# Hidden spatiotemporal symmetries and intermittency

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# Symmetries

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p, \quad \nabla \cdot \mathbf{u} = 0$$

Euler system

(valid in the inertial interval)

$$\begin{aligned}
\text{temporal translation:} \quad & \mathbf{u}(\mathbf{r}, t) \mapsto \mathbf{u}(\mathbf{r}, t' + t), && t' \in \mathbb{R}; \\
\text{spatial translation:} \quad & \mathbf{u}(\mathbf{r}, t) \mapsto \mathbf{u}(\mathbf{r} + \mathbf{r}', t), && \mathbf{r}' \in \mathbb{R}^d; \\
\text{rotation:} \quad & \mathbf{u}(\mathbf{r}, t) \mapsto \mathbf{Q}^{-1}\mathbf{u}(\mathbf{Q}\mathbf{r}, t), && \mathbf{Q} \in \mathrm{O}(d); \\
\text{Galilean transformation:} \quad & \mathbf{u}(\mathbf{r}, t) \mapsto \mathbf{u}(\mathbf{r} + \mathbf{v}t, t) - \mathbf{v}, && \mathbf{v} \in \mathbb{R}^d;
\end{aligned}$$

**Galilean group**

$$\begin{aligned}
\text{temporal scaling:} \quad & \mathbf{u}(\mathbf{r}, t) \mapsto \mathbf{u}(\mathbf{r}, t/a)/a, && a > 0; \\
\text{spatial scaling:} \quad & \mathbf{u}(\mathbf{r}, t) \mapsto b\mathbf{u}(\mathbf{r}/b, t), && b > 0,
\end{aligned}$$

**spatiotemporal scaling group**

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# Symmetries

$$\text{temporal translation:}\quad \mathbf{u}(\mathbf{r},t) \mapsto \mathbf{u}(\mathbf{r},t'+t), \qquad t' \in \mathbb{R};$$

$$\text{spatial translation:}\quad \mathbf{u}(\mathbf{r},t) \mapsto \mathbf{u}(\mathbf{r}+\mathbf{r}',t), \qquad \mathbf{r}' \in \mathbb{R}^d;$$

$$\text{rotation:}\quad \mathbf{u}(\mathbf{r},t) \mapsto \mathbf{Q}^{-1}\mathbf{u}(\mathbf{Q}\mathbf{r},t), \qquad \mathbf{Q} \in \mathrm{O}(d);$$

$$\text{Galilean transformation:}\quad \mathbf{u}(\mathbf{r},t) \mapsto \mathbf{u}(\mathbf{r}+\mathbf{v}t,t)-\mathbf{v}, \quad \mathbf{v} \in \mathbb{R}^d;$$

**Galilean group**

$$\text{temporal scaling:}\quad \mathbf{u}(\mathbf{r},t) \mapsto \mathbf{u}(\mathbf{r},t/a)/a, \qquad a > 0;$$

$$\text{spatial scaling:}\quad \mathbf{u}(\mathbf{r},t) \mapsto b\mathbf{u}(\mathbf{r}/b,t), \qquad b > 0,$$

**spatiotemporal scaling group**

# Dynamical system setting

Configuration space: $x = \mathbf{u}(\mathbf{r}) \in \mathcal{X}$ infinite-dimensional probability measure space $(\mathcal{X}, \Sigma, \mu)$

Flow (evolution) operator: $\Phi^t x = \mathbf{u}(\mathbf{r}, t)$ $\Phi^t : \mathcal{X} \mapsto \mathcal{X}$

Invariance of the measure: $\Phi^t_\sharp \mu = \mu$

Symmetry operators:

(at $t = 0$)

$$
\begin{aligned}
s_{\mathrm{s}}^{\mathbf{r}'} &: \ \mathbf{u}(\mathbf{r}) \mapsto \mathbf{u}(\mathbf{r} + \mathbf{r}'), && \mathbf{r}' \in \mathbb{R}^d, && \text{(spatial translation)} \\
s_{\mathrm{r}}^{\mathbf{Q}} &: \ \mathbf{u}(\mathbf{r}) \mapsto \mathbf{Q}^{-1}\mathbf{u}(\mathbf{Q}\mathbf{r}), && \mathbf{Q} \in \mathrm{O}(d), && \text{(rotation)} \\
s_{\mathrm{g}}^{\mathbf{v}} &: \ \mathbf{u}(\mathbf{r}) \mapsto \mathbf{u}(\mathbf{r}) - \mathbf{v}, && \mathbf{v} \in \mathbb{R}^d, && \text{(Galilean transformation)} \\
s_{\mathrm{ts}}^{a} &: \ \mathbf{u}(\mathbf{r}) \mapsto \mathbf{u}(\mathbf{r})/a, && a > 0, && \text{(temporal scaling)} \\
s_{\mathrm{ss}}^{b} &: \ \mathbf{u}(\mathbf{r}) \mapsto b\mathbf{u}(\mathbf{r}/b), && b > 0. && \text{(spatial scaling)}
\end{aligned}
$$

Symmetry: a one-to-one measurable map $s : \mathcal{X} \mapsto \mathcal{X}$ such that the invariance of $\mu$ implies the invariance of $s_\sharp \mu$

