

Discrete macroscopic fluctuation theory and duality properties

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The question

Can one formulate MFT
before taking space to be continuum?

A pedagogical talk, two parts:

- (I) The harmonic process and its NESS (via duality).
- (II) Large spin large deviations (in progress with K. Mallick, H. Suda, T. Sasamoto).

PART (I)

Outline of part (I)

- (I.1) MFT as universal theory for diffusive systems
(validated on the exclusion process)
- (I.2) KMP model, duality.
- (I.3) The harmonic process.
- (I.4) Non-equilibrium steady state.
- (I.5) Large deviations of the empirical energy and MFT.

(I.1) Introduction:

MFT and exclusion process

Macroscopic Fluctuation Theory (MFT)

[Bertini-De Sole-Gabrielli-Landim-Jona Lasinio, ≥ 2001]

- **Universal** theory for the large deviations (around hydrodynamic limit) of diffusive systems with **Diffusivity** $D(\rho)$ and **Mobility** $\sigma(\rho)$.
- Hydrodynamic limit

$$\partial_t \rho + \partial_x j = 0, \quad j = -D(\rho) \partial_x \rho$$

- Fundamental formula: joint **density-current LDP**

$$\mathbb{P}((\rho^N, j^N) \sim (\rho, j)) \asymp e^{-N \mathcal{I}_{[0, T]}(\rho, j)}$$

$$\mathcal{I}_{[0, T]}(\rho, j) = \mathcal{F}(\rho_0) + \frac{1}{2} \int_0^T dt \int dx \frac{[j + D(\rho) \partial_x \rho]^2}{\sigma(\rho)}$$

- Formally, it follows from

$$\partial_t \rho + \partial_x j = 0, \quad j = -D(\rho) \partial_x \rho + \sqrt{\frac{\sigma(\rho)}{N}} \xi$$

$$\langle \xi(x, t) \xi(x', t') \rangle = \delta(x - x') \delta(t - t')$$

Macroscopic Fluctuation Theory (MFT)

[Bertini-De Sole-Gabrielli-Landim-Jona Lasinio, $\gtrsim$ 2000]

- Density LDP

$$\mathbb{P}(\rho^N \sim \rho) \asymp e^{-N\mathcal{I}_{[0,T]}(\rho)}$$

$$\mathcal{I}_{[0,T]}(\rho) = \mathcal{F}(\rho_0) + \inf_{j: \partial_t \rho + \partial_x j = 0} \frac{1}{2} \int_0^T dt \int dx \frac{[j + D(\rho) \partial_x \rho]^2}{\sigma(\rho)}$$

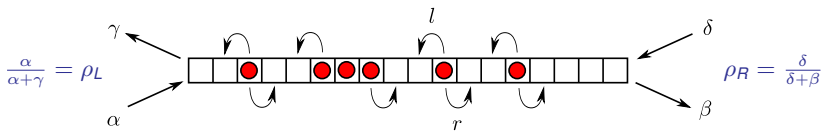
- Introducing the conjugate field $H(x, t)$

$$\mathcal{I}_{[0,T]}(\rho) = \mathcal{F}(\rho_0) + \sup_H \int_0^T dt \left[\int dx (\partial_t \rho) H - \mathcal{H}_{\text{MFT}}(\rho, H) \right]$$

$$\mathcal{H}_{\text{MFT}}(\rho, H) = \int dx \left(-D(\rho) (\partial_x \rho) (\partial_x H) + \frac{1}{2} \sigma(\rho) (\partial_x H)^2 \right)$$

- [Kipnis-Olla-Varadhan '89] provided a rigorous proof of density LDP for the symmetric exclusion process on the torus.

Open symmetric exclusion process



$\rho_L = \rho_R \implies$ Equilibrium

$\rho_L \neq \rho_R \implies$ Non-Equilibrium

The non-equilibrium steady state (NESS) is given by a **matrix product ansatz** [Derrida-Evans-Hakim-Pasquier '93].

Open XXX spin chain is **integrable** for spin 1/2.

Algebraic description of SEP

$\mathfrak{su}(2)$ Lie algebra of the compact Lie group $SU(2)$ with generators satisfying

$$[S^0, S^\pm] = \pm S^\pm \quad [S^+, S^-] = 2S^0$$

Representation with $(2s + 1)$ -dimensional matrices (spin $s \in \frac{\mathbb{N}}{2}$)

$$S^+|n\rangle = (2s - n)|n+1\rangle \quad S^-|n\rangle = n|n-1\rangle \quad S^0|n\rangle = (n - s)|n\rangle$$

$$|n\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad S^+ = \begin{pmatrix} 0 & & & & & \\ & 2s & & & & \\ & & \ddots & & & \\ & & & 2s-1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ & & & & & & 0 \end{pmatrix} \quad S^- = \begin{pmatrix} 0 & 1 & & & & \\ & \ddots & 2 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & 2s \\ & & & & & & 0 \end{pmatrix} \quad S^0 = \begin{pmatrix} -s & 0 & & & & \\ & -s+1 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & s-1 & \\ & & & & & s \end{pmatrix}$$

Spin chains and (Partial) Exclusion Process

In a representation with $(2s+1)$ -dimensional matrices

$$S_1^- S_2^+ + S_1^+ S_2^- + 2S_1^0 S_2^0 - 2s^2$$

leads to Partial Exclusion Process [Schutz 94] with edge generator

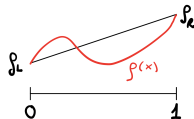
$$\begin{aligned} L_{12}^{PEP} f(\eta_1, \eta_2) &= \eta_1(2s - \eta_2) \left(f(\eta_1 - 1, \eta_2 + 1) - f(\eta_1, \eta_2) \right) \\ &+ \eta_2(2s - \eta_1) \left(f(\eta_1 + 1, \eta_2 - 1) - f(\eta_1, \eta_2) \right) \end{aligned}$$

Reversible measures: product of Binomial($2s, \theta$).

Density large deviations for the boundary-driven SEP

- SEP density large deviations [Derrida-Lebowitz-Speer 2001]

$$\mathbb{P}\text{rob}\left[\rho^N \sim \rho\right] \sim e^{-N I(\rho)}$$



$$I(\rho) = \sup_{\substack{F:[0,1] \rightarrow \mathbb{R}_+ \\ \text{monotone} \\ F(0)=\rho_L \\ F(1)=\rho_R}} \int_0^1 dx \left[\rho(x) \ln \frac{\rho(x)}{F(x)} + (1 - \rho(x)) \ln \frac{1 - \rho(x)}{1 - F(x)} + \ln \frac{F'(x)}{\rho_R - \rho_L} \right]$$

- SEP transport coefficients $D(\rho) = 1, \sigma(\rho) = \rho(1 - \rho)$.

[Bertini-De Sole-Gabrielli-Landim-Jona Lasinio, JSP 2002, MPAG 2003]

Current large deviations for the SEP on $\mathbb{Z}$ (step initial condition)

- Integrated current at bond $(0, 1)$ up to time $T > 0$

$$C(T) = \sum_{i \geq 1} (\eta_i(T) - \eta_i(0))$$

- Micro: [Derrida, Gelshenfeld (2009)], [Imamura, Mallick, Sasamoto (2021)]

$$\mathbb{E}(e^{\lambda C(T)}) \sim e^{\sqrt{T}\Phi(\lambda)}$$

$$\Phi(\lambda) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3/2}} \omega^n$$

$$\omega = \rho_L(e^{-\lambda} - 1) + \rho_R(e^{\lambda} - 1) + \rho_L \rho_R (e^{-\lambda} - 1)(e^{\lambda} - 1)$$

- Macro: [Moriya, Mallick, Sasamoto (2022)]: they could reproduce $\Phi(\lambda)$ by solving MFT equations. Connection to integrable classical systems.

How about other models?

(I.2) KMP and duality

KMP process

Heat Flow in an Exactly Solvable Model

C. Kipnis,¹ C. Marchioro,² and E. Presutti³

Received November 15, 1980

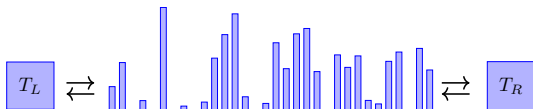
A chain of one-dimensional oscillators is considered. They are mechanically uncoupled and interact via a stochastic process which redistributes the energy between nearest neighbors. The total energy is kept constant except for the interactions of the extremal oscillators with reservoirs at different temperatures. The stationary measures are obtained when the chain is finite; the thermodynamic limit is then considered, approach to the Gibbs distribution is proven, and a linear temperature profile is obtained.



