Lagrangians, nodal curves and mirror symmetry on the Hitchin system

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Joint work in progress with E. Franco, R. Hanson, J. Horn.

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What's going on?

I. Hecke Lagrangians in the Hitchin system

Higgs bundles and their moduli

- $X \rightsquigarrow$ smooth projective curve over $\mathbb{C}$, genus $g \geq 2$.
- $K \rightsquigarrow$ canonical bundle of X.
- Higgs bundle $\rightsquigarrow (E, \varphi)$, with
  - $E \rightsquigarrow$ vector bundle over X of rank n.
  - $\varphi : E \longrightarrow E \otimes K \rightsquigarrow$ Higgs field.
- $G = \mathrm{GL}(n, \mathbb{C})$.
- Take Higgs bundles of degree $-n(n-1)(g-1)$.

Hitchin '87, Simpson '92

$\mathcal{M} \rightsquigarrow$ moduli space of semistable Higgs bundles.

- $\mathcal{M} \rightsquigarrow$ quasi-projective of dimension $n^2(2g-2)+2$.
- $\Omega \rightsquigarrow$ holomorphic symplectic form on (the smooth locus of) $\mathcal{M}$.

The Hitchin map and spectral curves

Hitchin '87

Hitchin map $\rightsquigarrow$

$$h : \mathcal{M} \longrightarrow B := \bigoplus_{i=1}^{n} H^0(X, K^i)$$
$$(E, \varphi) \longmapsto (\mathrm{tr}(\varphi), \ldots, \mathrm{tr}(\wedge^i \varphi), \ldots, \det(\varphi)).$$

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$$\lambda^n + \cdots + a_i \lambda^{n-i} + \cdots + a_n \in H^0(|K|, \pi^* K^n).$$

- $\pi : |K| \longrightarrow X$, $\lambda \in H^0(|K|, \pi^* K)$, $\lambda(q) = q$.
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- $a \in B$ generic $\Longrightarrow C_a$ smooth.
- $\mathrm{Jac}^0_{C_a} \rightsquigarrow$ (generalised) Jacobian of degree of C_a.
- $\overline{\mathrm{Jac}}^0_{C_a} \rightsquigarrow$ compactified Jacobian $\rightsquigarrow$ projective variety, parameterising semistable rank 1 torsion-free sheaves of degree 0 on C_a.

Its fibres

Hitchin '87; Beauville–Narasimhan–Ramanan '87; Schaub '98

$\overline{\mathrm{Jac}}^0_{C_a} \cong h^{-1}(a)$, the isomorphism given by $\mathcal{F} \mapsto (E, \varphi) = (\pi_{a,*}\mathcal{F}, \pi_{a,*}\lambda)$.

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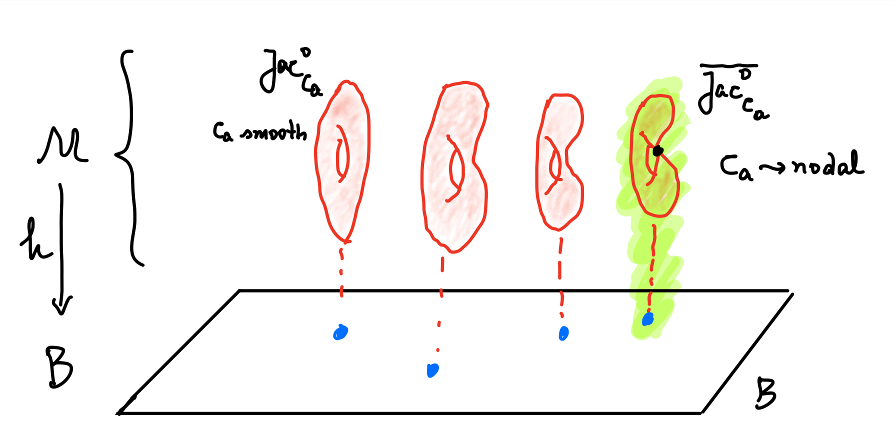
- $\left\{ \begin{array}{c} \text{semistable Higgs} \\ \text{bundles on } X \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{semistable rank 1 torsion-free} \\ \text{sheaves supported on some} \\ \text{spectral curve in } |K| \end{array} \right\}$

The integrable system

- $a \in B$ generic $\Longrightarrow$ generic fibres are abelian varieties.