The measure is symmetric with respect to $s$ if: $s_\sharp \mu = \mu$

# Commutation relations

| | $\Phi^t$ | $s_{\mathrm{s}}^{\mathbf{r}}$ | $s_{\mathrm{r}}^{\mathbf{Q}}$ | $s_{\mathrm{g}}^{\mathbf{v}}$ | $s_{\mathrm{ts}}^{a}$ | $s_{\mathrm{ss}}^{b}$ |
|---|---|---|---|---|---|---|
| $\Phi^t$ | $\Phi^{t_1+t_2}$ | $s_{\mathrm{s}}^{\mathbf{r}} \circ \Phi^t$ | $s_{\mathrm{r}}^{\mathbf{Q}} \circ \Phi^t$ | $s_{\mathrm{s}}^{\mathbf{v}t} \circ s_{\mathrm{g}}^{\mathbf{v}} \circ \Phi^t$ | $s_{\mathrm{ts}}^{a} \circ \Phi^{t/a}$ | $s_{\mathrm{ss}}^{b} \circ \Phi^t$ |
| $s_{\mathrm{s}}^{\mathbf{r}}$ | $\Phi^t \circ s_{\mathrm{s}}^{\mathbf{r}}$ | $s_{\mathrm{s}}^{\mathbf{r}_1+\mathbf{r}_2}$ | $s_{\mathrm{r}}^{\mathbf{Q}} \circ s_{\mathrm{s}}^{\mathbf{Q}\mathbf{r}}$ | $s_{\mathrm{g}}^{\mathbf{v}} \circ s_{\mathrm{s}}^{\mathbf{r}}$ | $s_{\mathrm{ts}}^{a} \circ s_{\mathrm{s}}^{\mathbf{r}}$ | $s_{\mathrm{ss}}^{b} \circ s_{\mathrm{s}}^{\mathbf{r}/b}$ |
| $s_{\mathrm{r}}^{\mathbf{Q}}$ | $\Phi^t \circ s_{\mathrm{r}}^{\mathbf{Q}}$ | $s_{\mathrm{s}}^{\mathbf{Q}^{-1}\mathbf{r}} \circ s_{\mathrm{r}}^{\mathbf{Q}}$ | $s_{\mathrm{r}}^{\mathbf{Q}_1\mathbf{Q}_2}$ | $s_{\mathrm{g}}^{\mathbf{Q}^{-1}\mathbf{v}} \circ s_{\mathrm{r}}^{\mathbf{Q}}$ | $s_{\mathrm{ts}}^{a} \circ s_{\mathrm{r}}^{\mathbf{Q}}$ | $s_{\mathrm{ss}}^{b} \circ s_{\mathrm{r}}^{\mathbf{Q}}$ |
| $s_{\mathrm{g}}^{\mathbf{v}}$ | $s_{\mathrm{s}}^{-\mathbf{v}t} \circ \Phi^t \circ s_{\mathrm{g}}^{\mathbf{v}}$ | $s_{\mathrm{s}}^{\mathbf{r}} \circ s_{\mathrm{g}}^{\mathbf{v}}$ | $s_{\mathrm{r}}^{\mathbf{Q}} \circ s_{\mathrm{g}}^{\mathbf{Q}\mathbf{v}}$ | $s_{\mathrm{g}}^{\mathbf{v}_1+\mathbf{v}_2}$ | $s_{\mathrm{ts}}^{a} \circ s_{\mathrm{g}}^{a\mathbf{v}}$ | $s_{\mathrm{ss}}^{b} \circ s_{\mathrm{g}}^{\mathbf{v}/b}$ |
| $s_{\mathrm{ts}}^{a}$ | $\Phi^{at} \circ s_{\mathrm{ts}}^{a}$ | $s_{\mathrm{s}}^{\mathbf{r}} \circ s_{\mathrm{ts}}^{a}$ | $s_{\mathrm{r}}^{\mathbf{Q}} \circ s_{\mathrm{ts}}^{a}$ | $s_{\mathrm{g}}^{\mathbf{v}/a} \circ s_{\mathrm{ts}}^{a}$ | $s_{\mathrm{ts}}^{a_1 a_2}$ | $s_{\mathrm{ss}}^{b} \circ s_{\mathrm{ts}}^{a}$ |
| $s_{\mathrm{ss}}^{b}$ | $\Phi^t \circ s_{\mathrm{ss}}^{b}$ | $s_{\mathrm{s}}^{b\mathbf{r}} \circ s_{\mathrm{ss}}^{b}$ | $s_{\mathrm{r}}^{\mathbf{Q}} \circ s_{\mathrm{ss}}^{b}$ | $s_{\mathrm{g}}^{b\mathbf{v}} \circ s_{\mathrm{ss}}^{b}$ | $s_{\mathrm{ts}}^{a} \circ s_{\mathrm{ss}}^{b}$ | $s_{\mathrm{ss}}^{b_1 b_2}$ |

Symmetries not commuting with the flow

$$\Phi^t \circ s_{\mathrm{g}}^{\mathbf{v}} = s_{\mathrm{s}}^{\mathbf{v}t} \circ s_{\mathrm{g}}^{\mathbf{v}} \circ \Phi^t$$

Galilean transformation

$$\Phi^t \circ s_{\mathrm{ts}}^{a} = s_{\mathrm{ts}}^{a} \circ \Phi^{t/a}$$

temporal scaling

Existence of the flow operator and commutation relations are our central **assumptions**.

# Part 1:

## Quotient construction with respect to temporal scalings

## (not taking into account Galilean transformations)

# Symmetries

Symmetry group: $\mathcal{S} = \mathcal{H}_{\rm ts} + \mathcal{G}$

$h^a = s^a_{\rm ts} \in \mathcal{H}_{\rm ts} = \left\{ s^a_{\rm ts} : \ a > 0 \right\}$ (temporal scalings) $\mathbf{u}(\mathbf{r}) \mapsto \mathbf{u}(\mathbf{r})/a$

$g \in \mathcal{G}$ (spatial scalings and rotations) $\mathbf{u}(\mathbf{r}) \mapsto b\mathbf{u}(\mathbf{r}/b)$

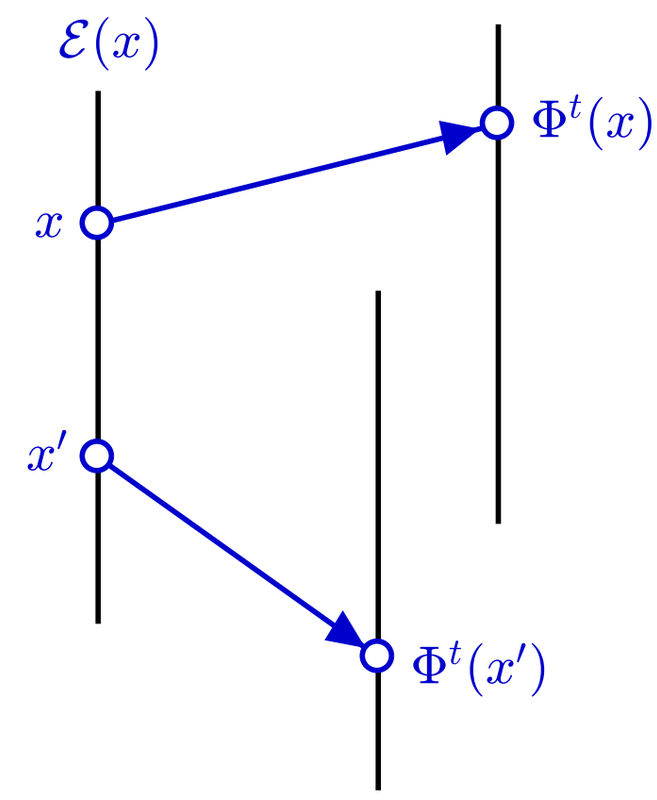
Commutation relations:

$$h^{a_1} \circ h^{a_2} = h^{a_1 a_2},$$

$$\Phi^t \circ g = g \circ \Phi^t, \quad g \circ h^a = h^a \circ g,$$

$$\Phi^t \circ h^a = h^a \circ \Phi^{t/a}.$$

quotient space




Equivalence relation: $x \sim x' \quad \text{if} \quad x' = h^a(x), \ a > 0$

Equivalence class: $\mathcal{E}_{\rm ts}(x) = \{x' \in \mathcal{X} : x' \sim x\}$

Equivalence is not flow-invariant!