E. Presutti (1942–2024)

[H. Spohn, JPhysA '82]:“ Long range correlations for stochastic lattice gases”

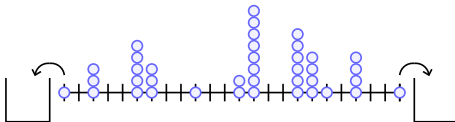
KMP process [Kipnis, Marchioro, Presutti, '82]



$$(X_i, X_{i+1}) \longrightarrow \left((X_i + X_{i+1}) \textcolor{red}{U}, (X_i + X_{i+1}) \textcolor{red}{(1 - U)} \right)$$

$$U \sim \textit{Uniform}(0, 1)$$

↓ Duality ↓



$$\begin{aligned} (\eta_i, \eta_{i+1}) &\longrightarrow \left(\textcolor{red}{K}, \eta_i + \eta_{i+1} - \textcolor{red}{K} \right) \\ K &\sim \textit{Uniform}\{0, 1, \dots, \eta_i + \eta_{i+1}\} \end{aligned}$$

Generator and Semigroup

For an Ω -valued Markov process $(\eta_t)_{t \geq 0}$, i.e.

$$\mathbb{E}[f(\eta_{t+s}) \mid \mathcal{F}_t] = \mathbb{E}[f(\eta_{t+s}) \mid \eta_t], \quad f \in B(\Omega)$$

the semigroup $(\mathcal{S}_t)_{t \geq 0}$ gives the expectation

$$\mathcal{S}_s f(\eta_t) := \mathbb{E}[f(\eta_{t+s}) \mid \mathcal{F}_t]$$

the Markov generator $\mathcal{L}$ gives the infinitesimal change

$$\mathbb{E}[f(\eta_{t+\Delta t}) \mid \mathcal{F}_t] = f(\eta_t) + \mathcal{L}f(\eta_t)\Delta t + o(\Delta t)$$

KMP process

Markov process $\{X(t), t \geq 0\}$ taking values on $(0, \infty)^N$ with generator

$$L = L_1 + \left(\sum_{i=1}^{N-1} L_{i,i+1} \right) + L_N$$

$$L_{i,i+1}f(x) = \int_0^1 du \left[f(x_1, \dots, u(x_i + x_{i+1}), (1-u)(x_i + x_{i+1}), \dots, x_N) - f(x) \right]$$

$$L_1f(x) = \int_0^\infty du \beta_L e^{-\beta_L u} \left[f(u, x_2, \dots, x_N) - f(x) \right] \quad \beta_L = \frac{1}{T_L}$$

$$L_Nf(x) = \int_0^\infty du \beta_R e^{-\beta_R u} \left[f(x_1, \dots, x_{N-1}, u) - f(x) \right] \quad \beta_R = \frac{1}{T_R}$$

KMP-dual process

Markov process $\{\eta(t), t \geq 0\}$ taking values on $\mathbb{N}_0^{N+2}$ with generator

$$\hat{L} = \hat{L}_1 + \left(\sum_{i=1}^{N-1} \hat{L}_{i,i+1} \right) + \hat{L}_N$$

$$\hat{L}_{i,i+1}f(\eta) = \sum_{k=0}^{\eta_i + \eta_{i+1}} \frac{1}{\eta_i + \eta_{i+1} + 1} \left[f(\eta_0, \dots, k, \eta_i + \eta_{i+1} - k, \dots, \eta_{N+1}) - f(\eta) \right]$$

$$\hat{L}_1 f(\eta) = f(\eta_0 + \eta_1, 0, \eta_2, \dots, \eta_N, \eta_{N+1}) - f(\eta)$$

$$\hat{L}_N f(\eta) = f(\eta_0, \dots, \eta_{N-1}, 0, \eta_N + \eta_{N+1}) - f(\eta)$$

KMP duality

Theorem: [Kipnis, Marchioro, Presutti, '82]

- ▶ Let $\{X(t), t \geq 0\}$ be the KMP process and let $\{\eta(t), t \geq 0\}$ be the dual-KMP process. Then $\eta(t) \xrightarrow{D} X(t)$ with

$$D(\eta, x) = T_L^{\eta_0} \left[\prod_{i=1}^N \frac{x_i^{\eta_i}}{\eta_i!} \right] T_R^{\eta_{N+1}}$$

Namely,

$$\mathbb{E}_x \left[D(\eta, x(t)) \right] = \mathbb{E}_\eta \left[D(\eta(t), x) \right]$$

- ▶ Stationary moments: limit $t \rightarrow \infty$

$$\mathbb{E} \left[D(\eta, x) \right] = \sum_{k=0}^{|\eta|} T_L^k T_R^{|\eta|-k} p_\eta(k)$$

where

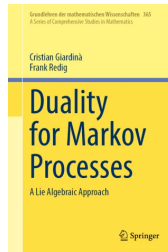
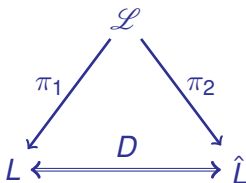
$$p_\eta(k) = \mathbb{P} \left[\eta(\infty) = k\delta_0 + (|\eta| - k)\delta_{N+1} \mid \eta(0) = \eta \right]$$

absorption probabilities

How can we understand the KMP duality?

Duality: a Lie algebraic approach

Abstract generator: an element of the universal enveloping algebra of a Lie algebra



Dual models come in families associated to Lie algebras

XXX spin chain

► Partial exclusion process:

$$\mathcal{H} = \sum_i \left[S_i^- S_{i+1}^+ + S_i^+ S_{i+1}^- + 2S_i^0 S_{i+1}^0 - 2s^2 \right]$$

$\mathfrak{su}(2)$ Lie algebra

$$[S^0, S^\pm] = \pm S^\pm,$$

$$[S^+, S^-] = 2S^0$$

► KMP: [G., Kurchan, Redig $\geq$ 2006]

$$\mathcal{H} = \sum_i \left[S_i^- S_{i+1}^+ + S_i^+ S_{i+1}^- - 2S_i^0 S_{i+1}^0 + 2s^2 \right]$$

$\mathfrak{su}(1, 1)$ Lie algebra

$$[S^0, S^\pm] = \pm S^\pm,$$

$$[S^+, S^-] = -2S_0$$

$$\mathcal{H}_{i,i+1}^{\text{KMP}} = \lim_{t \rightarrow \infty} \exp(t \mathcal{H}_{i,i+1}) - 1$$

XXX spin chain

► Partial exclusion process:

$$\mathcal{H} = \sum_i \left[S_i^- S_{i+1}^+ + S_i^+ S_{i+1}^- + 2S_i^0 S_{i+1}^0 - 2s^2 \right] \quad [s = \frac{1}{2} \text{ integrable}]$$

$$\mathfrak{su}(2) \text{ Lie algebra} \quad [S^0, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = 2S^0$$

► KMP: [G., Kurchan, Redig $\geq$ 2006]

$$\mathcal{H} = \sum_i \left[S_i^- S_{i+1}^+ + S_i^+ S_{i+1}^- - 2S_i^0 S_{i+1}^0 + 2s^2 \right] \quad [\text{not integrable}] !!$$

$$\mathfrak{su}(1, 1) \text{ Lie algebra} \quad [S^0, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = -2S_0$$

$$\mathcal{H}_{i,i+1}^{\text{KMP}} = \lim_{t \rightarrow \infty} \exp(t \mathcal{H}_{i,i+1}) - 1$$

$\mathfrak{su}(1, 1)$ Lie algebra:

representation with differential operators

$$\mathcal{S}^+ f(x) = x f(x)$$

$$\mathcal{S}^- f(x) = \left(x \frac{d^2}{dx^2} + 2s \frac{d}{dx} \right) f(x)$$

$$\mathcal{S}^0 f(x) = \left(x \frac{d}{dx} + s \right) f(x)$$

These operators satisfy:

$$[\mathcal{S}^0, \mathcal{S}^\pm] = \pm \mathcal{S}^\pm \quad [\mathcal{S}^+, \mathcal{S}^-] = -2\mathcal{S}^0$$

Brownian Energy Process

In a representation with differential operators

$$S_1^- S_2^+ + S_1^+ S_2^- - 2S_1^0 S_2^0 + 2s^2$$

leads to the Brownian energy process with edge generator

$$\begin{aligned} L_{12}^{BEP} f(x_1, x_2) \\ = -2s(x_1 - x_2) \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) f(x_1, x_2) + x_1 x_2 \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right)^2 f(x_1, x_2) \end{aligned}$$

KMP process

- ▶ BEP Reversible measures: product of Gamma $(2s, \theta)$.
- ▶ A “thermalization limit” of the BEP process yields family of KMP process:

$$L_{12}^{KMP} = \lim_{t \rightarrow \infty} \left(e^{tL_{12}^{BEP}} - 1 \right)$$

X_1, X_2 independent $\Gamma(2s, \theta)$ r.v.

$\Downarrow$

$$\frac{X_1}{X_1 + X_2} \sim \beta(2s, 2s)$$

- ▶ $s = 1/2 \rightarrow$ original KMP

$\mathfrak{su}(1, 1)$ Lie algebra

representation with ∞ -dimensional matrices

$$S^+|n\rangle = (2s + n)|n+1\rangle$$

$$S^-|n\rangle = n|n-1\rangle$$

$$S^0|n\rangle = (n + s)|n\rangle$$

$$|n\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \quad S^+ = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ 2s & & \ddots & & \\ & & & \ddots & \\ & & 2s+1 & & \ddots \\ & & & & \ddots \end{pmatrix} \quad S^- = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & & & \\ & & 2 & & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix} \quad S^0 = \begin{pmatrix} s & 0 & & & \\ 0 & s+1 & & & \\ & & \ddots & & \\ & & & s+2 & \\ & & & & \ddots \end{pmatrix}$$

These matrices satisfy:

$$[S^0, S^\pm] = \pm S^\pm$$

$$[S^+, S^-] = -2S^0$$

Symmetric Inclusion Process and $\mathfrak{su}(1, 1)$

In a representation with ∞ -dimensional matrices

$$S_1^- S_2^+ + S_1^+ S_2^- - 2S_1^0 S_2^0 + 2s^2$$

leads to the symmetric inclusion process with edge generator

$$\begin{aligned} L_{12}^{SIP} f(\eta_1, \eta_2) &= \eta_1(2s + \eta_2) \left(f(\eta_1 - 1, \eta_2 + 1) - f(\eta_1, \eta_2) \right) \\ &+ \eta_2(2s + \eta_1) \left(f(\eta_1 + 1, \eta_2 - 1) - f(\eta_1, \eta_2) \right) \end{aligned}$$

- ▶ SIP and MFT: [Bettelheim, Meerson, PRE 2024]
- ▶ Q-SIP and free probability: [D. Bernard, T. Jin, S. Scopa, S. Wei, PRE 2025]
- ▶ ASIP with duality: [Carinci, G., Redig, Sasamoto, JSP 2016]
- ▶ ASIP and KPZ : [Minoguchi, Huber, Garbe, Gambassi, Rabl, PRL 2025]

Dual KMP process

- ▶ Reversible measure: product of Negative Binomial($2s, \theta$).
- ▶ A “thermalization limit” of the SIP process yields a family of dual KMP process:

$$L_{12}^{dKMP} = \lim_{t \rightarrow \infty} \left(e^{tL_{12}^{SIP}} - 1 \right)$$

N_1, N_2 independent Neg. Bin. ($2s, \theta$) r.v.

