# Lectures on Hydrodynamics

(Abhishek Dhar)

## Preparatory Lectures
## BSSP 2025.

Part (I) ASEP – Burgers Equation.

Part (II) Simple Fluids.

Part (III) Quantum Systems

Part (IV) Connection to Boltzmann's Equation.

Part (V) A microscopic derivation of Euler equations of a fluid.

$\boxed{\text{General references}}$ (for Part II)

* Landau and Lifschitz – Fluid Mechanics
* Blandford and Thorne – Classical Physics
* Chaikin and Lubensky – Principles of Condensed matter Physics.

# MICRO — MACRO Description 1

**Microscopic picture** $X = \{q_i, p_i\}$

$i = 1, 2 \cdots dN$

$d$ = dimensions.

Hamiltonian: $H = H(X)$.

Time evolution: $\dot{q}_i = \partial H / \partial p_i$ ; $\dot{p}_i = -\partial H / \partial q_i$.

$$\{q_i(0), p_i(0)\} \xrightarrow[\text{FIXES.}]{} \{q_i(t), p_i(t)\}.$$

QM : $|\psi_t\rangle$ $\quad i\hbar |\dot{\psi}_t\rangle = H |\psi_t\rangle$.

$$|\psi_0\rangle \longrightarrow |\psi_t\rangle.$$

**Macroscopic picture** :

Equilibrium. : We care only about a few variables → the conserved quantities.

Conserved quantities :

Most systems → Energy $H(x) \equiv E$

Number of particles $N$

Total Momentum $P$

Equilibrium properties : given by $S\Big(\underbrace{E - \frac{P^2}{2M}}_{U}, N\Big)$

Boltzmann–Gibbs theory of ensembles.

Integrable Models : Generalized Gibbs Ensembles.

$(I_1, I_2, \ldots I_N) \rightarrow$ Extensive # of Conserved quantities.

- Quantum Newton's Cradle Nature 2006 Weiss et. al. / Science 2021. (1D Bose Gas)
- KPZ in Quantum Spin Chain – Nat. Phys. 2021 (Moore/Tennant)

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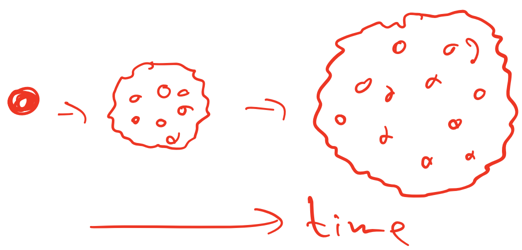
Out of equilibrium.

Water flowing in a tube

- Nature 2019 Ilani et al – Poiseuille flow in Graphene



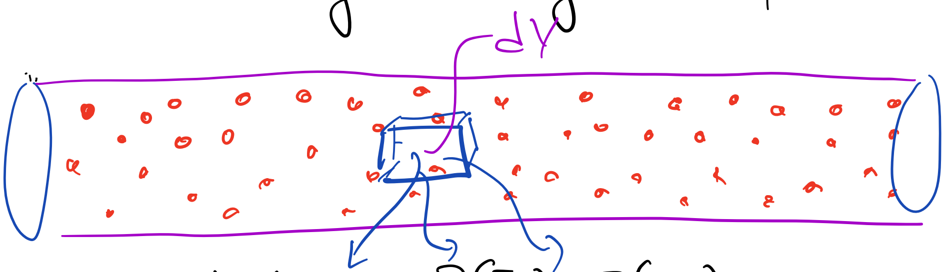
time

Evolution of a blast.

3

Extension of equilibrium.

Local equilibrium: the conserved fields vary slowly in space and time.




$N(\bar{x},t)$, $\mathbb{P}(\bar{x},t)$, $E(\bar{x},t)$.

Density field $\rho(\bar{x},t) = \dfrac{N(\bar{x},t)}{dV}$

Momentum field $p(\bar{x},t) = \dfrac{\mathbb{P}(\bar{x},t)}{dV}$

$$e(\bar{x},t) = \frac{E(\bar{x},t)}{dV}$$

OR Formally : $\rho(\bar{x},t) = \sum_{i=1}^{N} \langle \delta(\bar{x} - \bar{x}_i(t)) \rangle$ $\quad \underbrace{\langle\ \rangle}_{\text{ensemble.}}$

$$p(\bar{x},t) = \sum_{i=1}^{N} \langle \bar{p}_i(t)\, \delta(\bar{x} - \bar{x}_i(t)) \rangle$$

$$e(\bar{x},t) = \sum_{i=1}^{N} \langle e_i\, \delta(\bar{x} - \bar{x}_i(t)) \rangle$$

$$e_i = \frac{\bar{p}_i^2}{2} + \frac{1}{2}\sum_j V(r_{ij})$$

The equations of hydrodynamics tells us how these fields evolve with time :

* Because they are conserved locally, therefore they evolve slowly in space and in time.

* Typical form is

$$\partial_t u^a(\bar{x},t) = -\nabla \bar{j}^a(\bar{x},t)$$

$$\bar{j}^a(\bar{x},t) = \underset{\text{EULER}}{\bar{j}^a(\bar{x},t)} + \underset{\text{NS}}{\bar{j}^a(\bar{x},t)}$$

EULER ↓ Reversible, No entropy production

NS ↓ Dissipative, Positive entropy production.

* All the complicated microscopic details go into :

A. Equation of state. (EULER PART)

B. Transport Coefficients

$\eta$, $\underline{\mu}$, $\kappa$ → Viscosity, thermal conductivity

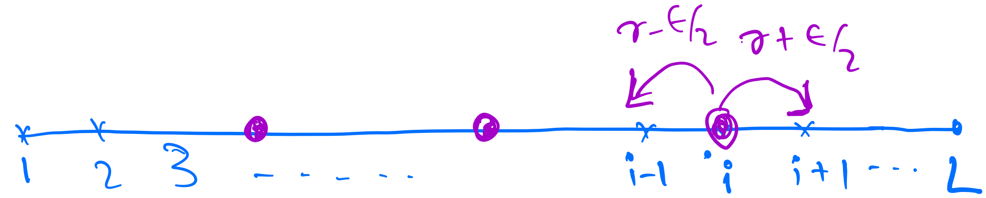
# PART – I

Simple example – Burgers equation.
Simpler → Diffusion equation.

Microscopic model : ASEP

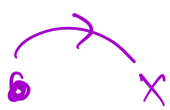
Asymmetric Simple Exclusion Process.

* models cars on a single lane,
* ions in cellular ion channels
* Confined Colloidal particles.