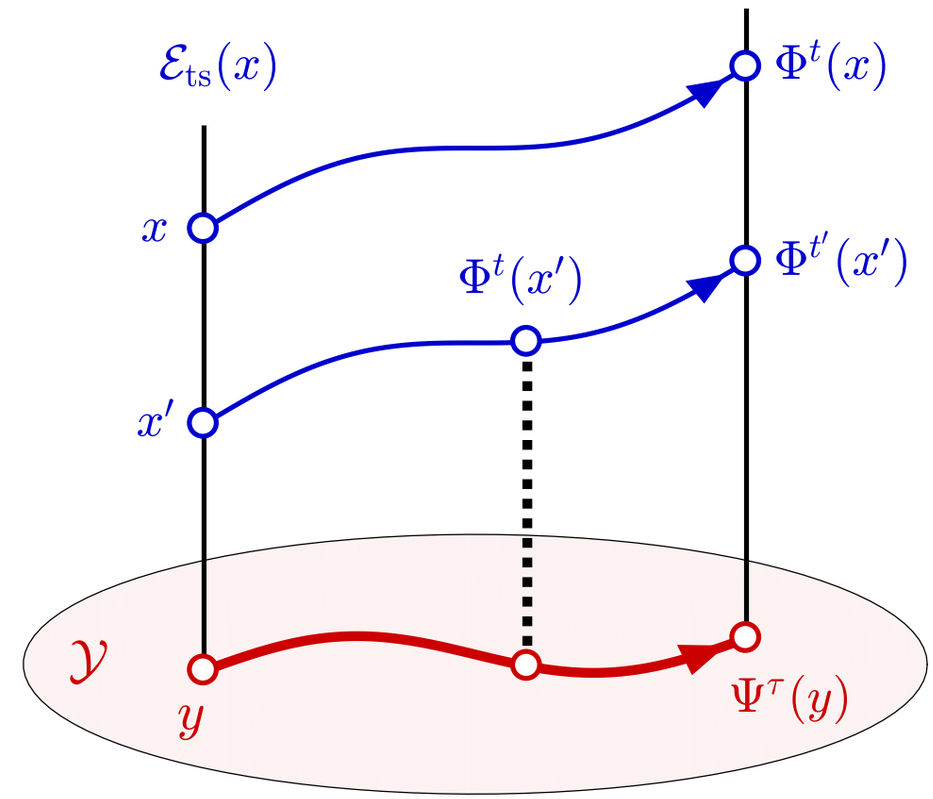
# Representative set

Equivalence of evolved states:

$$\boxed{\Phi^t \circ h^a = h^a \circ \Phi^{t/a}}$$

$$\Downarrow$$

$$\Phi^{t'}(x') \sim \Phi^t(x) \qquad t' = at$$




time synchronization with respect to the representative set

**Definition 2.** *We call $\mathcal{Y} \subset \mathcal{X}$ a representative set (with respect to the group $\mathcal{H}_{\mathrm{ts}}$), if the following properties are satisfied. For any $x \in \mathcal{X}$, there exists a unique value $a = A(x) > 0$ such that $h^a(x) \in \mathcal{Y}$. The function $A : \mathcal{X} \mapsto \mathbb{R}_+$ is measurable with $\int A \, d\mu < \infty$.*

Properties:

$$A \circ h^a(x) = \frac{A(x)}{a}, \quad A(y) = 1$$

Projector:

$$y = P(x) = h^{A(x)}(x)$$

# Normalized system

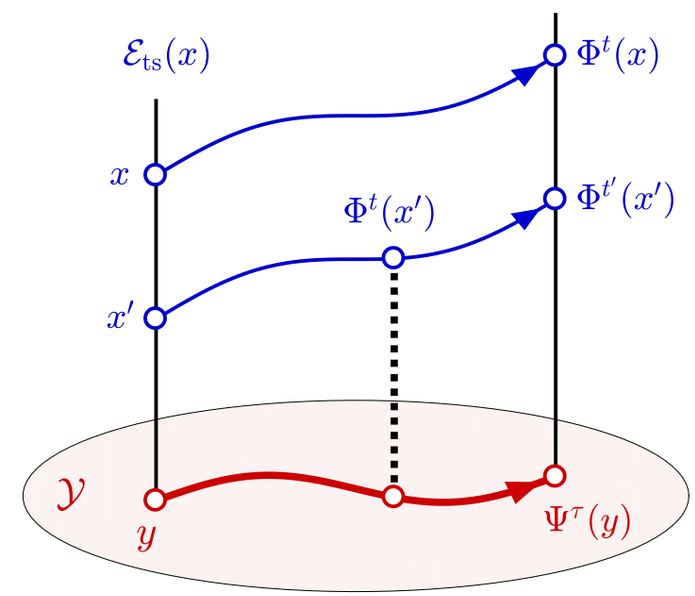
## Change of time with the relative speed $A(x)$ (ergodic theory)

**Proposition 1** ([14]). *For a positive measurable function $A(x)$, one can introduce a new flow $\Phi_A^\tau$ with a new time $\tau \in \mathbb{R}$ defined by the relations*

$$\Phi_A^\tau(x) = \Phi^t(x), \quad \tau = \int_0^t A \circ \Phi^s(x) ds.$$

*The flow $\Phi_A^\tau$ has the invariant measure $\mu_A$, which is absolutely continuous with respect to $\mu$ as*

$$\frac{d\mu_A}{d\mu} = \frac{A(x)}{\int A d\mu}.$$




Normalized flow and invariant measure

**Theorem 1.** *The map*

$$\Psi^\tau(y) = P \circ \Phi_A^\tau(y)$$

*with $y \in \mathcal{Y}$ defines a flow in the representative set. It has the invariant probability measure*

$$\nu = P_\sharp \mu_A.$$

The proof uses commutation relations and properties of push-forward measures.