Hitchin '87

$(\mathcal{M}, h) \rightsquigarrow$ integrable system $\rightsquigarrow$ the Hitchin system.



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  For each $a \in B^1$, take those Higgs bundles whose spectral data lies in the closure of a fibre of ν_a^*. Then take the closure of this i.e. allow $a \in \overline{B^1} \subset B$.

Their (rough) definition

- Take $a \in B^1$. Set $\widetilde{\pi}_a = \pi_a \circ \nu_a$.

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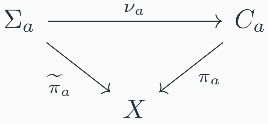

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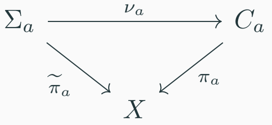

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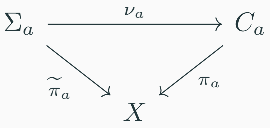

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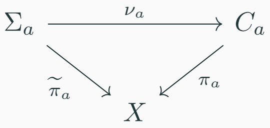

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- $k = 0 \implies \mathcal{L} =$ Hitchin section.

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Proof.

$P = (E, \varphi) \in \mathcal{L}$ generic i.e. $P \in \mathcal{L} \cap \mathrm{Jac}^0_{C_a}$ for some $a \in B^1$.

$p =$ node of C_a; $x = \pi_a(p) \in X$.

$$0 \longrightarrow \mathcal{O}_{C_a} \longrightarrow \nu_{a,*}\mathcal{O}_{\Sigma_a} \longrightarrow \mathcal{O}_p \longrightarrow 0 \quad \Longrightarrow \quad 0 \longrightarrow \pi_{a,*}\mathcal{O}_{C_a} \longrightarrow \widetilde{\pi}_{a,*}\mathcal{O}_{\Sigma_a} \longrightarrow \mathcal{O}_x \longrightarrow 0$$

$$\begin{array}{ccccc}
T_P(\nu_a^*)^{-1}(\mathcal{O}_{\Sigma_a}) \cong H^0(X, \mathcal{O}_x) \cong \mathbb{C} & \hookrightarrow & \boxed{T_P\mathcal{L}} & \twoheadrightarrow & H^0(X, (\widetilde{\pi}_{a,*}\mathcal{O}_{\Sigma_a})^* \otimes K) \cong T_{h(P)}B^1 \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
T_P\overline{\mathrm{Jac}}^0_{C_a} \cong H^1(X, \pi_{a,*}\mathcal{O}_{C_a}) & \hookrightarrow & T_P\mathcal{M} & \xrightarrow{dh_P} & H^0(X, (\pi_{a,*}\mathcal{O}_{C_a})^* \otimes K) = T_{h(P)}B \\
\big\downarrow & & & & \big\downarrow \\
T_{\mathcal{O}_{\Sigma_a}}\mathrm{Jac}^0_{\Sigma_a} \cong H^1(X, \widetilde{\pi}_{a,*}\mathcal{O}_{\Sigma_a}) & & & & H^0(X, \mathcal{O}_x)^*
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  at $x \in X$.

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- $\mathcal{L} \rightsquigarrow$ Lagrangian of Hecke cycles (the $\mathbb{P}^1$'s).

II. Branes and mirror symmetry in $\mathcal{M}$

BBB and BAA-branes in $\mathcal{M}$

- Lagrangian in $\mathcal{M} \rightsquigarrow$ brane in $\mathcal{M}$...
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    But the flatness condition seems to be 'oversimplified'...

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"...a rank 1 A-brane on $\mathcal{L}$ is usually described as a complex line bundle $L \to \mathcal{L}$ with a flat unitary connection (...). This is in general oversimplified (...). The correct global statement is that L is a flat spin$_c$ structure over $\mathcal{L}$."

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$$w_2(M) = c_1(L) \bmod 2.$$

- Spin-structure $\Longrightarrow$ flat Spin_c structure.

