\chi'_{at}(\delta) = \frac{nd^2}{\epsilon_0\hbar}\,\frac{\rho_{22}^{(0)} - \rho_{11}^{(0)}}{1 + \frac{I}{I_{sat}}}\,\frac{\delta}{\delta^2 + \Gamma^2}$$

$$\chi''_{at}(\delta) = -\frac{nd^2}{\epsilon_0\hbar}\,\frac{\rho_{22}^{(0)} - \rho_{11}^{(0)}}{1 + \frac{I}{I_{sat}}}\,\frac{\Gamma}{\delta^2 + \Gamma^2}$$

Maxwell-Bloch Equations:

Start with Bloch eqns, describing the atom in the active medium.

Assumptions

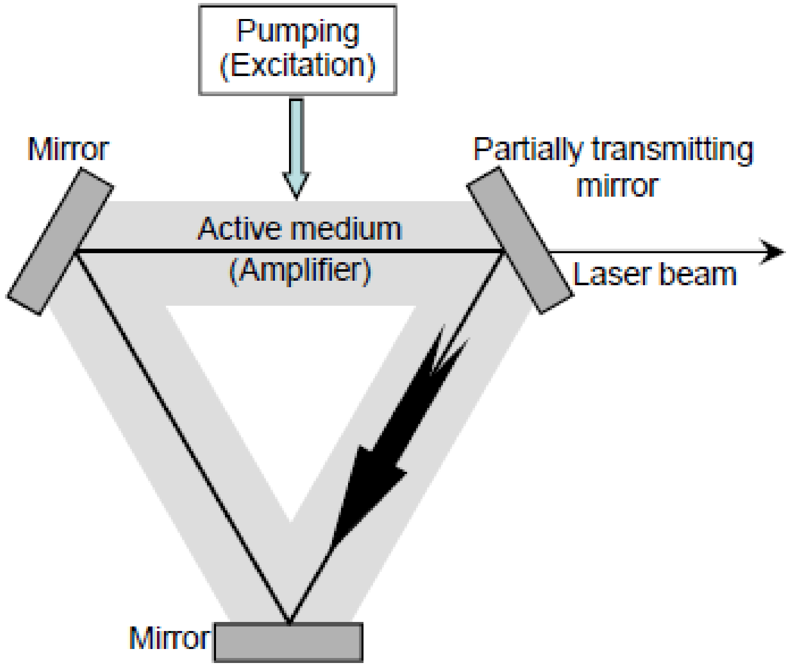
(a) Active medium fills the whole cavity

(b) Intracavity field can be treated as a plane wave.

$$E(z,t) = A(z,t)e^{-i(\omega t - kz)} + c.c = 2\text{Re}[A(z,t)e^{-i(\omega t - kz)}]$$

(c) Polarization is fixed $\Rightarrow$ hence scalar expression

(d) $A(z,t)$ slowly varying function of both z, t.



Unidirectional ring laser cavity.

Equation for the evolution of polarization

$$P_{at}(z,t) = \left(P(z,t)e^{-i(\omega t - kz)} + c.c\right) = 2\text{Re}\left[P(z,t)e^{-i(\omega t - kz)}\right]$$

$$P(z,t) = nd\sigma_{21}(z,t)$$

$$\frac{d\sigma_{21}}{dt} = -(\Gamma - i\delta)\sigma_{21} - i\frac{\Omega}{2}(\rho_{22} - \rho_{11})$$

$$\Omega = \frac{dE_0}{\hbar} = \frac{2dA}{\hbar}$$

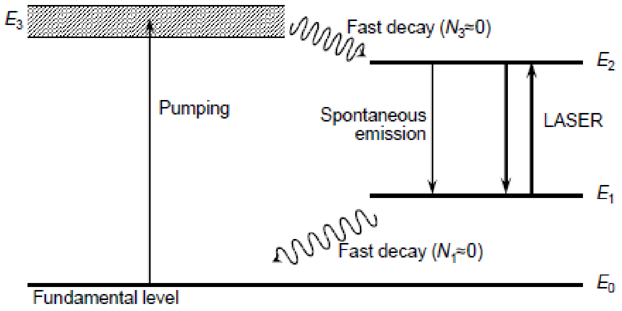
$$\Delta n = n(\rho_{22} - \rho_{11})$$

$$\boxed{\frac{dP}{dt} = -(\Gamma - i\delta)P - i\frac{d^2}{\hbar}\Delta n A}$$

Now, A is no longer given and liable to change. Atoms can change the field.