# Statistical symmetries in the normalized system

**Theorem 2.** *Consider invariant measures $\mu$ and $g_\sharp\mu$ of the flow $\Phi^t$ for some $g \in \mathcal{G}$. We denote by $\nu$ and $g_\star\nu$ the corresponding invariant measures of the flow $\Psi^\tau$ given by Theorem 1. Then,*

$$\boxed{g_\star\nu = (P \circ g)_\sharp \nu_C, \quad C = A \circ g,}$$

*where $\nu_C$ is an absolutely continuous measure with respect to $\nu$ such that*

$$\frac{d\nu_C}{d\nu} = \frac{C(y)}{\int C d\nu}.$$

*For any $g$ and $g' \in \mathcal{G}$ we have $(g' \circ g)_\star \nu = g'_\star(g_\star \nu)$.*

**Hidden statistical scaling symmetry:**

$$(h \circ g)_\sharp \mu = \mu \quad \Longrightarrow \quad g_\star \nu = \nu$$

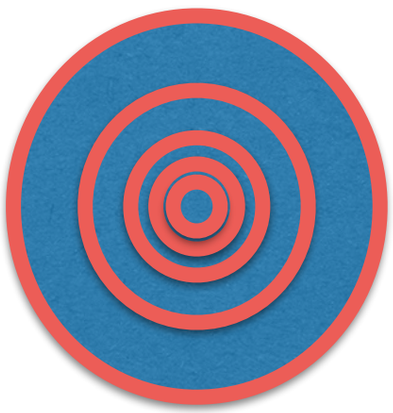
original $\quad \not\Longleftarrow \quad$ normalized

# Example: Shell model of turbulence

shells $n = 0, 1, 2, \ldots$

wavenumber $k_n = 2^n$

shell velocities $u_n(t) \in \mathbb{C}$



Full system:

$$\frac{du_n}{dt} = B_n - \mathrm{Re}^{-1} k_n^2 u_n + f_n, \quad n \geq 0$$

viscosity forcing

$$B_n = \begin{cases} i(k_{n+1}u_{n+2}u_{n+1}^* - k_{n-1}u_{n+1}u_{n-1}^* + k_{n-2}u_{n-1}u_{n-2}), & n > 1; \\ i(k_2 u_3 u_2^* - k_0 u_2 u_0^*), & n = 1; \\ i k_1 u_2 u_1^*, & n = 0, \end{cases}$$

Ideal (inviscid, unforced) system:

$$\frac{du_n}{dt} = i\left(k_{n+1}u_{n+2}u_{n+1}^* - k_{n-1}u_{n+1}u_{n-1}^* + k_{n-2}u_{n-1}u_{n-2}\right), \quad n \in \mathbb{Z}$$

Symmetries of the ideal system:

temporal scaling: $u_n(t) \mapsto u_n(t/a)/a, \quad a > 0;$ $\longrightarrow$ $h^a = s_{\mathrm{ts}}^a$

spatial scaling: $u_n(t) \mapsto k_m u_{n+m}(t), \; m \in \mathbb{Z}$ $\longrightarrow$ $g \in \mathcal{G}$

# Normalized system and scaling symmetry

Representative states (generalized multipliers) and change of time:

$$U_n = \frac{u_n}{A(x)} \qquad A(x) = \sqrt{\sum_{n<0} k_n^2 |u_n|^2} \qquad \tau = \int_0^t A \circ \Phi^s(x)\, ds$$

Equation of motion for the normalized system:

$$\begin{aligned} \frac{dU_n}{d\tau} &= i\left(k_{n+1}U_{n+2}U_{n+1}^* - k_{n-1}U_{n+1}U_{n-1}^* + k_{n-2}U_{n-1}U_{n-2}\right) \\ &\quad + U_n \sum_{j<0} k_j^3 \left(2\pi_{j+1} - \frac{\pi_j}{2} - \frac{\pi_{j-1}}{4}\right), \quad \pi_j = \mathrm{Im}\left(U_{j-1}^* U_j^*\, U_{j+1}\right) \end{aligned}$$

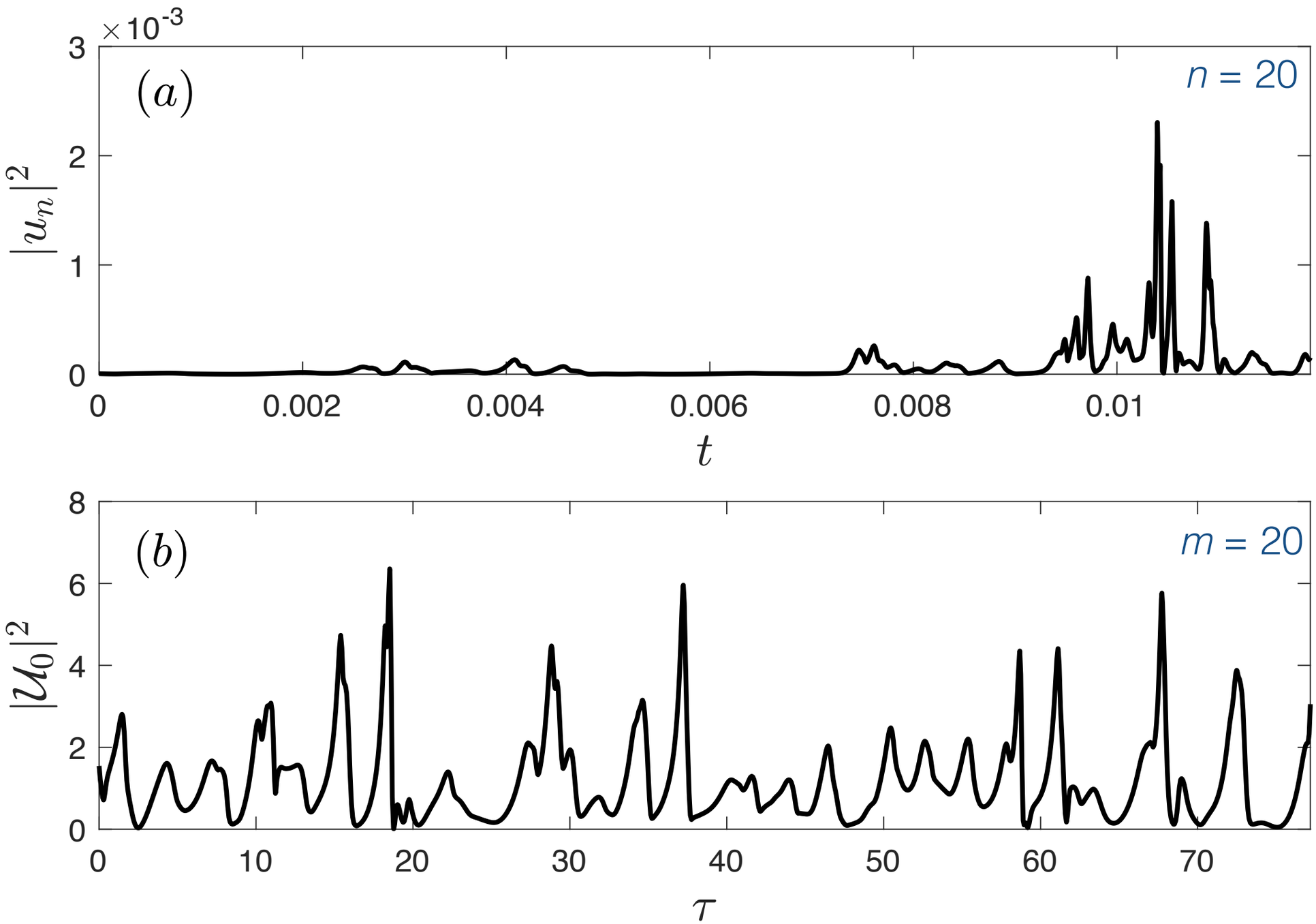
Statistical scaling symmetries: $\nu \mapsto g_\star^m \nu = (P \circ g^m)_\sharp \nu_C \qquad C = A \circ g^m$