$\Downarrow$

Given $N_1 + N_2 = n$, $N_1 \sim \beta\text{-Bin.}(n, 2s, 2s)$

- ▶ $s = 1/2 \longrightarrow$ dual KMP

$\mathfrak{su}(1, 1)$ algebra duality

$$\left\{ \begin{array}{l} S^+ D(\cdot, x)(n) = S^+ D(n, \cdot)(x) \\ S^- D(\cdot, x)(n) = S^- D(n, \cdot)(x) \\ S^0 D(\cdot, x)(n) = S^0 D(n, \cdot)(x) \end{array} \right. \quad D(n, x) = x^n \frac{\Gamma(2s)}{\Gamma(2s+n)}$$

$$\begin{aligned} \text{E.g.:} \quad S^+ D(n, \cdot)(x) &= x \cdot x^n \frac{\Gamma(2s)}{\Gamma(2s+n)} = x^{n+1} (2s+n) \frac{\Gamma(2s)}{\Gamma(2s+n+1)} \\ &= (2s+n) D(n+1, x) = S^+ D(\cdot, x)(n) \end{aligned}$$

MFT & KMP model

Large deviations (NESS)

- $D_{\text{KMP}}(\rho) = 1, \sigma_{\text{KMP}}(\rho) = \rho^2$. For the stationary large deviations, MFT predicts [Bertini-Gabrielli-Lebowitz, 2005]

$$I(\rho) = \inf_{\substack{\theta: [0,1] \rightarrow \mathbb{R}_+ \\ \text{monotone} \\ \theta(0)=T_L \\ \theta(1)=T_R}} \int_0^1 dx \left[\frac{\rho(x)}{\theta(x)} - 1 - \ln \frac{\rho(x)}{\theta(x)} - \ln \frac{\theta'(x)}{T_R - T_L} \right]$$

- However, this could not be checked from microscopic computations: the stationary measure (NESS) is not explicit (no closed-form for the absorption probabilities of dual particles).
- [Bertini-De Sole-Gabrielli-Landim-Jona Lasinio, JSTAT 2007] noticed that the blue term is the large dev. function for the empirical profile of the order statistics of uniform i.i.d.. However, the ansatz

$$\mu(dx) = \mathbb{E} \left[\prod_{i=1}^N \mu_{T_i}(dx_i) \right] \qquad T_i = T_L + (T_R - T_L) U_{i,N}$$

did not work for KMP [more on this later].

Dynamical large deviations

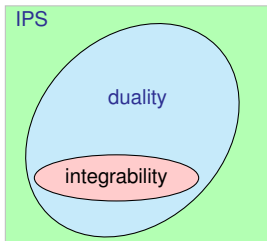
- ▶ [Bettelheim-Smith-Meerson, PRL 22] considered MFT equations and applied ISM to solve the problem of non-stationary heat transfer, starting from a localized heat pulse: $\rho(x, 0) = w\delta(x)$

$$J_T = \int_0^\infty \rho(x, T) dx - \frac{w}{2} \quad \mathbb{P}(J_T \sim j) \sim e^{-\sqrt{T}s(j/w)}$$

Excellent agreement with numerical simulations.

- ▶ However, we miss a proof of dynamical large deviation principle for KMP: [Kipnis-Olla-Varadhan '89] could not be extended to KMP.
- ▶ We miss the super-exponential estimate:
 - ▶ FEM condition of [Kipnis Landim 99] does not apply, as the reversible measure has exponential tails
 - ▶ [Heydecker 26] proves the MFT lower bound for smooth path. He also finds that a finite-cost trajectory can develop a Dirac mass.

Can we find **integrable** KMP-like models?



Guideline: the duality of KMP arises from $\mathfrak{su}(1, 1)$ symmetry.

(I.3) Harmonic process

Frassek, G., Kurchan, *Non-compact quantum spin chains as integrable stochastic particle processes*, J. Stat. Phys. 180, 366–397 (2020)

Integrable quantum spin chains

► Integrable XXX_s chain with non-compact spins

$$\mathcal{H} = \sum_i 2 \left(\psi(S_{i,i+1}) - \psi(2s) \right) \quad [s > 0 \text{ integrable}]$$

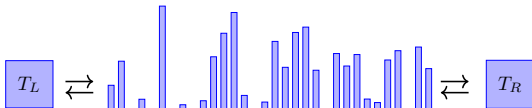
$$\psi(z) = \frac{\partial}{\partial z} \log \Gamma(z) \quad \text{Digamma function}$$

$$S_{i,i+1}(S_{i,i+1} - 1) = S_i^0 S_{i+1}^0 - \frac{1}{2}(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+)$$

[Faddeev, Lipatov, Korchemsky, Derkachov, Beisert, ...]

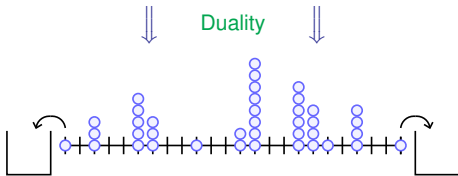
This spin chain provides the
“integrable version of the KMP process”.

Integrable heat conduction model



$$(X_i, X_{i+1}) \longrightarrow \begin{cases} (X_i - B_1, X_{i+1} + B_1) & \text{probability } \frac{1}{2} \\ (X_i + B_2, X_{i+1} - B_2) & \text{probability } \frac{1}{2} \end{cases}$$

$B_1, B_2 \sim$ “improper” $\text{Beta}(0^+, 2s)$ taking values on $(0, X_i)$, *resp.* on $(0, X_{i+1})$



$$(\eta_i, \eta_{i+1}) \longrightarrow \begin{cases} \left(\eta_i - \mathbf{K}_1, \eta_{i+1} + \mathbf{K}_1 \right) & \text{probability } \frac{1}{2} \\ \left(\eta_i + \mathbf{K}_2, \eta_{i+1} - \mathbf{K}_2 \right) & \text{probability } \frac{1}{2} \end{cases}$$

$K_1, K_2 \sim$ “improper” Beta-Binomial($0^+, 2s$) taking values on $\{0, \dots, \eta_i\}$, *resp.* on $\{0, \dots, \eta_{i+1}\}$

Integrable heat conduction model arises from the Integrable XXX_s Hamiltonian

(using the representation of $\mathfrak{su}(1, 1)$
with differential operators)

Integrable heat conduction model

Markov process $\{X(t), t \geq 0\}$ taking values on $(0, \infty)^N$ with generator

$$L = L_1 + \left(\sum_{i=1}^{N-1} L_{i,i+1} \right) + L_N$$

$$\begin{aligned} L_{i,i+1}f(x) &= \int_0^{x_i} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{x_i}\right)^{2s-1} \left[f(x - \alpha\delta_i + \alpha\delta_{i+1}) - f(x) \right] \\ &+ \int_0^{x_{i+1}} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{x_{i+1}}\right)^{2s-1} \left[f(x + \alpha\delta_i - \alpha\delta_{i+1}) - f(x) \right] \end{aligned}$$

$$\begin{aligned} L_1f(\eta) &= \int_0^{x_1} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{x_1}\right)^{2s-1} \left[f(x - \alpha\delta_1) - f(x) \right] \\ &+ \int_0^\infty \frac{d\alpha}{\alpha} e^{-\alpha/T_L} \left[f(x + \alpha\delta_1) - f(x) \right] \end{aligned}$$

$$\begin{aligned} L_Nf(\eta) &= \int_0^{x_1} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{x_1}\right)^{2s-1} \left[f(x - \alpha\delta_1) - f(x) \right] \\ &+ \int_0^\infty \frac{d\alpha}{\alpha} e^{-\alpha/T_R} \left[f(x + \alpha\delta_1) - f(x) \right] \end{aligned}$$

Equilibrium Reversible measures for $T_L = T_R = T$

- ▶ $s = 1/2$: product of Exponential distributions with mean T

$$\mu^{eq}(dx) = \prod_{i=1}^N \frac{1}{T} \exp \left[-\frac{x_i}{T} \right] dx_i$$

- ▶ $s > 0$: product of Gamma distributions with shape parameter $2s$ and mean $2sT$

$$\mu^{eq}(dx) = \prod_{i=1}^N \frac{1}{\Gamma(2s) T^{2s}} x_i^{2s-1} \exp \left[-\frac{x_i}{T} \right] dx_i$$

Harmonic process arises from the Integrable XXX_s Hamiltonian

(using representation of $\mathfrak{su}(1, 1)$)

with ∞ -dimensional matrices)