Evolution of population inversion

In order to have a simpler description, we restrict ourselves to '4-level' system



Lower level is not pumped $\Lambda_1 << \Lambda_2$, $\gamma_1 >> \gamma_2$

We can assume that lower level is always empty $\rho_{11} = 0$

Population inversion per unit volume is given by $\Delta n = n_2 = \rho_{22} n$

$$\frac{d\rho_{22}}{dt} = \Lambda_2 - \gamma_2 \rho_{22} - \frac{i}{2}(\Omega^* \sigma_{21} - \Omega \sigma_{12})$$

$$\frac{d\Delta n}{dt} = -\frac{1}{\tau}(\Delta n - \Delta n_0) - \frac{in}{2}\left(\frac{2A^* d}{\hbar}\sigma_{21} - \frac{2Ad}{\hbar}\sigma_{21}^*\right)$$

$$\frac{d\Delta n}{dt} = -\frac{(\Delta n - \Delta n_0)}{\tau} - \frac{i}{\hbar}(A^* P - A P^*)$$

where $\tau_1 = \tau_2 = \dfrac{1}{\gamma_2}$ and $\Delta n_0 = n \Lambda_2 \tau$

Evolution of field

EM field must be solution of Maxwell's equations

$$\frac{\partial^2 E}{\partial z^2} - \epsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2} - \zeta \mu_0 \frac{\partial E}{\partial t} = \mu_0 \frac{\partial^2 P}{\partial t^2}$$

$$P = P_{mat} + P_{at}$$

average conductivity ζ

Assume that the cavity has low loss in propagation

$\Rightarrow$ we neglect $\frac{c_0}{n_0} \left| \frac{\partial A}{\partial z} \right| << \frac{\partial A}{\partial t}$

$$\frac{dA}{dt} = -\frac{1}{2\tau_{cav}} A + i \frac{\omega}{2\epsilon} P \qquad \tau_{cav} = \frac{\epsilon}{\zeta}$$

In presence of cavity detuning $\delta_{cav} = \omega - \omega_q$

$$\Rightarrow \frac{dA}{dt} = -\left(\frac{1}{2\tau_{cav}} - i\delta_{cav} \right) A + i \frac{\omega}{2\epsilon} P$$

Maxwell-Bloch equations

$$\frac{dA}{dt} = -\left(\frac{1}{2\tau_{cav}} - i\delta_{cav}\right)A + i\frac{\omega}{2\epsilon}P$$

$$\frac{dP}{dt} = -(\Gamma - i\delta)P - i\frac{d^2}{\hbar}A\Delta n$$

$$\frac{d\Delta n}{dt} = -\frac{1}{\tau}(\Delta n - \Delta n_0) - \frac{i}{\hbar}(A^*P - AP^*)$$

Contain a relaxation term with lifetime τ_{cav}, τ or Γ^{-1}

Adiabatic elimination and laser classes

<u>Class C laser</u> : All lifetimes: same order of magnitude.

Example: NH_3 Maser (far infrared) exhibits deterministic chaos.