Rules

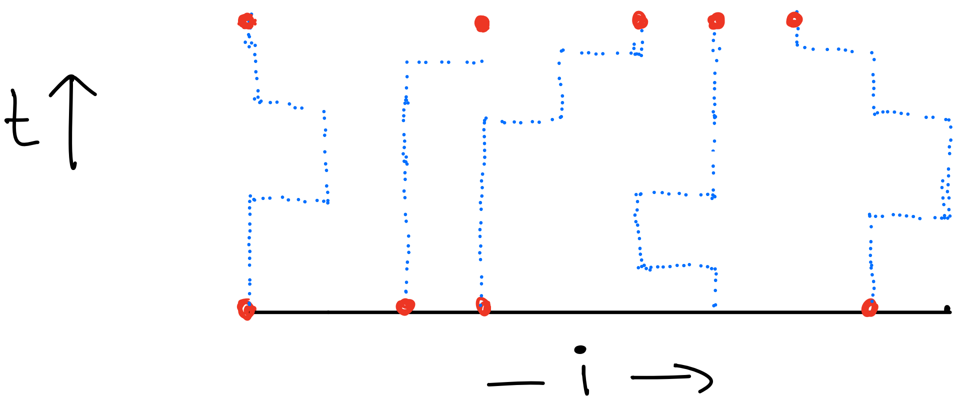
- $i = 1, 2, \ldots L$ sites.
- $N$ occupied sites $\{n_i = 0, 1\}$.

  $\sum n_i = N$.
- $X = \{n_1, n_2 \ldots n_L\}$ Complete Microscopic Description.
- Dynamics  particles hop to empty sites on the right with rate $\left(r + \frac{\epsilon}{2}\right)$

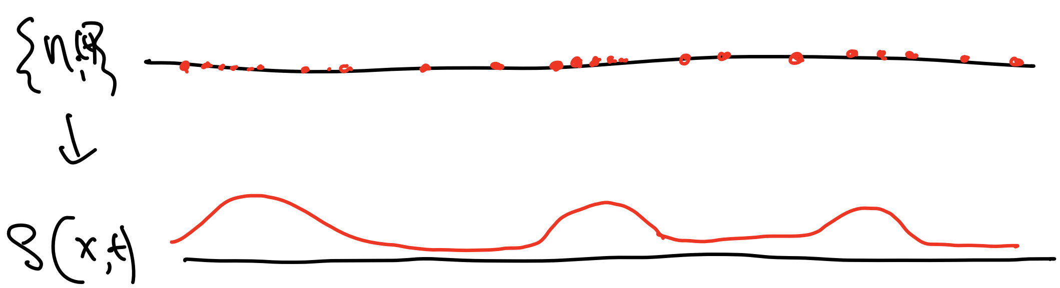
6


Particles hop to the left with rate $(r - \epsilon/2)$

Space time evolution.




- Only conserved quantity number of particles $\Rightarrow$ Hydrodynamic field $\rho(x,t)$.



Microscopic $\Rightarrow$ Hydrodynamic Eqn.

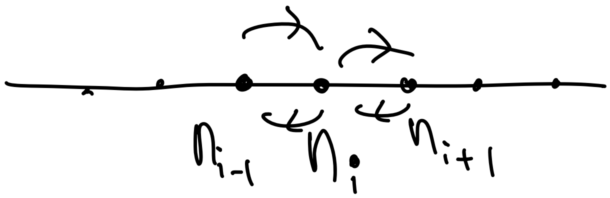
$\partial_t P(X) = W P(x)$.

$\parallel$

C.

Master equation description of stochastic dynamics

eqn. for $\rho(x,t)$.




$$\frac{d\langle n_i\rangle}{dt} = (r+\epsilon/2)\langle n_{i-1}(1-n_i)\rangle - (r-\epsilon/2)\langle n_i(1-n_{i-1})\rangle + (r-\epsilon/2)\langle n_{i+1}(1-n_i)\rangle - (r+\epsilon/2)\langle n_i(1-n_{i+1})\rangle$$

$$= r\left[\left(\langle n\rangle_{i-1} - \langle n_i\rangle\right) - \left(\langle n\rangle_i - \langle n_{i+1}\rangle\right)\right]$$

$$+ \frac{\epsilon}{2}\left[\langle n_{i-1}(1-n_i)\rangle + \langle n_i(1-n_{i-1})\rangle - \left\{\langle n_i(1-n_{i+1})\rangle + \langle n_{i+1}(1-n_i)\rangle\right\}\right]$$

$$= j_{i-1\to i} - j_{i\to i+1} = -\partial_i j$$

where $j_{i\to i+1} = -r\left(\langle n\rangle_{i+1} - \langle n_i\rangle\right)$

$$+ \frac{\epsilon}{2}\left[\langle n_i(1-n_{i+1})\rangle + \langle n_{i+1}(1-n_i)\rangle\right]$$

So far this is exact.

Now we make approximations.

$$\langle n_i\, n_{i+1}\rangle = \rho_i \rho_{i+1}$$

Then $j_i = \frac{\epsilon}{2}\left[\rho_i + \rho_{i+1} - 2\rho_i\rho_{i+1}\right] - r(\rho_{i+1} - \rho_i)$

$$= \epsilon\, \rho_i(1-\rho_i) + \frac{\epsilon}{2}\left(\frac{\partial\rho}{\partial i} - 2\rho_i\frac{\partial\rho}{\partial i}\right) - r\frac{\partial\rho}{\partial i}$$

Let $x = ia$

$$j(x,t) = -ra^2\partial_x\rho + \epsilon a\rho(1-\rho) + \underbrace{\frac{\epsilon a^2}{2}\partial_x[\rho(1-\rho)]}_{\underbrace{\epsilon a}_{\text{finite}} \times \underbrace{a}_{\text{small}} \to 0.}$$

$= a j_i$

$\therefore\ \partial_t\rho = -\partial_x j$

$$\partial_t\rho = D\,\partial_x^2\rho - \lambda\,\partial_x\,\rho(1-\rho)$$

$$D = ra^2 \qquad \lambda = \epsilon a$$

9

$$j(x,t) = -D\,\partial_x \rho + \lambda \rho(1-\rho)$$

NS Euler

$$\partial_t \rho + \partial_x \underbrace{\lambda \rho(1-\rho)}_{\text{Drift Current}} = \underbrace{D\,\partial_x^2 \rho}_{\text{Diffusive Current}}$$

$x' = x - \lambda t \qquad t' = t$

$\rho'(x',t') = \rho(x,t)$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial x} = \frac{\partial}{\partial x'}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial t}$$

$$= -\lambda \frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}$$

$$\therefore -\lambda \frac{\partial \rho'}{\partial x'} + \frac{\partial \rho'}{\partial t'} + \lambda \frac{\partial \rho'}{\partial x'} - \lambda \frac{\partial}{\partial x'}\rho'^2$$

$$= D \frac{\partial \rho'}{\partial x'^2}$$

$$\equiv \frac{\partial \rho}{\partial t} - \lambda \frac{\partial}{\partial x}\rho^2 = D \frac{\partial \rho}{\partial x^2}$$

$\rho' \to \rho$
$x' \to x$
$t' \to t.$

$\rho \longrightarrow -\rho$

$$\frac{\partial \rho}{\partial t} + \lambda \frac{\partial}{\partial x}\rho^2 = D \frac{\partial^2 \rho}{\partial x^2}$$

$\lambda = \frac{1}{2}$ : Standard Burgers Equation. 10

$S = \partial_x h$

$\partial_t h + \lambda (\partial_x h)^2 = D \partial_x^2 h.$ → KPZ equation.