Harmonic process

Markov process $\{\eta(t), t \geq 0\}$ taking values on $\mathbb{N}_0^{N+2}$ with generator

$$\hat{L} = \hat{L}_1 + \left(\sum_{i=1}^{N-1} \hat{L}_{i,i+1} \right) + \hat{L}_N$$

$$\hat{L}_{i,i+1} f(\eta) = \sum_{k=1}^{\eta_i} \varphi_s(k, \eta_i) \left[f(\eta - k\delta_i + k\delta_{i+1}) - f(\eta) \right] + \sum_{k=1}^{\eta_{i+1}} \varphi_s(k, \eta_{i+1}) \left[f(\eta + k\delta_i - k\delta_{i+1}) - f(\eta) \right]$$

$$\hat{L}_1 f(\eta) = \sum_{k=1}^{\eta_1} \varphi_s(k, \eta_1) \left[f(\eta - k\delta_1 + k\delta_0) - f(\eta) \right]$$

$$\hat{L}_N f(\eta) = \sum_{k=1}^{\eta_N} \varphi_s(k, \eta_N) \left[f(\eta - k\delta_N + k\delta_{N+1}) - f(\eta) \right]$$

$$\varphi_s(k, n) = \frac{1}{k} \frac{\Gamma(n+1)\Gamma(n-k+2s)}{\Gamma(n+2s)\Gamma(n-k+1)}$$

$$\sum_{k=1}^n \varphi_s(k, n) = \psi(n+2s) - \psi(2s) = \sum_{k=1}^n \frac{1}{k+2s-1}$$

shifted harmonic numbers

Theorem: [Frassek, G, Kurchan, JSP 2020]

- ▶ Let $\{X(t), t \geq 0\}$ be the integrable heat conduction process and let $\{\eta(t), t \geq 0\}$ be the harmonic process. Then $\eta(t) \xrightarrow{D} X(t)$ with

$$D(\eta, x) = T_L^{\eta_0} \left[\prod_{i=1}^N x_i^{\eta_i} \frac{\Gamma(2s)}{\Gamma(2s + \eta_i)} \right] T_R^{\eta_{N+1}}$$

Namely,

$$\mathbb{E}_x [D(\eta, x(t))] = \mathbb{E}_\eta [D(\eta(t), x)]$$

- ▶ Stationary moments: limit $t \rightarrow \infty$

$$\mathbb{E} [D(\eta, x)] = \sum_{k=0}^{|\eta|} T_L^k T_R^{|\eta|-k} p_\eta(k)$$

$$p_\eta(k) = \mathbb{P} [\eta(\infty) = k\delta_0 + (|\eta| - k)\delta_{N+1} \mid \eta(0) = \eta]$$

absorption probabilities

Relation to previous models (spin $s = \frac{1}{2}$)

- The bulk of the harmonic is the $q \rightarrow 1$ limit of the **MADM model**

[Sasamoto-Wadati '98]

$$L^{\text{MADM}} = \sum_{i \in \mathbb{Z}} L_{i,i+1}$$

$$\begin{aligned} L_{i,i+1} f(\eta) &= \sum_{k=1}^{\eta_i} \frac{1}{[k]_q} \left[f(\eta - k\delta_i + k\delta_{i+1}) - f(\eta) \right] \\ &+ \sum_{k=1}^{\eta_{i+1}} \frac{q^k}{[k]_q} \left[f(\eta + k\delta_i - k\delta_{i+1}) - f(\eta) \right] \end{aligned}$$

$$q\text{-number} \quad [k]_q = \frac{1 - q^k}{1 - q} \rightarrow k \quad \text{as } q \rightarrow 1$$

Relation to previous models (spin s)

- The bulk of the harmonic is the $q \rightarrow 1$ limit of the q -Hahn model

[Povolotsky, 2013], [Barraquand-Corwin, 2015]

$$\varphi^{r,q,\nu}(k, n) = \frac{1}{[k]_q} \frac{(q; q)_n (\nu; q)_{n-k}}{(\nu; q)_n (q, q)_{n-k}}$$

$$\varphi^{\ell,q,\nu}(k, n) = \frac{\nu^k}{[k]_q} \frac{(q; q)_n (\nu; q)_{n-k}}{(\nu; q)_n (q, q)_{n-k}}$$

$$(\nu; q)_n = \prod_{j=0}^{n-1} (1 - \nu q^j) \quad \lim_{q \rightarrow 1} \frac{(q^{2s}; q)_n}{(1 - q)^n} = \frac{\Gamma(2s + n)}{\Gamma(2s)}$$

$$\lim_{q \rightarrow 1} \varphi^{r/\ell, q, q^{2s}}(k, n) = \varphi_s(k, n)$$

(I.4) Non-equilibrium steady state harmonic process

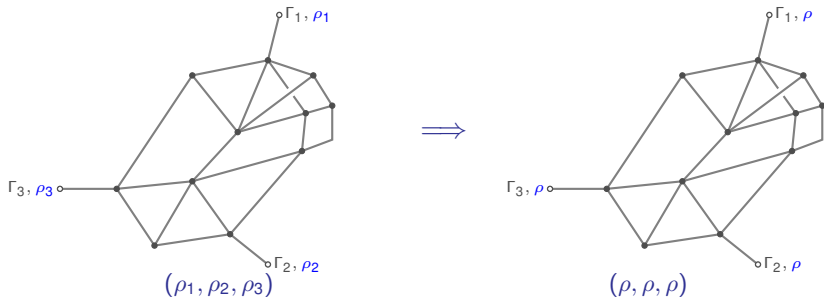
2 steps:

- (i) Use **duality** to simplify the problem and **integrability** to compute moments
[Frassek, G.]: J. Math. Phys. (2022)
- (ii) Use **generating functions** to recognize the structure of the stationary measure
[Carinci, Franceschini, Frassek, G., Redig]: Comm. Math. Phys (2025)

Step (i): computing moments in the NESS

As a consequence of duality, there exist a non-local mapping between non-equilibrium and equilibrium (isospectrality argument, [Alcaraz et al., '94])

$$L^{eq} = P^{-1}LP$$



- ▶ The mapping P is **non-local**, it holds on any graph, with multiple reservoirs [Frassek, G., Kurchan, 2020]
- ▶ The mapping was observed **macroscopically** in the context of the MFT by [Tailleur, Kurchan, Lecomte, 2008]

Step (i): computing moments in the NESS

$$1 \text{ dual particle: } \eta = \delta_i \quad \implies \quad D(\delta_i, x) = \frac{x_i}{2s}$$

$$\mathbb{E} \left[\frac{x_i}{2s} \right] = T_R p_{\delta_i}(0) + T_L p_{\delta_i}(1)$$

$$p_{\delta_i}(0) = \frac{i}{N+1} \qquad p_{\delta_i}(1) = 1 - \frac{i}{N+1}$$

$$\mathbb{E}[x_i] = 2s \left(T_L + \frac{T_R - T_L}{N+1} i \right) \qquad \text{Linear profile}$$

2 dual particles : $\eta = \delta_{i_1} + \delta_{i_2}$

$$\mathbb{C}ov(x_{i_1}, x_{i_2}) = \mathbb{E}[x_{i_1} x_{i_2}] - \mathbb{E}[x_{i_1}]\mathbb{E}[x_{i_2}]$$

$$D(\eta, \delta_{i_1} + \delta_{i_2}) = \frac{x_{i_1} x_{i_2}}{(2s)^2} \quad \text{if } i_1 < i_2$$

$$\mathbb{C}ov(x_{i_1}, x_{i_2}) = (2s)^2 \frac{i_1(N+1-i_2)}{(N+1)^2(N+2)} (T_L - T_R)^2$$

Remark: Long range correlations

$$\lim_{N \rightarrow \infty} N \mathbb{C}ov(x_{y_1 N}, x_{y_2 N}) = (2s)^2 y_1(1-y_2)(T_R - T_L)^2 \quad 0 \leq y_1 < y_2 \leq 1$$

3 dual particles : $\eta = \delta_{i_1} + \delta_{i_2} + \delta_{i_3}$

$$\begin{aligned}\kappa_3(x_{i_1}, x_{i_2}, x_{i_3}) &= \mathbb{E}[x_{i_1} x_{i_2} x_{i_3}] + 2\mathbb{E}[x_{i_1}]\mathbb{E}[x_{i_2}]\mathbb{E}[x_{i_3}] \\ &\quad - \mathbb{E}[x_{i_1} x_{i_2}]\mathbb{E}[x_{i_3}] - \mathbb{E}[x_{i_1} x_{i_3}]\mathbb{E}[x_{i_2}] - \mathbb{E}[x_{i_2} x_{i_3}]\mathbb{E}[x_{i_1}]\end{aligned}$$

Using three dual particles started at $1 \leq i_1 < i_2 < i_3 \leq N$

$$\kappa_3(x_{i_1}, x_{i_2}, x_{i_3}) = (2s)^3 \frac{2i_1(N+1-2i_2)(N+1-i_3)}{(N+1)^3(N+2)(N+3)} (T_R - T_L)^3$$

Remark: for $0 \leq y_1 < y_2 < y_3 \leq 1$

$$\lim_{N \rightarrow \infty} N^2 \kappa_3(x_{y_1 N}, x_{y_2 N}, x_{y_3 N}) = (2s)^3 2y_1(1-2y_2)(1-y_3)(T_R - T_L)^3$$

NESS moments

Theorem [Frassek, G., J.Math.Phys. (2022)]

- For the integrable heat conduction process:

$$\mathbb{E} \left[\prod_{i=1}^N x_i^{\eta_i} \frac{\Gamma(2s)}{\Gamma(\eta_i + 2s)} \right] = \sum_{n=0}^{|\eta|} T_R^{|\eta|-n} (T_L - T_R)^n g_\eta(n)$$

$$g_\eta(n) = \sum_{\substack{n_1, \dots, n_N \\ \sum_i n_i = n}} \prod_{i=1}^N \binom{\eta_i}{n_i} \prod_{j=1}^{2s} \frac{2s(N+2-i) - j}{2s(N+2-i) - j + \sum_{k=i}^N n_k}$$