Kapustin–Witten's Conjecture

Kapustin–Witten '07

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- The supports of dual branes have same image under h.
- One way to test this $\rightsquigarrow$ restrict to a generic fibre of h.

  For the Lagrangian $\mathcal{L}$ $\rightsquigarrow$ restrict to fibres corresponding to nodal spectral curves.

What we want to do

- Recall: $a \in B^1 \rightsquigarrow \mathcal{L} \cap h^{-1}(a) = \overline{(\nu_a^*)^{-1}(\mathcal{O}_{\Sigma_a})} \cong \mathbb{P}^1 \subset \overline{\mathrm{Jac}}^0_{C_a}$.

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> Compute the Fourier-Mukai transform of $\mathcal{O}(-1) \in \mathbf{D}^b(\overline{\mathrm{Jac}}^0_{C_a})$.

III. Fourier–Mukai transforms and normalisations

Fourier–Mukai on (compactified) Jacobians

- Fix $a \in B^1 \rightsquigarrow C_a = C \rightsquigarrow \nu : \Sigma \longrightarrow C$ normalisation. $\mathrm{Jac}^0_\Sigma \rightsquigarrow$ abelian variety.

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$\Phi_\Sigma : \mathbf{D}^b(\mathrm{Jac}^0_\Sigma) \longrightarrow \mathbf{D}^b(\mathrm{Jac}^0_\Sigma)$ given by

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is an equivalence of categories. $\Phi_\Sigma \rightsquigarrow$ Fourier–Mukai transform.

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- $(M, V) \in \mathrm{PMod}_C^0 \rightsquigarrow V$ records the gluing data to produce a rank 1 torsion-free sheaf on C. More precisely...

PMod_C^0 as an intermediate space between $\overline{\mathrm{Jac}}_C^0$ and Jac_Σ^0

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- The morphism ρ is finite and a resolution of singularities of $\overline{\mathrm{Jac}}_C^0$.
- In particular, PMod_C^0 is also a compactification of Jac_C^0.
- The morphism $\dot{\nu}$ is a $\mathbb{P}^1$-bundle.

$$\begin{array}{ccccc}
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- In particular, $\rho_* \mathcal{O}(-1) \cong \mathcal{O}(-1)$.

Universal parabolic module

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Universal parabolic module

- $\mathrm{PMod}^0_C \rightsquigarrow$ fine moduli space. Universal object:

$$((1 \times \dot{\nu})^* \mathcal{U}, \mathcal{V}),$$

where:

  - $\mathcal{U} \longrightarrow \Sigma \times \mathrm{Jac}^0_\Sigma \rightsquigarrow$ universal line bundle normalised at some y_0 i.e. $\mathcal{U}_{y_0} \cong \mathcal{O}_{\mathrm{Jac}^0_\Sigma}$.
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For any $\mathcal{E}^\bullet \in \mathbf{D}^b(\mathrm{Jac}^{-1}_\Sigma)$, the following isomorphisms hold in $\mathbf{D}^b(\overline{\mathrm{Jac}}^0_C)$:

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- Applying Φ_C yields

$$\boxed{\Phi_C(\mathcal{O}(-1)) \cong \mathcal{O}_{\mathrm{Im}(\check{\nu})}}$$

Lagrangians of Hecke cycles and critical loci

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For every $a \in B^1$, the Fourier–Mukai transform of $\mathcal{O}(-1) \in \mathbf{D}^b(h^{-1}(a))$, whose support is $\mathcal{L} \cap h^{-1}(a)$, is $\mathcal{O}_{\mathrm{Im}(\check{\nu}_a)} \in \mathbf{D}^b(h^{-1}(a))$, the trivial line bundle over $\mathrm{Im}(\check{\nu}_a) \subset \overline{\mathrm{Jac}}^0_{C_a} \cong h^{-1}(a)$. I.e. $\Phi_{C_a}(\mathcal{O}(-1)) \cong \mathcal{O}_{\mathrm{Im}(\check{\nu}_a)}$

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- $\mathcal{C} \subset \mathcal{M} \rightsquigarrow$ closure of loci where rank of h drops by $1 \rightsquigarrow 1^{st}$-critical loci of h.
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Franco–Gothen–O.–Peón-Nieto '21

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