$h = \frac{-D}{\lambda} \ln Z$ (EXACT SOLUTION)

$\partial_t h = \frac{-D}{\lambda} \frac{1}{Z} \partial_t Z \qquad \partial_x h = \frac{-D}{\lambda} \frac{1}{Z} \partial_x Z$

$\partial_x^2 h = +\frac{D}{\lambda} \frac{1}{Z^2} (\partial_x Z)^2 - \frac{D}{\lambda} \frac{\partial_x^2 Z}{Z}$

$$\therefore \frac{-D}{\lambda} \frac{1}{Z} \partial_t Z + \frac{D^2}{\lambda} \frac{1}{Z^2} (\partial_x Z)^2$$

$$= \frac{D^2}{\lambda} \frac{1}{Z^2} (\partial_x Z)^2 - \frac{D^2}{\lambda} \frac{\partial_x^2 Z}{Z}$$

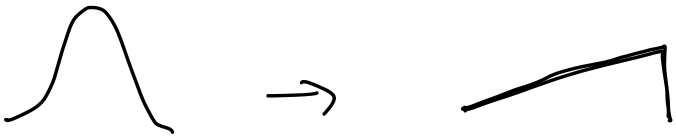
$\Rightarrow \boxed{\partial_t Z = D \partial_x^2 Z}$ → Diffusion equation.

Easy to solve.

- Euler equation (Inviscid Burgers Eqn.)

$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0.$$

Develops shocks in finite time



- Viscous Burgers equation.

$$u(x,t) = \frac{1}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right)$$

Self Similar Solution.

$$\partial_t u = -\frac{1}{2\, t^{3/2}} f - \frac{x}{2\, t^{1/2} t^{3/2}} f'$$

$$\boxed{\begin{aligned} \xi &= x/t^{1/2} \\ f' &= df/d\xi. \end{aligned}}$$

$$\partial_x u^2 = \frac{1}{t}\frac{1}{t^{1/2}} 2 f f'$$

$$\therefore -\frac{1}{2 t^{3/2}}\left(f + \xi f'\right) + \frac{1}{2 t^{3/2}} 2 f f' = 0$$

$$\Rightarrow -\left(f + \xi f'\right) + 2 f f' = 0 \quad *$$

11'

Viscous Case

$$\partial_x^2 u = \frac{1}{t^{3/2}} f''$$

$$\therefore \; -(f + \xi f') + 2 f f' = 2 f''$$

Self-Similar Scaling Solution

$$u(x,t) = \frac{1}{\sqrt{t}} f\left(\frac{x}{\sqrt{2 \nu t}}\right)$$

At long times, Solution forgets initial Conditions

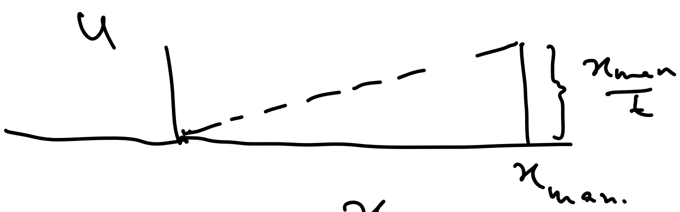
$$\int u \, dx = \int \frac{dx}{\sqrt{t}} f\left(\frac{x}{\sqrt{2\nu t}}\right) = \underset{\sim}{A}$$

Conserved.

Euler eqn. *

Solution $F = z$

$$\therefore \; u = \frac{1}{\sqrt{t}} \frac{x}{\sqrt{t}} = \frac{x}{t}$$

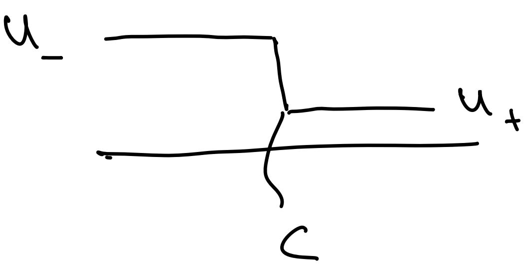



$$\int_0^{x_{max}} u \, dx = \frac{x_{max}^2}{2t} = A \quad \text{Conserved.}$$

$$\therefore \; x_{max} = \sqrt{2At}$$

Long time → forgets initial conditions except A


$\boxed{\text{Other interesting solutions}}$ : → Domain Wall



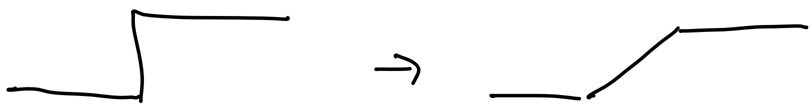

$$u(x,t) = g(x - ct)$$

$$-c g' + j' = 0$$

$$\Rightarrow j_+ - c g_+ = j_- - c g_-$$

$$\Rightarrow c = \frac{j_+ - j_-}{g_+ - g_-}$$

$$\therefore \quad c = \frac{u_+^2/2 - u_-^2/2}{u_+ - u_-} = \frac{1}{2}(u_+ + u_-)$$ 13



Summary.

- Microscopic model ASEP
- Single Conserved field $\rho(x,t)$
- Hydrodynamic description Burgers

Provides an easier way to understand observable behaviour of a macroscopic system.

- Self Similar solutions
  Domain Wall }.

# PART II

# FLUIDS

## Three conserved quantities

## Fluid equations (1D)

Fields: $\rho(x,t)$, $p(x,t)$, $e(x,t)$

Physical fields $u(x,t)$ $T(x,t)$

$$p(x,t) = \rho u$$

$$e = \rho\left(\frac{u^2}{2} + \epsilon(\rho,T)\right)$$

Flow energy.

internal energy per particle (in moving frame)

$\epsilon(\rho,T)$ from equilibrium physics.

# Equations of motion.

$$\partial_t \rho(x,t) + \partial_x(\rho u) = 0$$

$$\partial_t \rho u + \partial_x(\rho u^2 + P) = \partial_x(\eta \partial_x u)$$

$$\partial_t e + \partial_x (e+P)u = \partial_x(\eta u \partial_x u + \kappa \partial_x T)$$

$P = P(\rho, T)$ pressure

$$\left.\begin{array}{l}\eta \rightarrow \text{bulk modulus} \\ \kappa \rightarrow \text{thermal Conductivity}\end{array}\right\} \text{Transport Coefficients.}$$

# Fluids 3 D

16

5 Conserved Fields

$\rho(\bar{x},t)$ – Density 1

$\bar{p}(\bar{x},t) = \rho\,\bar{u}(\bar{x},t)$ = Momenta 3

$e(\bar{x},t) = \rho\left(\frac{\bar{u}^2}{2} + \epsilon\right)$ = Energy 1

Law of mass Conservation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\bar{u}) = 0$$

Momentum Conservation

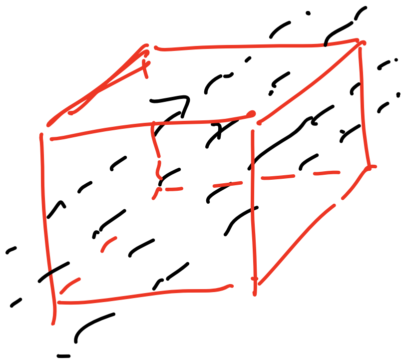
$$\frac{\partial(\rho\bar{u})}{\partial t} + \nabla\cdot T = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\rho u^\alpha + \frac{\partial}{\partial x^\beta}T^{\alpha\beta} = 0$$

$$T^{\alpha\beta} = \underbrace{\rho u^\alpha u^\beta + P\delta^{\alpha\beta}}_{\text{Euler}} + \underbrace{\Sigma^{\alpha\beta}}_{\text{NS}}$$

$\rho u^\alpha u^\beta \rightarrow$ Momentum

(17)



Flux due to Flow.