$$\boxed{\begin{aligned} d\tau \mapsto d\tau^{(m)} &= A \circ g^m(y)\, d\tau \\ U_n \mapsto U_n^{(m)} &= \frac{k_m U_{n+m}}{A \circ g^m(y)} \end{aligned}}$$

(exact nonlinear symmetry of the normalized system)

$$A \circ g^m(y) = \sqrt{\sum_{n<m} k_n^2 |U_n|^2} = \begin{cases} \left(1 + \sum_{0 \le n < m} k_n^2 |U_n|^2\right)^{1/2}, & m > 0; \\ 1, & m = 0; \\ \left(1 - \sum_{m \le n < 0} k_n^2 |U_n|^2\right)^{1/2}, & m < 0; \end{cases}$$

# Effect of normalization on the intermittent evolution



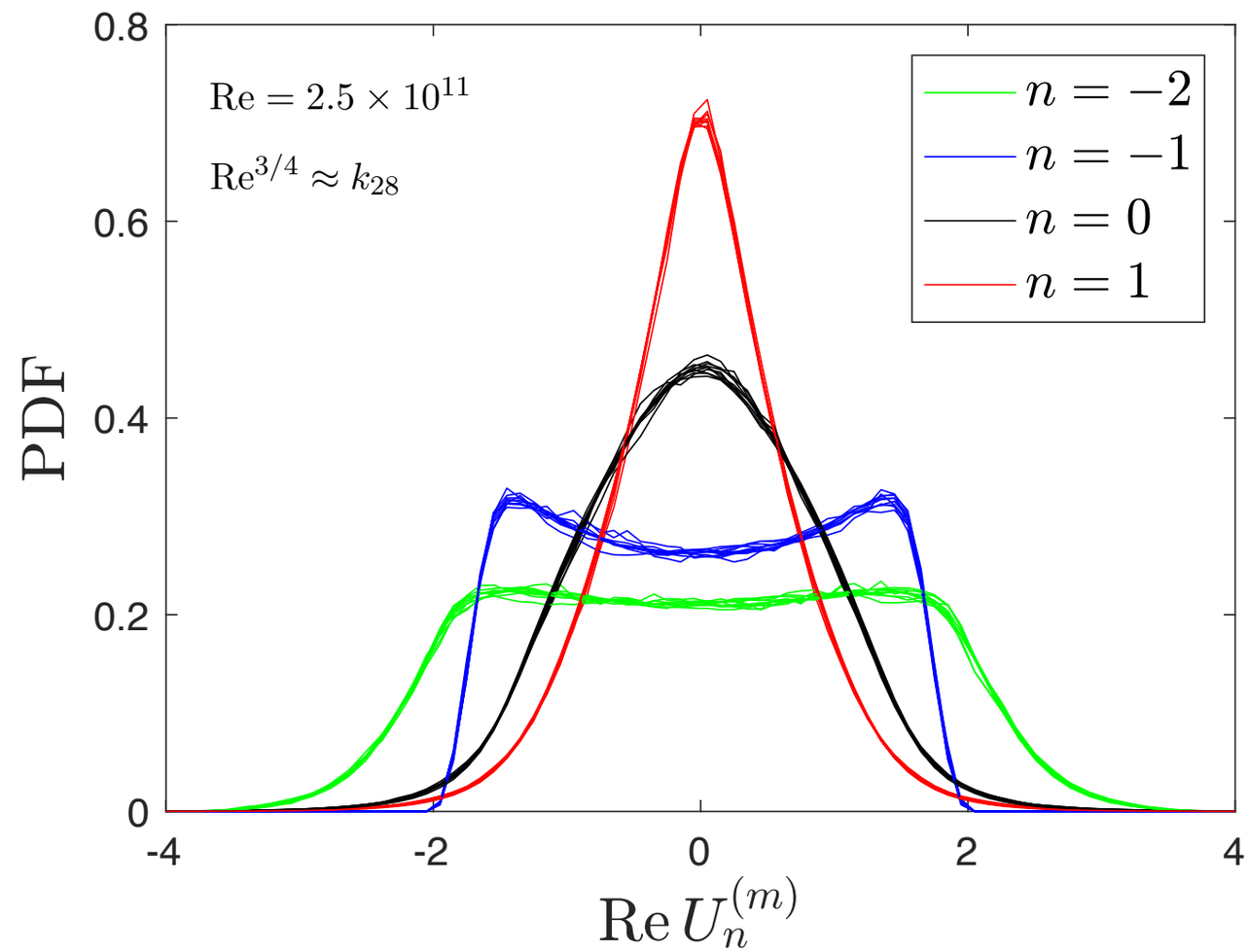
# Hidden symmetry: numerical tests

Scaling symmetries in the normalized system:

$$g_\star^m \nu : \quad U_n \mapsto U_n^{(m)} = \frac{k_m U_{n+m}}{A \circ g^m(y)}$$

$$d\tau \mapsto d\tau^{(m)} = A \circ g^m(y)\, d\tau$$

**Results for** $m = 12, \ldots, 21$




**Hidden scaling symmetry:**

$$g_\star^m \nu = \nu$$

the statistics of $U_n^{(m)}(\tau^{(m)})$ is universal:

independent of *m* in the inertial interval.