STEP (ii): identifying the stationary measure

$$\mathbb{E}\left[e^{\sum_{i=1}^N h_i x_i}\right] = \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}} \int_0^1 dv_1 \int_{v_1}^1 dv_2 \cdots \int_{v_{N-1}}^1 dv_N \prod_{i=1}^{N+1} (v_i - v_{i-1})^{2s-1} \\ \cdot \prod_{i=1}^N \left(\frac{1}{1 - h_i(T_L + (T_R - T_L)v_i)} \right)^{2s}$$

STEP (ii): identifying the stationary measure

$$\mathbb{E}\left[e^{\sum_{i=1}^N h_i x_i}\right] = \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}} \int_0^1 dv_1 \int_{v_1}^1 dv_2 \cdots \int_{v_{N-1}}^1 dv_N \prod_{i=1}^{N+1} (v_i - v_{i-1})^{2s-1} \\ \cdot \prod_{i=1}^N \left(\frac{1}{1 - h_i (T_L + (T_R - T_L)v_i)} \right)^{2s}$$

$$\blacktriangleright X \sim \text{Gamma}(2s, T) \implies \mathbb{E}(e^{hX}) = \left(\frac{1}{1-hT} \right)^{2s}$$

STEP (ii): identifying the stationary measure

$$\mathbb{E} \left[e^{\sum_{i=1}^N h_i x_i} \right] = \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}} \int_0^1 dv_1 \int_{v_1}^1 dv_2 \cdots \int_{v_{N-1}}^1 dv_N \prod_{i=1}^{N+1} (v_i - v_{i-1})^{2s-1} \cdot \prod_{i=1}^N \left(\frac{1}{1 - h_i (T_L + (T_R - T_L) v_i)} \right)^{2s}$$

- $X \sim \text{Gamma}(2s, T) \implies \mathbb{E}(e^{hX}) = \left(\frac{1}{1-hT}\right)^{2s}$
- Symmetric Dirichlet (2s): $\mathbf{R} = (R_1, R_2, \dots, R_{N+1})$

$$f_{\mathbf{R}}(\mathbf{r}) = \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}} \prod_{i=1}^{N+1} (r_i)^{2s-1} \mathbb{1}_{\{\sum_{i=1}^{N+1} r_i = 1\}}$$

- ordered Dirichlet process: $V_j = R_1 + \dots + R_j$

$$f_{v_1, v_2, \dots, v_N}(v_1, \dots, v_N) = \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}} \prod_{i=1}^{N+1} (v_i - v_{i-1})^{2s-1} \mathbb{1}_{\{0 \leq v_1 \leq \dots \leq v_N \leq 1\}}$$

STEP (ii): identifying the stationary measure

$$\mathbb{E}\left[e^{\sum_{i=1}^N h_i x_i}\right] = \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}} \int_0^1 dv_1 \int_{v_1}^1 dv_2 \cdots \int_{v_{N-1}}^1 dv_N \prod_{i=1}^{N+1} (v_i - v_{i-1})^{2s-1} \cdot \prod_{i=1}^N \left(\frac{1}{1 - h_i (T_L + (T_R - T_L)v_i)} \right)^{2s}$$

- $X \sim \text{Gamma}(2s, T) \implies \mathbb{E}(e^{hX}) = \left(\frac{1}{1-hT}\right)^{2s}$
- Symmetric Dirichlet (2s): $\mathbf{R} = (R_1, R_2, \dots, R_{N+1})$

$$f_{\mathbf{R}}(\mathbf{r}) = \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}} \prod_{i=1}^{N+1} (r_i)^{2s-1} \mathbb{1}_{\{\sum_{i=1}^{N+1} r_i = 1\}}$$

- ordered Dirichlet process: $V_j = R_1 + \dots + R_j$

$$f_{v_1, v_2, \dots, v_N}(v_1, \dots, v_N) = \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}} \prod_{i=1}^{N+1} (v_i - v_{i-1})^{2s-1} \mathbb{1}_{\{0 \leq v_1 \leq \dots \leq v_N \leq 1\}}$$

$$\mathbb{E}\left[\mathbf{e}^{\sum_{i=1}^N h_i x_i}\right] = \mathbb{E}\left[\mathbb{E}\left[\prod_{i=1}^N e^{h_i x_i} \mid \mathcal{T}\right]\right] \quad T_i = T_L + (T_R - T_L)\mathbf{V}_i$$

NESS of integrable heat conduction

Theorem [Carinci, Franceschini, Frassek, **G**, Redig, CMP (2025)]

$$\mu(dx) = \mathbb{E} \left[\prod_{i=1}^N \mu_{T_i}(dx_i) \right]$$

- ▶ μ_{T_i} Gamma distr. with shape $2s$ and mean $2sT_i$
- ▶ $T_i = T_L + (T_R - T_L)V_i$ $V_i = R_1 + \dots + R_i$ ordered Dirichlet

Explicitly

$$\begin{aligned} \mu(dx) = & \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}(T_R - T_L)^N} \int_{T_L}^{T_R} d\theta_1 \int_{\theta_1}^{T_R} d\theta_2 \cdots \int_{\theta_{N-1}}^{T_R} d\theta_N \left[\prod_{i=1}^{N+1} (\theta_i - \theta_{i-1})^{2s-1} \right] \\ & \cdot \prod_{i=1}^N \frac{1}{\Gamma(2s)(\theta_i)^{2s}} x_i^{2s-1} \exp \left[-\frac{x_i}{\theta_i} \right] dx_i \end{aligned}$$

- ▶ [Carinci, Franceschini, Gabrielli, **G**, Tsagarogiannis, JSP (2024)] $\longrightarrow s = 1/2$
- ▶ [**G**, Redig, van Tol, JSP (2025)] Propagation of mixtures in time

The hidden parameter process

Let $\Theta(t) = (\theta_1(t), \dots, \theta_N(t))$ be the process with generator

$$\mathcal{L} = \mathcal{L}_1 + \sum_{i=1}^{N-1} \mathcal{L}_{i,i+1} + \mathcal{L}_N$$

where

$$\mathcal{L}_1 f(\theta_1) = \int_0^1 \frac{du}{u} (1-u)^{2s-1} [f(\theta_1(1-u) + uT_L) - f(\theta_1)]$$

$$\begin{aligned} \mathcal{L}_{1,2} f(\theta_1, \theta_2) &= \int_0^1 \frac{du}{u} (1-u)^{2s-1} [f(\theta_1(1-u) + u\theta_2, \theta_2) - f(\theta_1, \theta_2)] \\ &+ \int_0^1 \frac{du}{u} (1-u)^{2s-1} [f(\theta_1, u\theta_1 + (1-u)\theta_2) - f(\theta_1, \theta_2)] \end{aligned}$$

$$\mathcal{L}_N f(\theta_N) = \int_0^1 \frac{du}{u} (1-u)^{2s-1} [f(\theta_N(1-u) + uT_R) - f(\theta_N)]$$

Propagation of mixtures

[G., Redig, van Tol JSP (2025)]

Start the integrable heat conduction process from an inhomogeneous product measure

$$\mu^{(0)} = \bigotimes_{i=1}^N \mu_{\theta_i} \qquad \mu_{\theta_i}(dx_i) = \frac{1}{\Gamma(2s)\theta_i^{2s}} x_i^{2s-1} \exp\left[-\frac{x_i}{\theta_i}\right] dx_i$$

At time $t > 0$ the evolved measure is

$$\mu^{(t)} = \mathbb{E}_{\theta} \left[\bigotimes_{i=1}^N \mu_{\Theta_i(t)} \right]$$

where $\Theta(t) = (\Theta_1(t), \dots, \Theta_N(t))$ is the hidden parameter model started from $\theta = (\theta_1, \dots, \theta_N)$.

(I.5) NESS large deviations and MFT

Large deviations

Theorem [Carinci, Franceschini, Frassek, **G**, Redig, CMP (2025)]

- The empirical energy profile of the integrable heat conduction process

$$\pi_N = \frac{1}{N} \sum_{i=1}^N x_i \delta_{\frac{i}{N}} \in \mathcal{M}_+([0, 1])$$

satisfies an LDP with the weak topology of finite positive measures

$$\mathbb{P}[\pi_N \sim \mu] \asymp e^{-N I(\mu)}$$

If $\mu(dy) = \rho(y)dy$ the rate function is

$$I(\rho) = \inf_{\substack{\theta: [0,1] \rightarrow \mathbb{R}_+ \\ \text{monotone} \\ \theta(0)=T_L \\ \theta(1)=T_R}} 2s \int_0^1 dy \left[\frac{\rho(y)}{\theta(y)} - 1 - \ln \frac{\rho(y)}{\theta(y)} - \ln \frac{\theta'(y)}{T_R - T_L} \right]$$

Comments

- **Macroscopic universality:** The integrable heat conduction has

$$D_{\text{IHC}}(\rho) = \frac{1}{2s} \quad \sigma_{\text{IHC}}(\rho) = \left(\frac{\rho}{2s} \right)^2$$

For $s = 1/2$ and **regular** profiles we recover the rate function predicted by MFT for the **KMP model** [Bertini-Gabrielli-Lebowitz, 2005]

$$D_{\text{KMP}}(\rho) = 1 \quad \sigma_{\text{KMP}}(\rho) = \rho^2$$

- **Singular measure:** [F. Sau, private communication] remarked that for a product of $\text{Gamma}(2s, \theta(i/N))$ r.v. , if μ has Lebesgue decomposition $\mu = \rho(y)dy + \mu^\perp$ then

$$I(\mu) = 2s \int_0^1 dy \left[\frac{\frac{\rho(y)}{2s}}{\theta(y)} - 1 - \ln \frac{\frac{\rho(y)}{2s}}{\theta(y)} \right] + \int_0^1 \frac{1}{\theta(y)} \mu^\perp(dy)$$