Similar to $j^\alpha = \rho u^\alpha$ (Mass flux in $\alpha$-direction)

But now $j^{\alpha\beta} = \underbrace{\rho u^\alpha}_{\text{Momentum Density in } \alpha\text{-direction}} u^\beta$ (Flux in $\beta$-direction)

Pressure Force $= -\dfrac{\partial P}{\partial x^\alpha}$

Force on volume V $= -\int P \, d\bar{s}$

$= -\int \bar{\nabla} P \, dv$

$\therefore$ Force per unit volume

$= -\bar{\nabla} P$

# Euler Equation

(18)

$$\frac{\partial}{\partial t}(\rho u^\alpha) + \frac{\partial}{\partial x^\beta}\left(\rho u^\alpha u^\beta + P\delta^{\alpha\beta}\right) = 0$$

$$\Rightarrow \rho\left(\frac{\partial u^\alpha}{\partial t} + u^\beta \frac{\partial u^\alpha}{\partial x^\beta}\right) = -\frac{\partial P}{\partial x^\alpha}$$

Since $u^\alpha\left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^\beta}(\rho u^\beta)\right) = 0$

(mass cons. eqn.)

## Convective derivative

$$dA(\bar{x},t) = \frac{\partial A}{\partial \bar{x}} \cdot d\bar{x} + \frac{\partial A}{\partial t} dt$$

Along fluid flow $d\bar{x} = \bar{u}\,dt$

$$\therefore dA = \frac{\partial A}{\partial \bar{x}} \cdot \bar{u}\,dt + \frac{\partial A}{\partial t} dt$$

$$\Rightarrow \frac{dA}{dt} = \bar{u} \cdot \nabla A + \frac{\partial A}{\partial t}.$$

∴ we can write Euler's equation as

$$\rho \frac{d\bar{u}}{dt} = -\bar{\nabla} P$$

or $\frac{d}{dt}(\rho \bar{u}\, dv) = -\bar{\nabla} P\, dv.$

Newton's law for a fluid Element

$\boxed{\text{Euler - 1755}}$

In the presence of gravity.

$$\rho \frac{d\bar{u}}{dt} = -\bar{\nabla} P + \rho \bar{g} \quad -\nabla \Phi$$

$$\frac{d\bar{u}}{dt} = \frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \nabla \bar{u}$$

$$= \frac{\partial \bar{u}}{\partial t} + \frac{1}{2}\nabla(u^2) - \bar{u} \times \underbrace{(\nabla \times \bar{u})}_{\bar{\omega} \to \text{Vorticity}}$$

$$\therefore \frac{\partial \bar{u}}{\partial t} + \nabla\left(\frac{1}{2}u^2 + \Phi\right) + \frac{\nabla P}{\rho} - \bar{u} \times \bar{\omega} = 0$$

Along streamlines : $\frac{1}{2}u^2 + \Phi + h = \text{Const.}$

Bernoulli.

$h = e + P/\rho$

$$\Sigma^{\alpha\beta} = -\zeta\,\Theta\,\delta^{\alpha\beta} - 2 \quad {}^{\alpha\beta} \rightarrow \cancel{*} \quad 20$$

$$\Theta = \nabla\cdot\bar{u} \qquad \sigma^{\alpha\beta} = \frac{1}{2}\left( \quad {}^{\alpha}_{\beta} + \frac{\partial u^\beta}{\partial x^\alpha}\right) - \frac{1}{3}\Theta\,\delta^{\alpha\beta}$$

Note → Velocity Strain t sor:

$$\frac{\partial u^\alpha}{\partial x^\beta} = \frac{1}{2}\left(\frac{\partial u^\alpha}{\partial x^\beta} + \frac{\partial u^\beta}{\partial x^\alpha}\right) \qquad \frac{1}{2}\left(\frac{\partial u^\alpha}{\partial x^\beta} - \frac{\partial u^\beta}{\partial x^\alpha}\right)$$

$$= \frac{1}{2}\left(\frac{\partial u^\alpha}{\partial x^\beta} + \frac{\partial u^\beta}{\partial x^\alpha}\right) \quad \frac{1}{3}\Theta\,\delta^{\alpha\beta} \rightarrow \sigma$$

$$+ \frac{1}{3}\Theta\,\delta^{\alpha\beta} \rightarrow \Theta$$

$$+ \frac{1}{2}\left(\frac{\partial u^\alpha}{\partial x^\beta} - \frac{\partial u}{\partial}\right) \rightarrow \gamma$$

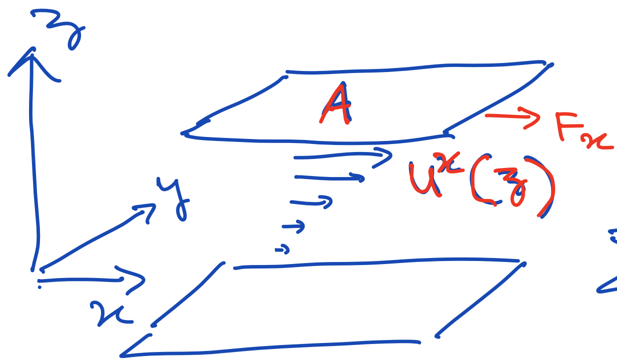
Stresses can only epend on $\sigma$ and $\Theta$

Most general relation for Isotropic Fluids: Eqn. $*$ above.

$\zeta$ → Bulk Viscosity $\eta$ → Shear Viscosity.