Universality of Kolmogorov multipliers

$$\frac{u_{m+1}}{u_m} = \frac{U_1^{(m)}}{U_0^{(m)}} = \frac{U_{j+1}^{(m-j)}}{U_j^{(m-j)}}$$

Hidden symmetry explains the universality of multipliers.

# Part 2: Intermittency

# Structure functions

Usual definition: $S_p(\ell) = \langle \|\delta_\ell \mathbf{u}\|^p \rangle$ $\quad \delta_\ell \mathbf{u} = \mathbf{u}(\mathbf{r}') - \mathbf{u}(\mathbf{r})$ $\quad \ell = \|\mathbf{r}' - \mathbf{r}\|$

Generalized definition: $S_p(\ell) = b^p \int F \circ s^b_{\mathrm{ss}} \, d\mu$ $\quad F \circ s^a_{\mathrm{ts}}(x) = \dfrac{F(x)}{a^p}$ $\quad b = \dfrac{1}{\ell} \gg 1$

spatial scaling

temporal scaling defines order of structure function

scaling factor

Usual definition follows for: $F(x) = \|\delta_1 \mathbf{u}\|^p$ $\quad \|\mathbf{r}' - \mathbf{r}\| = 1$

**We show that:**

- Hidden scaling symmetry yields asymptotic power law scaling: $S_p(\ell) \propto \ell^{\zeta_p}$
- Scaling exponents are obtained as Perron-Frobenius eigenvalues of linear operators based on the hidden symmetry of the normalized measure
- Scaling exponents can be anomalous
- Numerical test for the shell model

# Asymptotic power laws and scaling exponents

Hidden symmetry:

$$g_\star \nu = \nu$$

$\Longrightarrow$

Structure function are expressed as iterations of a positive operator in measure space:

$$\lambda^{(n+1)} = \mathcal{L}_p \quad [\lambda^{(n)}]$$

- Perron-Frobenius eigenvalue and eigenvector:

$$\mathcal{L}_p[\lambda_p] = R_p \lambda_p$$

**Corollary 3.** *Assuming limits (IV.32) and (IV.35) and a finite positive value of the integral*

$$\int F(y)\, \rho^\infty(y_\oplus | y_-)\, d\lambda_p\, dy_\oplus, \tag{IV.37}$$

*the structure function $S_p$ has the asymptotic power law scaling (IV.2) in the inertial interval with the exponent*

$$\zeta_p = -\log_2 R_p, \tag{IV.38}$$

*where $R_p$ is the Perron–Frobenius eigenvalue; see (IV.34).*

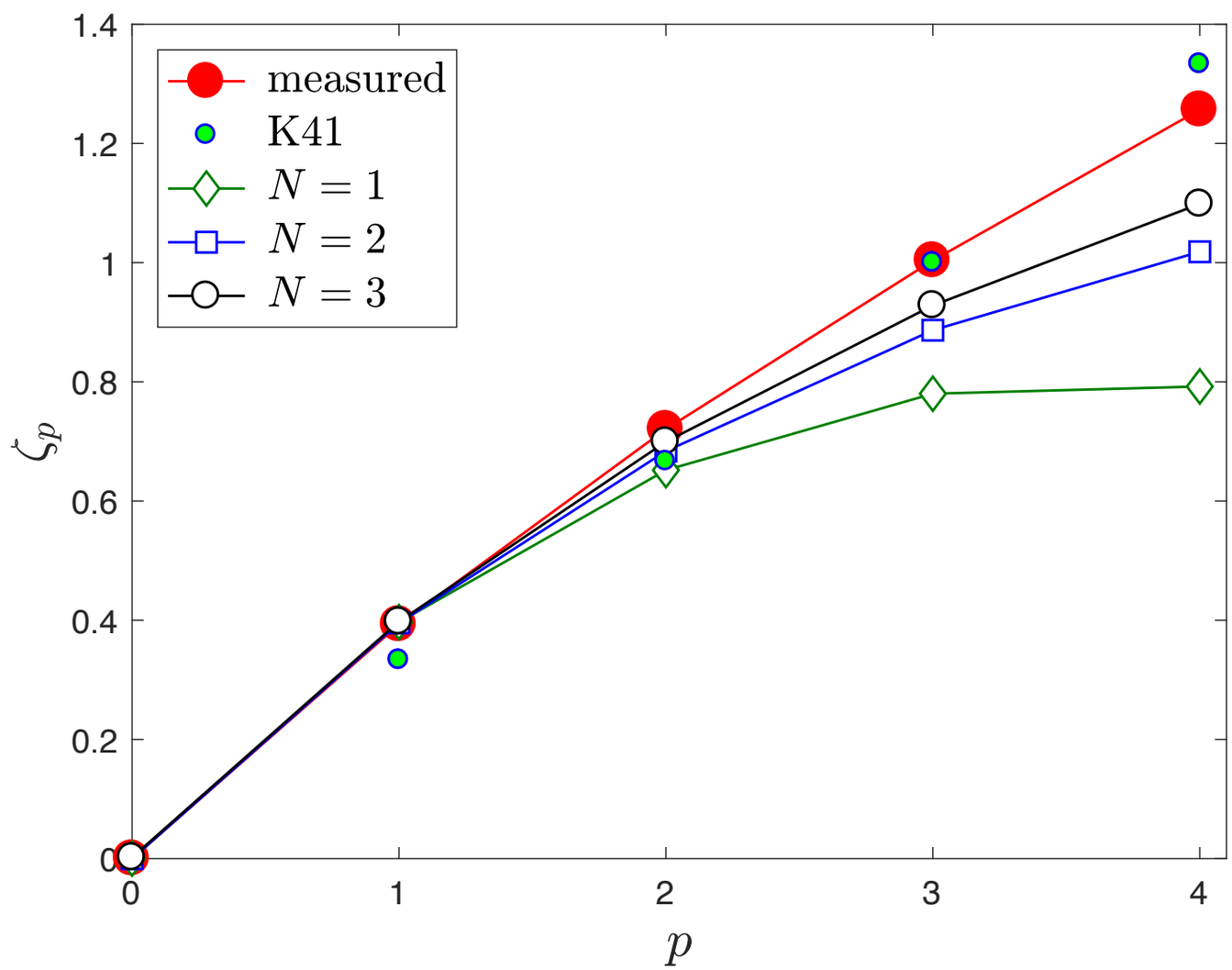
# Numerical test

Eigenvalue is obtained by integrating numerically the approximate operator $\mathcal{L}_p[\lambda]$ with arbitrary positive initial measure.