For the NESS mixture measure we thus expect

$$I(\mu) = \inf_{\substack{\theta: [0,1] \rightarrow \mathbb{R}_+ \\ \text{monotone} \\ \theta(0)=T_L \\ \theta(1)=T_R}} \left\{ 2s \int_0^1 dy \left[\frac{\frac{\rho(y)}{2s}}{\theta(y)} - 1 - \ln \frac{\frac{\rho(y)}{2s}}{\theta(y)} - \ln \frac{\theta'(y)}{T_R - T_L} \right] + \int_0^1 \frac{1}{\theta(y)} \mu^\perp(dy) \right\}$$

Pressure

- For $h \in C([0, 1])$ satisfying $\sup_{x \in [0, 1]} h(x) < \frac{1}{T_R}$ the pressure

$$P(h) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E} \left[e^{N \langle \pi_N, h \rangle} \right]$$

is given by

$$P(h) = \sup_{\substack{\theta: [0, 1] \rightarrow \mathbb{R}_+ \\ \text{monotone} \\ \theta(0) = T_L \\ \theta(1) = T_R}} 2s \int_0^1 dy \left[\ln \left(\frac{1}{1 - h(y)\theta(y)} \right) + \ln \frac{\theta'(y)}{T_R - T_L} \right]$$

- Remark: the Legendre transform of the pressure P gives the **convex hull** of I .
- The singular part of the weak-measure rate does not change P , because a singular mass contributes $\int (h - \frac{1}{\theta}) d\mu^\perp$, which is optimized by $\mu^\perp = 0$ when $h < \frac{1}{\theta}$.

Additivity principle

- The variational principle of MFT is equivalent to:

$$\tilde{P}_{T_L, T_R}^{[0,1]}(h) = \sup_{T_L \leq \theta_m \leq T_R} \left[\tilde{P}_{T_L, \theta_m}^{[0,y]}(h_1) + \tilde{P}_{\theta_m, T_R}^{[y,1]}(h_2) \right] \quad 0 < y < 1$$

where

$$\tilde{P}_{\theta_a, \theta_b}^{[a,b]}(h) := P_{\theta_a, \theta_b}^{[a,b]}(h) + 2s(b-a) \log \left(\frac{\theta_b - \theta_a}{b-a} \right)$$

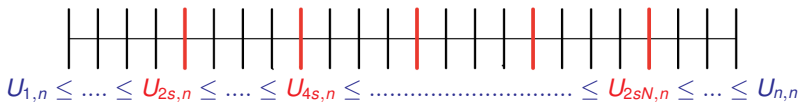
and

$$P_{\theta_a, \theta_b}^{[a,b]}(h) = \sup_{\theta} 2s \int_a^b \left[\ln \left(\frac{1}{1 - h(y)\theta(y)} \right) + \ln \left(\frac{(b-a)\theta'(y)}{\theta_b - \theta_a} \right) \right] dy$$

- Remark: this emerges as a consequence of the **Markovian structure** of the ordered Dirichlet.

Proof of large deviations

- Assume $s \in \mathbb{N}/2$ and represent the ordered Dirichlet in terms of gapped order statistics of $n = 2s(N + 1) - 1$ i.i.d. $\text{Uni}(0, 1)$



- NESS

$$\mu(dx) = \mathbb{E}\left[\prod_{i=1}^N \mu_{T_i}(dx_i)\right]$$

$$T_i = T_L + (T_R - T_L)U_{2si,n}$$

Proof of large deviations

1. Order statistics empirical profile $\pi'_N = \frac{1}{N} \sum_{i=1}^N T_i \delta_{\frac{i}{N}}$

$$\mathbb{P}\left[\pi'_N(dy) \approx \theta(y)dy\right] \sim e^{-N J(\theta)}$$

$$J(\theta) = 2s \int_0^1 dy \left[-\ln \frac{\theta'(y)}{T_B - T_l} \right]$$

$\theta(\cdot)$ monotone, $\theta(0) = T_L, \theta(1) = T_R$

[Duffy, Macci, Torrì, 2011]

- ## 2. Joint large deviations

$$\mathbb{P}\left[\pi_N(dy) \approx \rho(y)dy, \pi'_N(dy) \approx \theta(y)dy\right] \sim e^{-N\left[S\left(\frac{\rho}{2s} \mid \theta\right) + J(\theta)\right]}$$

with

$$S\left(\frac{\rho}{2s} \mid \theta\right) = 2s \int_0^1 dy \left[\frac{\frac{\rho(y)}{2s}}{\theta(y)} - 1 - \ln \frac{\frac{\rho(y)}{2s}}{\theta(y)} \right]$$

- ### 3. Energy large deviations

$$\mathbb{P}\left[\pi_N(dy) \approx \rho(y)dy\right] \sim e^{-N I(\rho)}$$

with

$$I(\rho) = \inf_{\substack{\theta: [0,1] \rightarrow \mathbb{R}_+ \\ \text{monotone} \\ \theta(0)=T_L \\ \theta(1)=T_B}} \left[S\left(\frac{\rho}{2s} \mid \theta\right) + J(\theta) \right]$$

contraction principle

Additivity principle

1. The order statistics of i.i.d $\text{Uni}(0,1)$ has a special Markovian property:

Conditional on $U_{k,n} = a$:

- ▶ $\{U_{j,n}\}_{1 \leq j \leq k-1}$ are distributed as $k-1$ ordered uniforms on $[0, a]$
- ▶ $\{U_{j,n}\}_{k+1 \leq j \leq n}$ are distributed as $n-k$ ordered uniforms on $[a, 1]$

2. Applying this property and Varadhan lemma:

$$\tilde{P}_{T_L, T_R}^{[0,1]}(h) = \sup_{T_L \leq \theta_m \leq T_R} \left[\tilde{P}_{T_L, \theta_m}^{[0,y]}(h_1) + \tilde{P}_{\theta_m, T_R}^{[y,1]}(h_2) \right] \quad 0 < y < 1$$

Summary of part (I)

- For the boundary-driven integrable heat-conduction model ($\equiv$ **integrable version of KMP model**) we characterized the stationary measure as

$$\mu(dx) = \mathbb{E} \left[\prod_{i=1}^N \mu_{T_i}(dx_i) \right] \quad T_1 \leq \dots \leq T_N \quad \text{ordered Dirichlet}$$

$$\mu_{T_i}(dx_i) = \frac{1}{\Gamma(2s)(T_i)^{2s}} x_i^{2s-1} \exp \left[-\frac{x_i}{T_i} \right] dx_i$$

- For the **KMP model**, the stationary state is also a mixture [Ferrari, Gabrielli, De Masi, JSP 2024].

$$\mu(dx) = \mathbb{E} \left[\prod_{i=1}^N \mu_{T_i}(dx_i) \right]$$

Question: What is the mixing law of KMP?

NESS large deviations

- For the boundary-driven integrable heat-conduction model the regular-profile formula agrees with MFT prediction with

$$D_{\text{IHC}}(\rho) = \frac{1}{2s} \quad \sigma_{\text{IHC}}(\rho) = \left(\frac{\rho}{2s}\right)^2$$

If $\mu(dy) = \rho(y)dy$ the rate function is

$$I(\rho) = \inf_{\substack{\theta: [0,1] \rightarrow \mathbb{R}_+ \\ \text{monotone} \\ \theta(0)=T_L \\ \theta(1)=T_R}} 2s \int_0^1 dy \left[\frac{\rho(y)}{\theta(y)} - 1 - \ln \frac{\rho(y)}{\theta(y)} - \ln \frac{\theta'(y)}{T_R - T_L} \right]$$

- But classical MFT, understood as a functional only of **smooth density and current fields**, may not be not the complete LD theory for KMP-type models.

The complete theory must be its measure-valued lower-semicontinuous relaxation, thus including **singular profiles with finite cost**.

The failure of the required superexponential estimate is a **structural warning** of precisely this phenomenon.

BREAK

PART (II)

Outline of part (II)

(II.1) Introduction

(II.2) Large deviations around hydrodynamic limit
(validated on the exclusion process)

(II.3) Large spin large deviations (around "fluid" limit)

(II.4) Proof of LDP

(II.5) Comments on dMFT

(II.1) Introduction

“Be wise, discretize!” (Mark Kac)

- ▶ Sometimes, the discrete theory is useful to make the continuum theory intellegible. Simple example:
 - A random walk has genuine jumps and an exponential Hamiltonian.
 - Brownian motion retains only its quadratic, Gaussian shadow.
- ▶ Discrete MFT plays the same role. It keeps the full **structure of the microscopic currents**, while ordinary MFT emerges in the diffusive limit, when only **drift and variance** survive.
- ▶ So discrete MFT is not simply MFT written on a mesh: it is a **prelimit theory** that explains why continuous MFT is quadratic – and may tell us what replaces it when the local fluctuations are not Gaussian.