## Usual notion of Viscous Force

21




$$\Sigma_{xz} = \frac{F_x}{A} = \eta \frac{\partial u^x}{\partial z}$$

$$\text{or } \Sigma_{xz} = 2\eta\left[\frac{1}{2}\left(\frac{\partial u^x}{\partial z} + \frac{\partial u^z}{\partial x}\right) - \frac{1}{3}\theta\delta^{xz}\right]$$

$$\text{if } \frac{\partial u^z}{\partial x} = 0 \quad \theta = 0.$$

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## Energy conservation equation.

$$T\,ds = de + P\,d(1/\rho) \qquad s = \text{entropy per particle.}$$

$$= de - \frac{P}{\rho^2}\,d\rho$$

$$\therefore T\frac{ds}{dt} = \frac{de}{dt} - \frac{P}{\rho^2}\frac{d\rho}{dt}$$

Ideal Euler fluid
no dissipation

hence $\frac{ds}{dt} = 0$ (Adiabatic motion along stream line)

$$\therefore \quad \frac{d\epsilon}{dt} = \frac{P}{\rho^2}\frac{d\rho}{dt} = \frac{-P}{\rho^2}\rho\,\nabla\cdot\bar{u} = -\frac{P}{\rho}\nabla\cdot\bar{u}$$

$$\therefore \quad \rho\frac{d\epsilon}{dt} = -P\,\nabla\cdot\bar{u}$$

$$\Rightarrow \rho\left(\frac{\partial\epsilon}{\partial t} + \bar{u}\cdot\nabla\epsilon\right) = -P\,\nabla\cdot\bar{u}$$

Not in conservative form.

$$\frac{\partial}{\partial t}(\rho\epsilon) + \epsilon\,\nabla\cdot\rho\bar{u} + \rho\bar{u}\cdot\nabla\epsilon = -P\,\nabla\cdot\bar{u}$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho\epsilon) + \nabla\cdot(\rho\epsilon\bar{u}) = -P\,\nabla\cdot\bar{u}$$

Not yet in Conservative form! 23

Need to add Convective energy term

$e = \rho\left(\epsilon + \frac{1}{2}u^2\right)$ → Recall total energy density Formula.

$$\frac{\partial}{\partial t}\left(\rho\frac{u^2}{2}\right) = \frac{u^2}{2}\frac{\partial \rho}{\partial t} + \rho\bar{u}\cdot\frac{\partial \bar{u}}{\partial t}$$

$$= -\frac{u^2}{2}\nabla\cdot(\rho\bar{u}) + \rho\bar{u}\cdot\left[-\bar{u}\cdot\nabla\bar{u} - \frac{\nabla P}{\rho}\right]$$

$$= -\nabla\cdot\rho\frac{u^2}{2}\bar{u} - \bar{u}\cdot\nabla P$$

$$\therefore \frac{\partial}{\partial t}\underbrace{\rho\left(\frac{u^2}{2}+\epsilon\right)}_{\text{energy density.}} + \nabla\cdot\underbrace{\left(\left[\rho\left(\epsilon+\frac{u^2}{2}\right)+P\right]\bar{u}\right)}_{\text{energy current.}} = 0$$

Work done on fluid.

So finally we have the Complete Euler equations for the 5 fields

$$\left.\begin{aligned}
&\partial_t \rho + \nabla\cdot(\rho \bar{u}) = 0 \\
&\rho(\partial_t \bar{u} + \bar{u}\cdot\nabla \bar{u}) = -\nabla P \\
&\rho(\partial_t \epsilon + \bar{u}\cdot\nabla \epsilon) = -P\nabla\cdot\bar{u} \\
&\epsilon = \epsilon(\rho, T) \qquad P = P(\rho, T)
\end{aligned}\right\}$$

CLOSED SET OF EQUATIONS

With Dissipation → In Conservative form.

$$\partial_t \rho + \partial_\alpha(\rho u_\alpha) = 0.$$

$$\partial_t(\rho u_\alpha) + \partial_\beta(\rho u_\alpha u_\beta) + \partial_\beta \sigma'_{\alpha\beta} = 0$$

$$\partial_t(\rho e) + \partial_\alpha(\rho e u_\alpha) + \partial_\beta(\sigma'_{\alpha\beta} u_\beta) + \partial_\alpha(-\kappa \partial_\alpha T) = 0$$

$$\sigma'_{\alpha\beta} = P\,\delta_{\alpha\beta} - \eta\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d}\nabla\cdot\bar{u}\,\delta_{\alpha\beta}\right) - \zeta(\nabla\cdot\bar{u})\,\delta_{\alpha\beta}$$

$\eta$ = Shear viscosity $\zeta$ = Bulk viscosity $\kappa$ = Heat Conductivity.

Refn. Hydrodynamics of Cold atomic gases: PRA 86, 033614

QUANTUM HYDRODYNAMICS (2012). 1

PART III

1D Bose Gas with $\delta$-interactions
Lieb-Liniger model.

$$H = \sum \frac{\hat{p}_i^2}{2m} + g \sum_{i<j} \delta(x_i - x_j)$$

~~En~~ Important parameter $\boxed{\gamma = \frac{m}{\hbar^2 \rho_0} g}$

I Weak Coupling or high density

$\gamma$ is small. This system is well described by the field

$$\langle \hat{\psi}(x,t) \rangle = \psi(x,t)$$

and the Gross-Pitaevski equation

$$i\hbar \partial_t \psi(x,t) = -\frac{\hbar^2}{2m} \partial_x^2 \psi + g |\psi|^2 \psi$$

Note $\psi^* \psi = \rho(x,t)$ ②

$\frac{1}{2}\left(-i\hbar \psi^* \partial_x \psi + i\hbar \psi \partial_x \psi^*\right) = p(x,t)$ momentum field.

~~So~~. Let $\psi = \sqrt{\rho}\, e^{i\phi}$

$$\partial_x \psi = \left(i\sqrt{\rho}\,\phi' + \frac{1}{2}\frac{\rho'}{\sqrt{\rho}}\right) e^{i\phi}$$

$$\therefore \quad p(x,t) = \hbar \rho \phi' = \frac{\hbar}{m}\phi' m\rho = \tilde{\rho}\, u$$

$$u(x,t) = \frac{\hbar}{m}\phi'$$

$$\partial_x^2 \psi = \left(\frac{i}{2\sqrt{\rho}}\rho'\phi' + i\sqrt{\rho}\,\phi'' - \frac{\rho'^2}{4\rho^{3/2}} + \frac{1}{2}\frac{\rho''}{\rho^{1/2}}\right) e^{i\phi}$$

$$+ \left(i\sqrt{\rho}\,\phi' + \frac{1}{2}\frac{\rho'}{\sqrt{\rho}}\right) i\phi' e^{i\phi}$$

$$= \left(\frac{i}{\sqrt{\rho}}\rho'\phi' + i\sqrt{\rho}\,\phi'' - \frac{\rho'^2}{4\rho^{3/2}} + \frac{1}{2}\frac{\rho''}{\rho^{1/2}} - \sqrt{\rho}\,\phi'^2\right) e^{i\phi}$$

$$\partial_t \psi = \left(i\sqrt{\rho}\,\dot{\phi} + \frac{\dot{\rho}}{2\sqrt{\rho}}\right) e^{i\phi}$$

$$\therefore -\hbar\sqrt{\rho}\,\dot{\phi} + \frac{i\hbar}{2\sqrt{\rho}}\dot{\rho} = \frac{-i\hbar^2}{2m\sqrt{\rho}}\rho'\phi' - \frac{i\hbar^2\sqrt{\rho}}{2m}\phi'' + \frac{\hbar^2}{8m}\frac{\rho'^2}{\rho^{3/2}} \quad (3)$$

$$-\frac{\hbar^2}{4m}\frac{\rho''}{\rho^{1/2}} + \frac{\hbar^2}{2m}\rho^{1/2}\phi'^2 + g\rho^{3/2}$$