Approximation of $\mathcal{L}_p[\lambda]$ of order *N* corresponds to

$$\rho_{\ominus}^{\infty}(y_0|y_-) \approx \rho_{\ominus}^{(19)}(y_0|y_{-1},\ldots,y_{-N})$$

approximated numerically using histograms.



# Part 3: General quotient construction (sweeping effect)

# Noncommutativity and equivalence relation

$$\Phi^t \circ s^a_{\mathrm{ts}} = s^a_{\mathrm{ts}} \circ \Phi^{t/a}$$

temporal scalings

$$\Phi^t \circ s^{\mathbf{v}}_{\mathrm{g}} = s^{\mathbf{v}t}_{\mathrm{s}} \circ s^{\mathbf{v}}_{\mathrm{g}} \circ \Phi^t$$

Galilean transformations

$$\mathcal{H} = \left\{ s^a_{\mathrm{ts}} \circ s^{\mathbf{v}}_{\mathrm{g}} : \ a > 0, \ \mathbf{v} \in \mathbb{R}^d \right\}$$

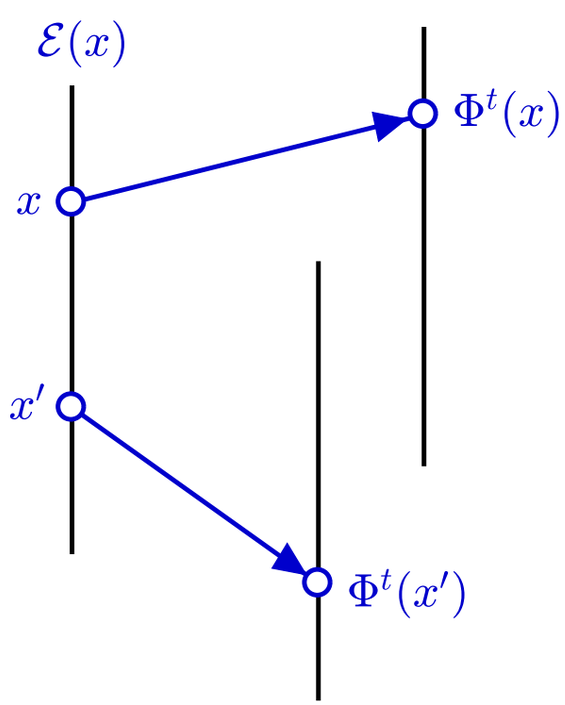
group of temporal scalings and Galilean transformations

Equivalence relation: $x \sim x' \quad \text{if} \quad x' = h(x), \ h \in \mathcal{H}$

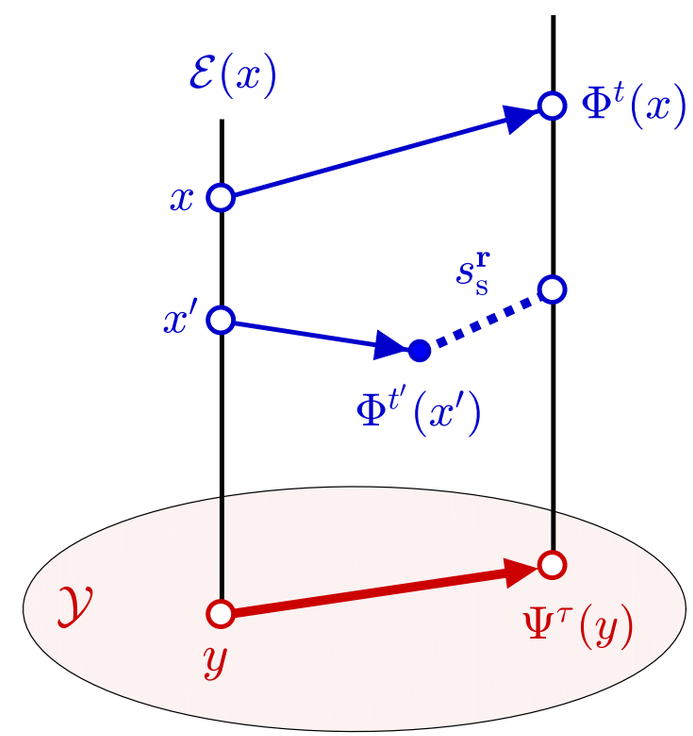
$$s^{\mathbf{r}}_{\mathrm{s}} \circ \Phi^{t'}(x') = h \circ \Phi^t(x)$$