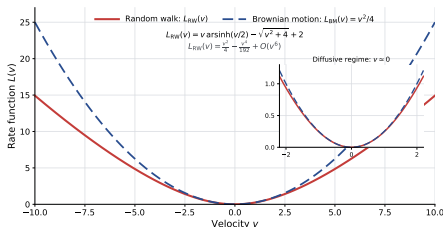
Path costs of random walk and of Brownian motion

For the continuous-time symmetric walk $X_t = N_t^+ - N_t^-$, with $N_t^\pm$ independent Poisson processes of rate 1, and Brownian motion B_t with generator ∂_{xx} , set

$$X^s(t) = \frac{X(st)}{s}, \quad B^s(t) = \frac{B(st)}{s}.$$

Both path-space LDPs have the form

$$\mathbb{P}(Z^s \sim x) \asymp \exp \left\{ -s \int_0^T L(\dot{x}(t)) dt \right\}$$



$$L_{RW}(v) = v \operatorname{arsinh}(v/2) - \sqrt{v^2 + 4} + 2$$

$$L_{BM}(v) = \frac{v^2}{4}$$

Near the diffusive regime,

$$L_{RW}(v) = \frac{v^2}{4} - \frac{v^4}{192} + O(v^6).$$

Random walk and Brownian motion: the correct scaling

For the continuous-time symmetric walk with jumps ± 1 at rate 1, define

$$Y_t^N := \frac{X_{Nt}}{\sqrt{N}} \implies B_t, \quad Y_\tau^{N,s} := \frac{Y_{s\tau}^N}{s} = \frac{X_{Ns\tau}}{\sqrt{N}s}$$

The scaled walk has Hamiltonian

$$\begin{aligned} H_N(p) &= N H_{\text{RW}}\left(\frac{p}{\sqrt{N}}\right) = N \left(e^{p/\sqrt{N}} + e^{-p/\sqrt{N}} - 2 \right) \\ &= p^2 + \frac{p^4}{12N} + O(N^{-2}) \xrightarrow{N \rightarrow \infty} H_{\text{BM}}(p) \end{aligned}$$

Its Legendre transform is

$$L_N(v) = N L_{\text{RW}}\left(\frac{v}{\sqrt{N}}\right) = \frac{v^2}{4} - \frac{v^4}{192N} + O(N^{-2}) \xrightarrow{N \rightarrow \infty} L_{\text{BM}}(v).$$

The path-space LDP is

$$\begin{aligned} \mathbb{P}(Y^{N,s} \sim x) &\asymp \exp \left\{ -sN \int_0^T L_{\text{RW}}\left(\frac{\dot{x}(t)}{\sqrt{N}}\right) dt \right\} = \exp \left\{ -s \int_0^T L_N(\dot{x}(t)) dt \right\}, \\ I_N(x) &:= \int_0^T L_N(\dot{x}) dt \xrightarrow{N \rightarrow \infty} I_{\text{BM}}(x) := \frac{1}{4} \int_0^T \dot{x}(t)^2 dt. \end{aligned}$$

The gap between the two curves measures finite- N information.
The invariance principle concerns the limit $N \rightarrow \infty$.

Information retained by the discrete theory at finite N

Non-Gaussian cumulants

Since H_N is the scaled cumulant generating function,

$$H_N(p) = p^2 + \frac{p^4}{12N} + O(N^{-2}),$$

$$\frac{\kappa_2(Y_t^N)}{t} = 2,$$

$$\frac{\kappa_4(Y_t^N)}{t} = \frac{2}{N} \rightarrow 0.$$

The Brownian limit retains only the variance.

Two microscopic walks can have the same Brownian limit:

jumps	$H_N(p)$	κ_4/t
± 1 at rate 1	$p^2 + \frac{p^4}{12N} + \dots$	$\frac{2}{N}$
± 2 at rate 1/4	$p^2 + \frac{p^4}{3N} + \dots$	$\frac{8}{N}$

Both Hamiltonians converge to p^2 .

Microscopic realization of the optimal drift

The exponential tilt producing velocity v gives the optimal rates

$$r_{\pm}^N(v) = N\sqrt{1 + \frac{v^2}{4N}} \pm \frac{v\sqrt{N}}{2}.$$

They determine both the drift and the activity:

$$\frac{r_+^N - r_-^N}{\sqrt{N}} = v,$$

$$a_N(v) := r_+^N + r_-^N = 2N\sqrt{1 + \frac{v^2}{4N}}.$$

For $N = 1$ and $v = 10$,

$$(r_+, r_-, a_1) \simeq (10.099, 0.099, 10.198).$$

Brownian optimal control:

$$dB_t^{(v)} = v dt + \sqrt{2} dW_t,$$

but contains no jump rates and no microscopic activity.

The continuum gives the leading action and drift.

The discrete theory also gives $1/N$ corrections and the jump mechanism.

(II.2) Dynamical large deviations

Recap of models

$su(2)$

- partial exclusion process
(integrable $s=1/2$)

$su(1, 1)$

- inclusion process
 - Brownian energy process
(not integrable for any s)
-
- harmonic process
 - integrable heat conduction process
(integrable for all s)

SEP Hydrodynamic limit

- The empirical density of the symmetric exclusion process converges

$$\mu^N(t, x) = \frac{1}{N} \sum_i \eta_i(N^2 t) \delta_{\frac{i}{N}} \longrightarrow \rho(t, x) \quad \text{as } N \rightarrow \infty$$

where $\rho(t, x)$ is the solution of

$$\frac{\partial \rho}{\partial t}(t, x) = \frac{\partial^2 \rho}{\partial x^2}(t, x), \quad \rho(0, x) = \rho^{(0)}(x).$$

- More precisely: let $\rho^{(0)}$ be an initial macroscopic profile and assume that the process is started from an initial measure s.t. for smooth test function ϕ

$$\lim_{N \rightarrow \infty} \mathbb{P}_N \left(\left| \frac{1}{N} \sum_{i=1}^N \phi\left(\frac{i}{N}\right) \eta_i(0) - \int_0^1 \phi(x) \rho^{(0)}(x) dx \right| > \delta \right) = 0$$

Then

$$\lim_{N \rightarrow \infty} \mathbb{P}_N \left(\left| \frac{1}{N} \sum_{i=1}^N \phi\left(\frac{i}{N}\right) \eta_i(tN^2) - \int_0^1 \phi(x) \rho(t, x) dx \right| > \delta \right) = 0$$

SEP Large deviations (around hydrodynamic limit)

► Kipnis-Olla-Varadhan [KOV89] proved a large deviation principle:

- M_1 : set of measurable functions $\gamma : [0, 1] \rightarrow [0, 1]$
- $D([0, T], M_1)$: cadlag paths on M_1
- For $\rho \in D([0, T], M_1)$ and $G \in C^{2,1}([0, 1] \times [0, T])$ introduce the linear functional

$$\begin{aligned}\ell(\rho; G) &= \int dx G(x, T) \rho(x, T) - \int dx G(x, 0) \rho(x, 0) \\ &\quad - \int_0^T dt \int dx \rho(x, t) \left(\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} \right) G(x, t)\end{aligned}$$

Kipnis-Olla-Varadhan

- Theorem [KOV89]: LDP for the empirical density of SEP on the torus $\mathbb{T}_N$, started from a local equilibrium measure with density profile $\gamma \in M_1$:

$$\mathbb{P}[\mu^N \sim \rho] \approx e^{-N\mathcal{I}(\rho)}$$

with rate function

$$\mathcal{I}(\rho) = \mathcal{I}_0(\rho(\cdot, 0); \gamma) + \mathcal{I}_{\text{dyn}}(\rho)$$

with

$$\mathcal{I}_0(\rho(\cdot, 0); \gamma) = \int dx \left\{ \rho(x, 0) \log \frac{\rho(x, 0)}{\gamma(x)} + (1 - \rho(x, 0)) \log \frac{1 - \rho(x, 0)}{1 - \gamma(x)} \right\}$$

and

$$\mathcal{I}_{\text{dyn}}(\rho) = \sup_{G \in \mathcal{C}^{2,1}} \left\{ \ell(\rho; G) - \frac{1}{2} \int_0^T dt \int dx \rho(x, t)(1 - \rho(x, t))(\partial_x G)^2(x, t) \right\}$$

How about other models?

- ▶ MFT Hamiltonian

$$\mathcal{H}_{\text{MFT}}(\rho, H) = \frac{1}{2} \sigma(\rho) (\partial_x H)^2 - D(\rho) (\partial_x \rho) (\partial_x H)$$

- ▶ While hydrodynamic limit is proved for many models, path space LDP are more subtle. The proof in [KOV89] is rather involved and is difficult to generalize it.
- ▶ A crucial point is the so-called *super-exponential estimate* which is needed to close the equation for the density. This has been extended to zero-range models with reversible product measures having finite moment generating function [Kipnis-Landim 89].
- ▶ Another issue is the finite cost of singular profile, as recently shown by [Heydecker 26] for KMP.

Path space LDP has not been established for the processes with $\mathfrak{su}(1, 1)$ symmetry

(inclusion, BEP, KMP, harmonic, integrable heat)

(II.3) “Large spin”

large deviations

Large spin

- ▶ For a class of models with spin s , we propose to consider LDP for large s , while keeping the lattice structure.
- ▶ This avoids some subtle points of large deviations around hydrodynamic limit: no super-exponential estimate, no coarse-graining, etc.
- ▶ We could proceed with s fixed and introducing another scaling parameter, say $\epsilon \rightarrow 0$. We prefer to use spin $s \rightarrow \infty$ because is reminiscent of "classical limit" which emerges for large spins.