Equating real and imaginary parts

$$\frac{\hbar}{2\sqrt{\rho}}\dot{\rho} = -\frac{\hbar^2}{2m}\frac{\rho'\phi'}{\sqrt{\rho}} - \frac{\hbar^2\sqrt{\rho}}{2m}\phi''$$

$$-\sqrt{\rho}\,\dot{\phi} = \frac{\hbar}{8m}\frac{\rho'^2}{\rho^{3/2}} - \frac{\hbar}{4m}\frac{\rho''}{\rho^{1/2}} + \frac{\hbar}{2m}\rho^{1/2}\phi'^2 + \frac{g\rho^{3/2}}{\hbar}$$

$$\Rightarrow \dot{\rho} = -\frac{\hbar}{m}\rho'\phi' - \frac{\hbar}{m}\rho\phi''$$

$$= -\rho' u - \rho u' = -\partial_x(\rho u)$$

$$\therefore \partial_t\rho + \partial_x(\rho u) = 0$$

$$\dot{\phi} = -\frac{\hbar}{8m}\frac{\rho'^2}{\rho^2} + \frac{\hbar}{4m}\frac{\rho''}{\rho} - \frac{\hbar}{2m}\phi'^2 - \frac{g\rho}{\hbar}$$

Taking a derivative and noting that $u = \frac{\hbar}{m}\phi'$

9

We get

$$\partial_t u + \partial_x \left( \frac{\hbar^2}{8m^2} \frac{\rho'^2}{\rho^2} - \frac{\hbar^2}{4m^2} \frac{\rho''}{\rho} + \frac{1}{2} u^2 + \frac{g\rho}{m} \right) = 0$$

Now note that

$$\partial_x \sqrt{\rho} = \frac{1}{2} \frac{\rho'}{\sqrt{\rho}} \qquad \partial_x^2 \sqrt{\rho} = \frac{1}{2} \frac{\rho''}{\sqrt{\rho}} - \frac{1}{4} \frac{\rho'^2}{\rho^{3/2}}$$

$$\boxed{\begin{aligned} &\partial_t u + \partial_x \left( \frac{u^2}{2} + \frac{g}{m} \rho - \frac{\hbar^2}{2m^2} \frac{\partial_x^2(\sqrt{\rho})}{\sqrt{\rho}} \right) = 0. \\ &\partial_t \rho + \partial_x (\rho u) = 0 \end{aligned}}$$

Hydrodynamic equations for mass and momentum fields

$$\cancel{\rho} \downarrow \rho \qquad\qquad \downarrow \underbrace{\rho v.}_{\downarrow}$$

NOTE: This can also be written in the form
$\partial_t(\rho v) + \partial_x j = 0.$

## II $\boxed{\text{Strong Coupling limit } \gamma >> 1}$ (5)

This becomes essentially like free Fermions (Since the wave function vanishes for $x_i = x_{i+1}$)

Look at zero temperature equation of state.

$$E = \sum_{n=1}^{N} \frac{\hbar^2 k_n^2}{2m} \qquad k_n = \frac{n\pi}{L}$$

$$= \frac{L}{\pi}\int_0^{k_F} dk \, \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m\pi} L \frac{k_F^3}{3} \qquad k_F = \frac{N\pi}{L} = \rho\pi$$

$$= \frac{\hbar^2 \pi^3}{6m\pi} \frac{N^3}{L^2}$$

$$P = -\frac{\partial E}{\partial L} = \frac{\hbar^2 \pi^3}{3m\pi} \rho^3$$

Euler Eqn. $\Rightarrow$ $\partial_t u + u\partial_x u = \dfrac{-\partial_x P}{m\rho} = \dfrac{-\hbar^2 \pi^2}{m^2} \partial_x \rho^2$

$\rho$: # density.

Thus, in this limit, the hydrodynamic equations are:

6

$$\boxed{\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0 \\ &\partial_t u + \partial_x\left(\frac{u^2}{2} + \frac{\hbar^2 \pi^2}{m^2}\rho^2\right) = 0 \end{aligned}}$$

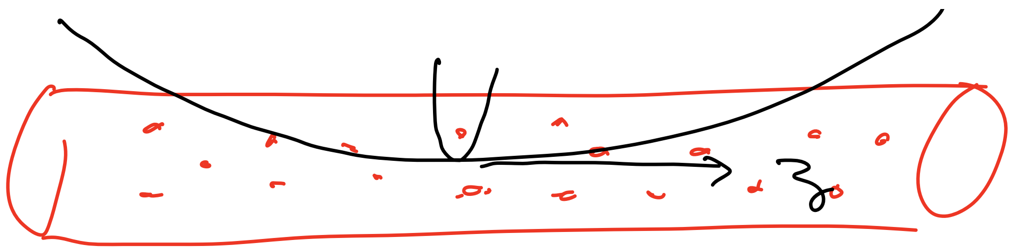
## Another example.

Ideal Fermi Gas (quasi 1D)

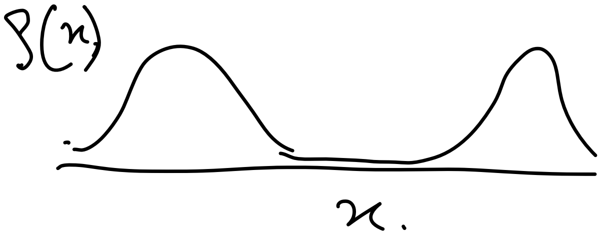
REFNS.

- Observation of Shock waves in a Strongly Interacting Fermi Gas
  PRL 106 150401 (2011)

Cold atoms in a Cigar-Shaped trap with $\omega_x, \omega_y \gg \omega_z$




Initial state




• Zero temperature

$$\left.\begin{aligned} &\partial_t \rho_{3D} + \nabla\cdot(\rho_{3D}\,\bar{u}) = 0 \\ &\partial_t \bar{u} + \bar{\nabla}\cdot\left(\frac{\bar{u}^2}{2} + \omega + V_{trap}\right) = 0 \end{aligned}\right\}$$

What is $\omega$ ?  Assum

Recall $\dfrac{d\bar{u}}{dt} = \dfrac{-\nabla P}{\rho}$

Zero temperature Fermi Gas

$$P = C'\rho^{5/3} \quad \therefore \quad \frac{\nabla P}{\rho} = \frac{5/3\, C'\rho^{2/3}\,\nabla\rho}{\rho}$$

$$= C\,\nabla(\rho^{2/3})$$

$$\therefore \quad \omega = c\rho^{2/3}$$

$$U_{trap} = \frac{1}{2}\omega_\perp^2 r^2 + \frac{1}{2}\omega_z^2 z^2$$

Since $\omega_\perp >> \omega_z$, the system relaxes fast in the $x, y$ direction and we can "integrate out" these degrees of freedom $(u_x, u_y)$ and write equations only for $\rho(z,t)$ and $u = u_z(z,t)$.