<u>Class B laser</u>: $\tau_{cav}, \tau >> \Gamma^{-1}$ and τ_{cav} comparable to τ

$$\frac{dP}{dt} = 0.$$ Eliminate P. Ex: CO_2 laser

<u>Class A laser</u>: $\tau_{cav} >> \tau, \Gamma^{-1}$

Eliminate P and Δn. Ex: most gas and dye lasers

Adiabatic elimination of P

$$\frac{dP}{dt} = 0 \qquad \Rightarrow -(\Gamma - i\delta)P - i\frac{d^2}{\hbar}A\Delta n = 0$$

$$\Rightarrow P = -i\frac{d^2}{\hbar}\frac{1}{(\Gamma - i\delta)}A\Delta n$$

Substitute in eqn for Δn. $$\frac{d\Delta n}{dt} = -\frac{1}{\tau}(\Delta n - \Delta n_0) - \frac{i}{\hbar}(A^*P - AP^*)$$

Use, $$I = 2\epsilon_0 n_0 c_0 |A|^2$$

$$I_{sat} = \frac{\epsilon_0 c_0 n_0 \hbar^2}{d^2\Gamma}\frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2 - A}(\Gamma^2 + \delta^2)$$

Rate equation approximation

$$\frac{d\Delta n}{dt} = \frac{1}{\tau}\left(\Delta n_0 - \Delta n - \frac{I}{I_{sat}}\Delta n\right)$$

$$\frac{dI}{dt} = \frac{I}{\tau_{cav}}\left(\frac{\Delta n}{\Delta n_{th}} - 1\right)$$

Number of photons (intercavity) $F = \frac{I}{\hbar\omega}\frac{n_0 L_{cav}}{c_0} S$

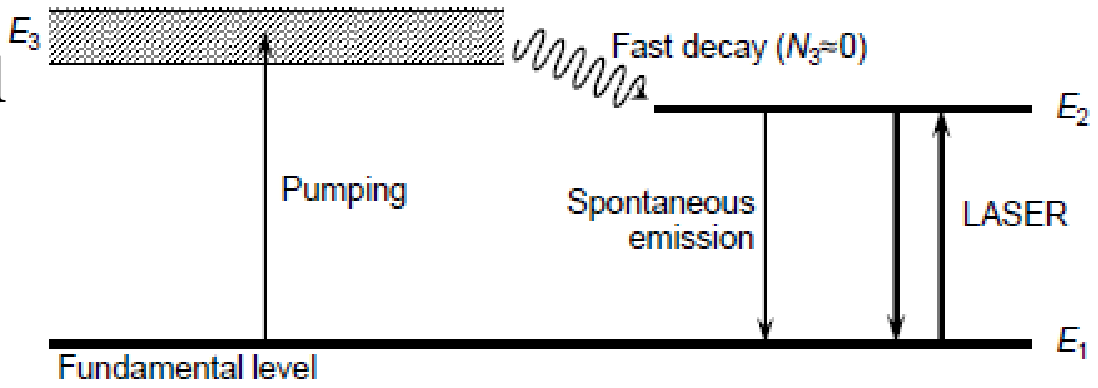
Population inversion $\Delta N = V_{cav}\Delta n = L_{cav} S \Delta n$

$$\frac{dF}{dt} = -\frac{F}{\tau_{cav}} + \kappa F \Delta N$$

$$\frac{d}{dt}\Delta N = -\frac{1}{\tau}(\Delta N - \Delta N_0) - \kappa F \Delta N$$

$$\kappa = \frac{c_0}{n_0}\frac{\sigma}{V_{cav}}$$

Three level systems



Level 1- fundamental level of considered atom.

Level 2- often (but not always) the first excited level

Level 3- intermediate level to pump level 2.

W_p- pumping prob./unit time

A- spt. emsn. prob. / unit vol.

3-level system closed. $N_1 + N_2 = N$

We suppose that decay of level 3 fast enough to make $N_3 = 0$.