Equivalence class: $\mathcal{E}(x) = \{x' \in \mathcal{X} : x' \sim x\}$

$$t' = at, \quad \mathbf{r} = -\mathbf{v}t$$

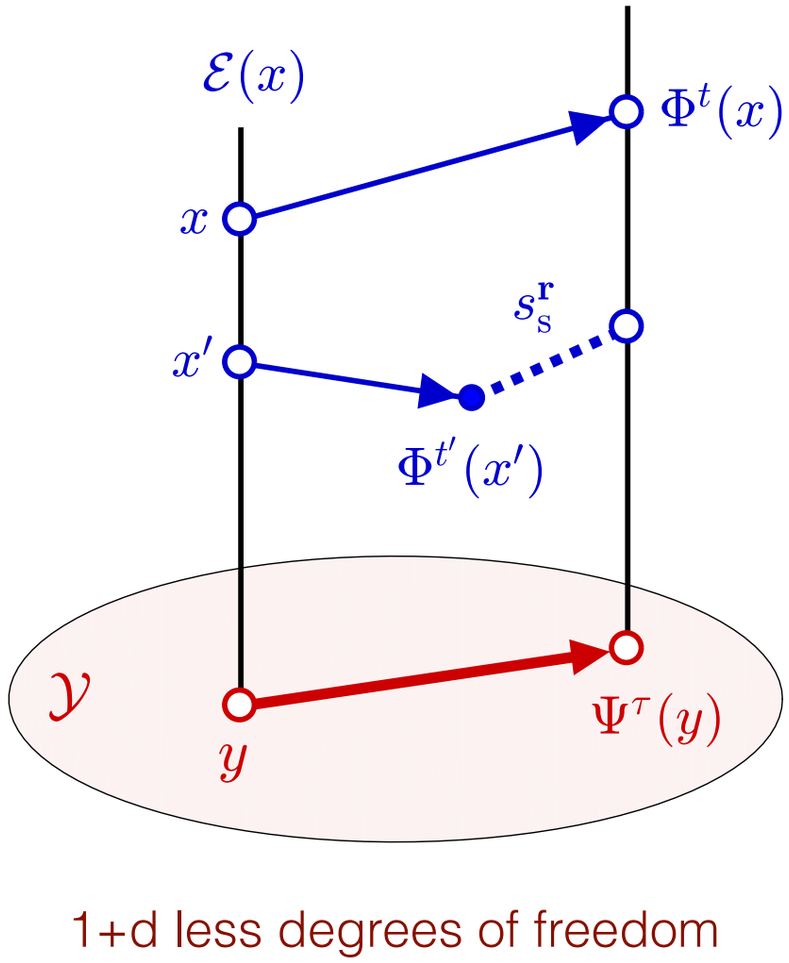



Equivalence is not flow-invariant!




Equivalence can be repaired
by a time change and a space shift

# Quotient-like construction




Equivalence is restored through time change and space shift synchronized with respect to a specific **representative state** in the equivalence class.

This can be done globally in configuration space by choosing a **representative set** $\mathcal{Y} \subset \mathcal{X}$ containing a single element in each equivalence class.

Reducing the dynamics to the representative set yields the **normalized system**.

- There is a normalized flow $\Psi^\tau : \mathcal{Y} \mapsto \mathcal{Y}$ on the representative set, which is induced by $\Phi^t$
- The normalized flow $\Psi^\tau$ has the invariant measure $\nu$, which is explicitly related to the original invariant measure $\mu$.

# Statistical symmetries in the normalized system

$$\mathcal{G} = \left\{ s_{\mathrm{r}}^{\mathbf{Q}} \circ s_{\mathrm{ss}}^{b} : \ \mathbf{Q} \in \mathrm{O}(d),\, b > 0 \right\}$$ group of rotations and spatial scalings

- The group $\mathcal{G}$ defines statistical symmetries in the normalized system. We introduce a transformation $\boxed{\nu \mapsto g_\star \nu}$ for any $g \in \mathcal{G}$, akin to the push-forward. This transformation preserves the group structure and the invariance of a measure with respect to $\Psi^\tau$.

- Using Galilean transformations in the quotient construction requires extra conditions:
    - **spatial homogeneity** (required by the symmetry condition)
    - **incompressibility** (required for the existence of the normalized invariant measure)

# Normalized Euler system

Normalized solution:

$$\widetilde{\mathbf{u}}(\mathbf{r},t) = \mathbf{u}\left(\mathbf{R}^t + \mathbf{r}, t\right) - \mathbf{u}\left(\mathbf{R}^t, t\right), \qquad \frac{d\mathbf{R}^t}{dt} = \mathbf{u}(\mathbf{R}^t, t), \quad \mathbf{R}^0 = \mathbf{0}.$$

(quasi-Lagrangian representation removes the sweeping effect)

$$\mathbf{U}(\mathbf{r},\tau) = \frac{\widetilde{\mathbf{u}}(\mathbf{r},t)}{a_z(t)}, \quad \tau = \int_0^t a_z(s)\,ds, \quad a_z(t) = A \circ \Omega^t(z),$$

intrinsic solution-dependent time

$$A(z) = \left(\int K(r)\,\|\widetilde{\mathbf{u}}(\mathbf{r})\|^2\,d\mathbf{r}\right)^{1/2}$$

Kolmogorov (1962) multipliers:

$$w_{ij}(\mathbf{r};\ell,\ell') = \frac{\delta_i u_j(\mathbf{r},\ell)}{\delta_i u_j(\mathbf{r},\ell')}, \quad \delta_i \mathbf{u}(\mathbf{r},\ell) = \mathbf{u}(\mathbf{r}+\ell\mathbf{e}_i) - \mathbf{u}(\mathbf{r}),$$

$$w_{ij}(\mathbf{r};\ell,\ell') = \frac{U_j^{(m)}(\mathbf{e}_i)}{U_j^{(m)}(\gamma\mathbf{e}_i)}, \quad \ell = 2^{-m}, \quad \gamma = \frac{\ell'}{\ell}.$$

# Conclusion

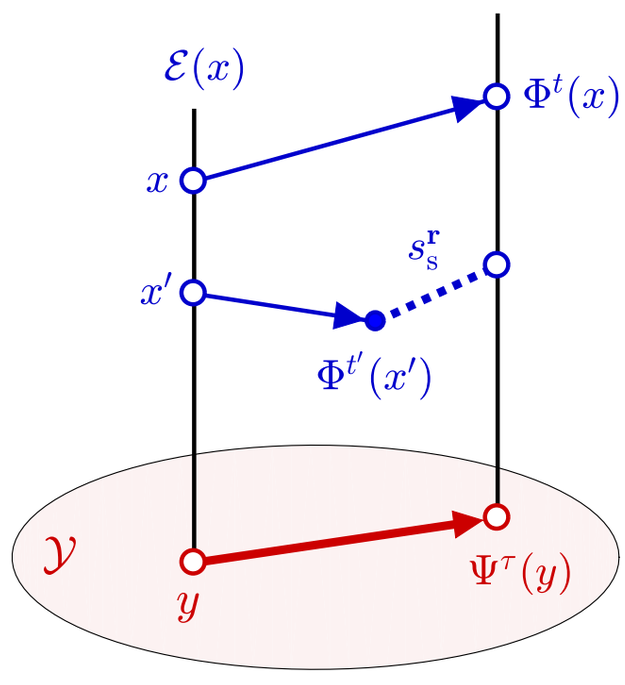
**symmetries not commuting with the flow** → **Normalized system on the representative set** → **Hidden symmetry**

Temporal scaling:

$$\Phi^t \circ s^a_{\mathrm{ts}} = s^a_{\mathrm{ts}} \circ \Phi^{t/a}$$

Galilean transformations:

$$\Phi^t \circ s^{\mathbf{v}}_{\mathrm{g}} = s^{\mathbf{v}t}_{\mathrm{s}} \circ s^{\mathbf{v}}_{\mathrm{g}} \circ \Phi^t$$




Power law asymptotic for structure functions (can be anomalous).

Scaling exponents as Perron-Frobenius eigenvalues.

Possible applications: shell model, Navier-Stokes turbulence, etc.

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Thank you!

arXiv:2010.13089

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