Note : In equilibrium $\partial_t \rho = 0$ $\bar{u} = 0$

So $\omega + U_{trap} = K$ (Constant)

$$\Rightarrow c\rho^{2/3} + \frac{\omega_\perp^2}{2} r^2 + \frac{\omega_z^2}{2} z^2 = K$$

$$\Rightarrow \rho_{eq.}^{2/3} = \left[\left(K - \frac{\omega_\perp^2}{2} r^2 - \frac{\omega_z^2}{2} z^2\right)/c\right]^{3/2}$$

$$= \bar{n}\left[1 - \frac{r^2}{R_1^2} - \frac{z^2}{R_z^2}\right]^{3/2} \quad 9$$

Integrating over $x, y$ gives

$$\int 2\pi r\, \rho_{eq.}^{2/3}\, dr = \frac{2\pi}{5} R_1^2 \bar{n} \left[1 - \frac{z^2}{R_z^2}\right]^{5/2}$$

This means that we can take the effective 1D $W_{eff}$ to have the form

$$W_{eff} = \bar{c}\, \rho^{2/5}$$

And the effective 1D Hydrodynamic equations are :

$$\left.\begin{aligned} &\partial_t \rho + \partial_z(\rho u) = 0 \\ &\rho\left[\partial_t u + \partial_z\left(\frac{u^2}{2} + c\,\rho^{2/5} + \frac{1}{2}\omega_z^2 z^2\right)\right] \\ &\qquad = \nu\, \partial_z(\underbrace{\rho\, \partial_z u}_{NS}) \end{aligned}\right\}$$

These equations were solved numerically and compared with experiments.

- Observation of Shock waves in a Strongly Interacting Fermi Gas
  PRL 106 150401 (2011)

See also

- Universal Shock-wave propagation in One-dimensional Bose fluids
  PRR 3 013098 (2021)

END OF QUANTUM HYDRODYNAMICS

# CONNECTIONS WITH BOLTZMANN'S EQN. — PART IV

Classical dilute weakly interacting gas.

# Boltzmann's Equation.



$$\partial_t f + v_\alpha \partial_\alpha f = \partial_t f \Big|_{\boxed{Coll.}}$$

Interactions.

Weakly Interacting Gas.

$$\rho(\bar{x},t) = \int f(\bar{x},\bar{v},t)\, d\bar{v}$$

$$\rho\, \bar{u}(\bar{x},t) = \int m\bar{v}\, f(\bar{x},\bar{v},t)\, d\bar{v}$$

$$e(\bar{x},t) = \int \frac{m\bar{v}^2}{2} f(\bar{x},\bar{v},t)\, d\bar{v}$$

Relation between $f$ and the Conserved fields.

Let us assume

$$f(\bar{x},\bar{v},t) = \frac{\rho(\bar{x},t)}{[2\pi k_B T(\bar{x},t)]^{3/2}}\, e^{-\frac{(\bar{v}-\bar{u})^2}{2k_BT}}$$

Local Equilibrium Form

Then (a) $\int f\, d\bar{v} = \rho(\bar{x},t)$

(b) $\int d\bar{v}\, v_\alpha f = \rho u_\alpha$

$\int d\bar{v}\, v_\alpha v_\beta f$

2

$$= \int d\bar{v}\, (\delta v_\alpha + u_\alpha)(\delta v_\beta + u_\beta) f$$

where $\delta v_\alpha = v_\alpha - u_\alpha$.

$$= \int d\,\bar{\delta v} \left[ \delta v_\alpha \delta v_\beta + u_\alpha \delta v_\beta + u_\beta \delta v_\alpha + u_\alpha u_\beta \right] f$$

(c) $= \rho\, k_B T\, \delta_{\alpha\beta} + \rho\, u_\alpha u_\beta$

(d) Similar equation for

$$\int d\bar{v}\; \bar{v}^2 v_\alpha\, f$$

Using (a) - (d) we get exactly the Euler equations for an ideal gas with $\boxed{\epsilon = \frac{3 k_B T}{2}. \quad P = \rho k_B T}$

To obtain the Navier-Stokes-Fourier terms we need

(a) Corrections to local equilibrium
(b) Contribution of the collision terms in the Boltzmann-Equation.

For a simple and accessible derivation see the book

Mehran Kardar - Statistical Physics of Particles - $\boxed{\text{Chapter 3}}$

# Part (V)

A Microscopic Derivation of the Euler equations:

$$\boxed{\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0 \\ &\partial_t(\rho u) + \partial_x(\rho u^2 + P) = 0 \\ &\partial_t\left(\frac{\rho u^2}{2} + \rho\epsilon\right) + \partial_x\left[\left(\frac{\rho u^2}{2} + \rho\epsilon + P\right)u\right] = 0 \end{aligned}}$$

$P = P(\rho, T) \quad \epsilon = \epsilon(\rho, T)$ ] From equilibrium Physics.

Consider a 1D Hamiltonian

$$H = \sum_l \frac{p_l^2}{2m} + \sum_{l \neq j} V(x_l - x_j)$$

We can write this in the form

$$H = \sum h_l \qquad h_l = \frac{p_l^2}{2m} + \frac{1}{2}\sum_{j \neq l} V(x_l - x_j)$$

2

# Microscopic equations of motion:

$$\boxed{\begin{aligned} \dot{x}_\ell &= p_\ell / m \\ \dot{p}_\ell &= -\sum_{j \neq \ell} U'(x_\ell - x_j) = \sum_{j \neq \ell} f_{\ell j} \end{aligned}}$$

Force on $\ell$-th particle due to $j$-th particle.

We define the macroscopic hydrodynamic fields

$$\rho(x,t) = m \sum_\ell \langle \delta(x - x_\ell(t)) \rangle$$

$$p(x,t) = \sum_\ell \langle p_\ell(t) \delta(x - x_\ell(t)) \rangle$$

$$e(x,t) = \sum_\ell \langle h_\ell(t) \delta(x - x_\ell(t)) \rangle$$

Average $\langle \cdots \rangle$ over initial phase space

distribution $P(x_1 x_2 \ldots x_N, p_1, p_2 \ldots p_N, 0)$ . 3
corresponding to an ensemble of initial conditions which give $\rho(x,0)$, $p(x,0)$, $e(x,0)$.

Define velocity field $u(x,t)$ via

$$p(x,t) = \rho(x,t)\, u(x,t)$$

$$\begin{aligned}\partial_t \rho(x,t) &= \partial_t m \sum \langle \delta(x - x_\ell) \rangle \\ &= -\partial_x \sum m \langle \dot{x}_\ell\, \delta(x - x_\ell) \rangle \\ &= -\partial_x\, p(x,t) = -\partial_x (\rho u)\end{aligned}$$

Hence we get the first Euler equation

$$\partial_t \rho(x,t) + \partial_x (\rho u) = 0$$

Next we look at momentum conservation eqn:

$$\begin{aligned}\partial_t p(x,t) &= \partial_t \sum_\ell \langle p_\ell\, \delta(x - x_\ell) \rangle \\ &= -\partial_x \Big( \sum_\ell \langle p_\ell \dot{x}_\ell\, \delta(x - x_\ell) \rangle \Big) + \sum_\ell \langle \dot{p}_\ell\, \delta(x - x_\ell) \rangle \\ &= -\partial_x \Big[ \sum_\ell \langle m v_\ell^2\, \delta(x - x_\ell) \rangle - \sum_{\ell, m \neq \ell} \langle f_{\ell m}\, \Theta(x - x_\ell) \rangle \Big]\end{aligned}$$