$$\frac{dN_2}{dt} = W_p N_1 - AN_2 - \kappa F \Delta N$$

$$\frac{dN_1}{dt} = -W_p N_1 + AN_2 + \kappa F \Delta N$$

$$N_1 + N_2 = N$$

$$N_1 - N_2 = -\Delta N$$

$$N_1 = \frac{N - \Delta N}{2}$$

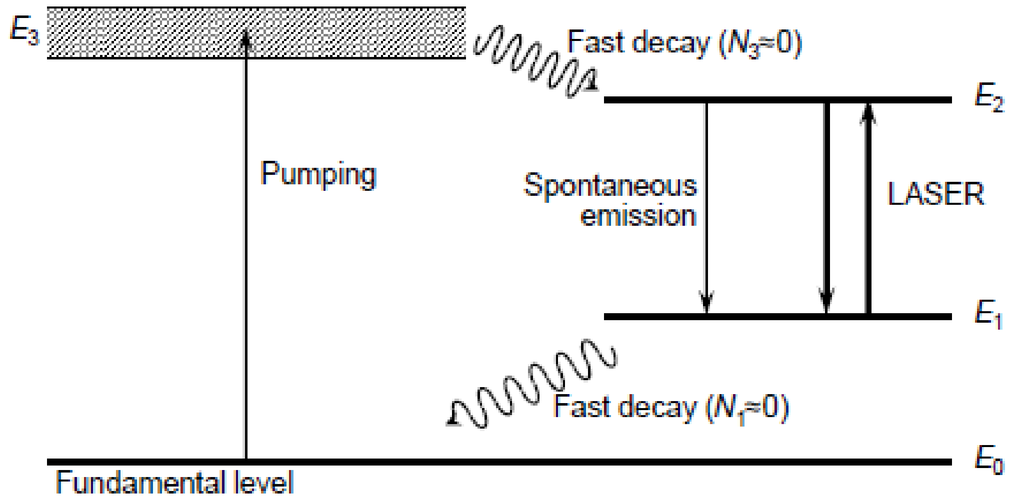
$$N_2 = \frac{N + \Delta N}{2}$$

Four level system

Assume : levels 3 and 1 decay fast enough $N_1 \sim N_3 = 0$

System is closed : $N_0 + N_2 = N$

$$\frac{dN_2}{dt} = W_p N_0 - AN_2 - \kappa F N_2$$
$$\frac{dN_0}{dt} = -W_p N_0 + AN_2 + \kappa F N_2$$

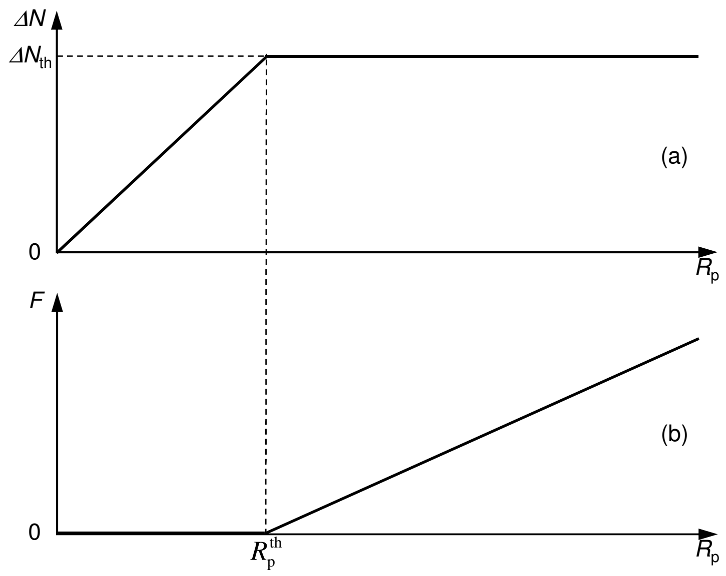


Both lead to Statz deMars equations

Steady state

$$\frac{dF}{dt} = -\frac{F}{\tau_{cav}} + \kappa F \Delta N$$

$$\frac{d}{dt}\Delta N = -\frac{1}{\tau}(\Delta N - \Delta N_0) - \kappa F \Delta N$$



$$\Delta N = \frac{1}{\kappa \tau_{cav}} = \Delta N_{th}$$

$$F = \frac{1}{\kappa \Delta N_{th}}[R_p - R_p^{th}], \quad R_p^{th} = \frac{1}{\kappa \tau \tau_{cav}}$$

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Thank you

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