Consider $\sum_\ell \langle m v_\ell^2 \, \delta(x-x_\ell) \rangle$

$$= \sum_\ell m \langle (v_\ell - u + u)^2 \, \delta(x-x_\ell) \rangle$$

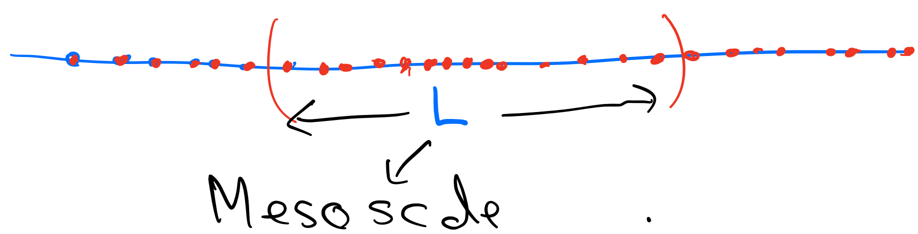
$$= \sum_\ell \langle m [ (v_\ell - u)^2 + 2(v_\ell - u)u + u^2 ] \, \delta(x-x_\ell) \rangle$$

$$= \rho u^2 + \sum_\ell \langle m (v_\ell - u)^2 \, \delta(x-x_\ell) \rangle$$

Hence we get :

$$\boxed{\partial_t \rho(x,t) = -\partial_x \left[ \rho u^2 + D(x) \right],} \to \text{PEQ}^*$$

Where $D(x) = \sum_\ell \langle m (v_\ell - u)^2 \, \delta(x-x_\ell) \rangle - \sum_{\ell, m \neq \ell} \langle f_{\ell m} \, \Theta(x - x_\ell) \rangle$




The Euler current is given by the equilibrium value of the current in a local-equilibrium cell of size L.

Let us thus evaluate.

$$\bar{D} = \frac{1}{L}\int_0^L dx\, D(x)$$

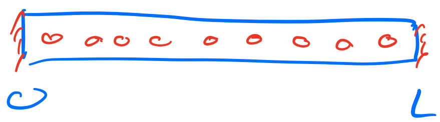
$$= \frac{1}{L}\left\{ \sum_l{}' \langle m(v_l-u)^2\rangle - \sum_{l,m\neq l}{}' \langle f_{lm}(\overbrace{L}^{\text{Vanishes}}-x_l)\rangle \right\}$$

Where the sum is over all particles in the cell of size $l$.

$$= \frac{1}{L}\left[ \sum_l{}' \langle m(v_l-u)^2\rangle + \sum_{l,m\neq l}{}' \langle f_{lm}x_l\rangle \right]$$

$= P$ using the Virial Theorem.

## Virial Theorem proof




$$\frac{d}{dt}\sum x_l p_l = \sum_l m_l v_l^2 + \sum_{l,m\neq l} x_l f_{lm} + \sum x_l f_l^{(e)}$$

↳ Eqn. (*)

External force from wall of box.

Do a time average on both sides

$\underset{T\to\infty}{Lt}\frac{1}{T}\int_0^T dt$ and use the fact that $x_\ell, p_\ell$ are bounded. Hence the left hand side of Eqn. (*), being a boundary term, vanishes. We get then

6

$$\sum_\ell \langle m_\ell v_\ell^2 \rangle + \sum_{\ell, m\neq \ell} \langle x_\ell f_{\ell m} \rangle = -\sum \left\langle x_\ell f_\ell^{(e)} \right\rangle.$$

We assume Time average = Ensemble average.

Now note that $f_\ell^{(e)}$ is non-zero only when $x_\ell = 0$ or $L$ and then the average of the impulsive wall force is precisely the pressure. Hence we get the desired formula:

$$\boxed{\sum_\ell \langle m_\ell v_\ell^2 \rangle + \sum_{\ell, m\neq \ell} \langle x_\ell f_{\ell m} \rangle = PL}$$

↳ Virial theorem.

# VIRIAL THEOREM PROOF - 2



Consider the constant pressure equilibrium ensemble

$$\hat{\rho}(\bar{x},\bar{p}) = \frac{e^{-\beta[H - P x_N]}}{Z(\beta,P,N)}$$


Then $$\sum_l \left\langle \frac{p_l^2}{m} \right\rangle + \sum_{l,m\neq l} \left\langle x_l f_{lm} \right\rangle$$

$$= \sum_l \left\langle p_l \frac{\partial H}{\partial p_l} - x_l \frac{\partial H}{\partial x_l} \right\rangle$$

Now $$\sum_l \frac{\partial}{\partial p_l}\left(p_l e^{-\beta[H-Px_N]}\right)$$

$$= \sum_l \left[e^{-(\ )} - \beta p_l \frac{\partial H}{\partial p_l} e^{-(\ )}\right].$$

$$\sum_l \frac{\partial}{\partial x_l}\left(x_l e^{-\beta[H-Px_N]}\right)$$

$$= \sum_l \left[e^{-(\ )} - \beta x_l \frac{\partial H}{\partial x_l} e^{-(\ )} + \beta P x_N e^{-(\ )}\right]$$

$$\therefore \beta \sum_l \left(p_l \frac{\partial H}{\partial p_l} - x_l \frac{\partial H}{\partial x_l}\right) e^{-(\ )} = \sum_l -\frac{\partial}{\partial p_l}\left(p_l e^{-(\ )}\right) + \frac{\partial}{\partial x_l}\left(x_l e^{-(\ )}\right) - \beta P x_N e^{-(\ )}$$

$$\Rightarrow \sum_{l} \left\langle p_l \frac{\partial H}{\partial p_l} - x_l \frac{\partial H}{\partial x_l} \right\rangle = -P\langle x_N \rangle = -PL$$ (8)

average Volume in Constant pressure ensemble.

(V) Page (4)

$\therefore$ PEQ$^*$ gives, on replacing $D(x)$ by the local equilibrium average $\overline{D}$

$$\partial_t \rho u + \partial_x(\rho u^2 + P) = 0$$

3D generalization of Virial Stress expression.

$$\frac{1}{V}\sum_{l \in V}\left\langle m(v_l^\alpha - u^\alpha)(v_l^\beta - u^\beta)\right\rangle + \frac{1}{V}\sum_{l,j \in V}\left\langle \frac{1}{2}(x_l^\alpha - x_j^\alpha) f_{lj}^\alpha \right\rangle$$

$$= \sigma_{\alpha\beta} = P\delta_{\alpha\beta} \quad \left(\text{isotropic system}\right)$$

$\Rightarrow$ Euler eqn. : $\boxed{\partial_t(\rho u_\alpha) + \partial_\beta(\rho u_\alpha u_\beta + P\delta_{\alpha\beta}) = 0}$

$\boxed{\underline{\text{Problem}}}$: Derive Euler eqn. for energy using a similar approach.

$$\partial_t e(x,t) = \partial_t \sum \langle h_\ell \, \delta(x-x_\ell) \rangle$$

$$= \ldots$$