Partial exclusion process

- ▶ Let $N \in \mathbb{N}$ be fixed and $s \in \mathbb{N}$, consider the partial exclusion process $\{\eta(t) : t \geq 0\}$ with “spin s ” on the discrete torus $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$.
- ▶ Introduce the sequence of scaled processes $\mathbf{X}^{(s)} = \{X^{(s)}(t) : t \geq 0\}$

$$X_i^{(s)}(t) := \frac{\eta_i(st)}{s}, \quad i = 1, \dots, N$$

with state space $\Omega^s = \{0, \frac{1}{s}, \frac{2}{s}, \dots, 1\}^N$ and with generator

$$\begin{aligned} (\mathcal{L}^s f)(x) &= s \sum_{i \in \mathbb{T}_N} x_i (1 - x_{i+1}) \left(f(x - \frac{1}{s} \delta_i + \frac{1}{s} \delta_{i+1}) - f(x) \right) \\ &\quad + s \sum_{i \in \mathbb{T}_N} x_{i+1} (1 - x_i) \left(f(x + \frac{1}{s} \delta_i - \frac{1}{s} \delta_{i+1}) - f(x) \right) \end{aligned}$$

- ▶ Initialization:

$$\nu_\theta^s(x) = \bigotimes_{i \in \mathbb{T}_N} \binom{s}{sx_i} \theta_i^{sx_i} (1 - \theta_i)^{s(1-x_i)}, \quad \theta_i \in (0, 1).$$

Inclusion process

- ▶ Let $N \in \mathbb{N}$ be fixed and $s \in \mathbb{N}$, consider the inclusion process $\{\eta(t) : t \geq 0\}$ with “spin s ” on the discrete torus $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$.
- ▶ Introduce the sequence of scaled processes $\mathbf{X}^{(s)} = \{X^{(s)}(t) : t \geq 0\}$

$$X_i^{(s)}(t) := \frac{\eta_i(st)}{s}, \quad i = 1, \dots, N$$

with state space $\Omega^s = \{0, \frac{1}{s}, \frac{2}{s}, \dots\}^N$ and with generator

$$\begin{aligned} (\mathcal{L}^s f)(x) &= s \sum_{i \in \mathbb{T}_N} x_i (1 + x_{i+1}) \left(f(x - \frac{1}{s} \delta_i + \frac{1}{s} \delta_{i+1}) - f(x) \right) \\ &\quad + s \sum_{i \in \mathbb{T}_N} x_{i+1} (1 + x_i) \left(f(x + \frac{1}{s} \delta_i - \frac{1}{s} \delta_{i+1}) - f(x) \right) \end{aligned}$$

- ▶ Initialization:

$$\nu_\theta^s(x) = \bigotimes_{i \in \mathbb{T}_N} \frac{1}{(sx_i)!} \frac{\Gamma(s + sx_i)}{\Gamma(s)} \left(\frac{\theta_i}{1 + \theta_i} \right)^{sx_i} \left(\frac{1}{1 + \theta_i} \right)^s, \quad \theta_i \in (0, \infty).$$

Harmonic process

- ▶ Let $N \in \mathbb{N}$ be fixed and $s \in \mathbb{N}$, consider the harmonic process $\{\eta(t) : t \geq 0\}$ with “spin s ” on the discrete torus $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$.
- ▶ Introduce the sequence of scaled processes $\mathbf{X}^{(s)} = \{X^{(s)}(t) : t \geq 0\}$

$$X_i^{(s)}(t) := \frac{\eta_i(st)}{s}, \quad i = 1, \dots, N$$

with state space $\Omega^s = \{0, \frac{1}{s}, \frac{2}{s}, \dots\}^N$ and with generator

$$\begin{aligned} (\mathcal{L}^s f)(x) &= s \sum_{i \in \mathbb{T}_N} \sum_{k=1}^{sx_{i+i}} \varphi_s(k, sx_i) (f(x - \frac{k}{s} \delta_i + \frac{k}{s} \delta_{i+1}) - f(x)) \\ &\quad + s \sum_{i \in \mathbb{T}_N} \sum_{k=1}^{sx_{i+1}} \varphi_s(k, sx_{i+1}) (f(x + \frac{k}{s} \delta_i - \frac{k}{s} \delta_{i+1}) - f(x)) \\ \varphi_s(k, n) &= \frac{1}{k} \frac{\Gamma(n+1) \Gamma(n-k+2s)}{\Gamma(n+2s) \Gamma(n-k+1)} \end{aligned}$$

- ▶ Initialization:

$$\nu_{\theta}^s(x) = \bigotimes_{i \in \mathbb{T}_N} \frac{1}{(sx_i)!} \frac{\Gamma(s + sx_i)}{\Gamma(s)} \left(\frac{\theta_i}{1 + \theta_i} \right)^{sx_i} \left(\frac{1}{1 + \theta_i} \right)^s, \quad \theta_i \in (0, \infty).$$

Law of large numbers (Kurtz theorem)

Theorem [G, Sasamoto, (2026+)]

- ▶ Let $\mathbf{X} = \{X(t) : t \geq 0\}$ be the deterministic process taking values on $\Omega = [0, 1]^N$ (partial exclusion) or $\Omega = [0, \infty)^N$ (inclusion / harmonic) evolving as

$$\frac{dX_i(t)}{dt} = X_{i-1}(t) - 2X_i(t) + X_{i+1}(t), \quad i = 1, \dots, N$$

with $X(0) = \theta \in \Omega$.

- ▶ Consider the scaled process $\mathbf{X}^{(s)} = \{X^{(s)}(t) : t \geq 0\}$ (either partial exclusion or inclusion or harmonic) started from the initial measure ν_θ^s .

For all $\epsilon > 0$, $T > 0$

$$\lim_{s \rightarrow \infty} \mathbb{P}_\theta \left(\sup_{t \in [0, T]} |X^{(s)}(t) - X(t)| > \epsilon \right) = 0.$$

Large deviation principle

Theorem [CG, T. Sasamoto, (2026+)]

- ▶ The sequence of scaled harmonic process $(\mathbf{X}^{(s)})_{s \in \mathbb{N}}$, (either **partial exclusion** or **inclusion** or **harmonic**) started from the initial measure ν_θ^s satisfies a large deviation principle with speed s and rate function $I : D([0, T], \Omega) \rightarrow [0, \infty]$

1. (Upper bound). For every closed set $\mathcal{C} \in D([0, T], \Omega)$

$$\limsup_{s \rightarrow \infty} \frac{1}{s} \log \mathbb{P}(\mathbf{X}^s \in \mathcal{C}) \leq - \inf_{\mathbf{x} \in \mathcal{C}} I(\mathbf{x})$$

2. (Lower bound). For every open set $\mathcal{O} \in D([0, T], \Omega)$

$$\liminf_{s \rightarrow \infty} \frac{1}{s} \log \mathbb{P}(\mathbf{X}^s \in \mathcal{O}) \geq - \inf_{\mathbf{x} \in \mathcal{O}} I(\mathbf{x})$$

LDP (cont'd)

- Rate function of **partial exclusion / inclusion**

$$I(\mathbf{x}) = I_0(x(0)) + I_{\text{dyn}}(\mathbf{x}),$$

$$I_0(x) = \sum_{i \in \mathbb{T}_N} \left[x_i \log \frac{x_i}{\theta_i} \pm (1 \mp x_i) \log \frac{1 \mp x_i}{1 \mp \theta_i} \right]$$

$$I_{\text{dyn}}(\mathbf{x}) = \sup_{h \in \mathcal{C}^1} \left[h(T) \cdot x(T) - h(0) \cdot x(0) - \int_0^T \left(\dot{h}(t) \cdot x(t) - \mathcal{H}(x(t), h(t)) \right) dt \right]$$

with Hamiltonian

$$\mathcal{H}(x, h) = \sum_{i \in \mathbb{T}_N} x_i(1 \mp x_{i+1})(e^{h_{i+1} - h_i} - 1) + (1 \mp x_i)x_{i+1}(e^{-(h_{i+1} - h_i)} - 1)$$

LDP (cont'd)

► Rate function of harmonic

$$I(\mathbf{x}) = I_0(x(0)) + I_{\text{dyn}}(\mathbf{x}),$$

$$I_0(x) = \sum_{i \in \mathbb{T}_N} \left[x_i \log \frac{x_i}{\theta_i} - (1 + x_i) \log \frac{1 + x_i}{1 + \theta_i} \right]$$

$$I_{\text{dyn}}(\mathbf{x}) = \sup_{h \in C^1} \left[h(T) \cdot x(T) - h(0) \cdot x(0) - \int_0^T \left(\dot{h}(t) \cdot x(t) - \mathcal{H}(x(t), h(t)) \right) dt \right]$$

with Hamiltonian

$$\begin{aligned} & \mathcal{H}(x, h) \\ &= \sum_{i \in \mathbb{T}_N} -\log \left(1 + x_i(1 + x_{i+1})(e^{h_{i+1} - h_i} - 1) + (1 + x_i)x_{i+1}(e^{-(h_{i+1} - h_i)} - 1) \right) \end{aligned}$$

Classical spin chains

- The harmonic Hamiltonian

$$\mathcal{H}(x, h) = - \sum_{i \in \mathbb{T}_N} \log \left[1 + x_i(1+x_{i+1})(e^{-(h_i-h_{i+1})}-1) + (1+x_i)x_{i+1}(e^{-(h_{i+1}-h_i)}-1) \right]$$

can be written as Ishimori chain with classical $\mathfrak{su}(1, 1)$ spins

$$\mathcal{H} = - \sum_i \log \left(\frac{1}{4} + \mathcal{S}_i \cdot \mathcal{S}_{i+1} \right)$$

$$\mathcal{S}_i^+ = (1+x_i)e^{h_i} \quad \mathcal{S}_i^- = x_i e^{-h_i} \quad \mathcal{S}_i^z = x_i + \frac{1}{2}$$

$$\{\mathcal{S}_i^+, \mathcal{S}_i^-\} = -2\mathcal{S}_i^z \quad \{\mathcal{S}_i^z, \mathcal{S}_i^+\} = \mathcal{S}_i^+ \quad \{\mathcal{S}_i^z, \mathcal{S}_i^-\} = -\mathcal{S}_i^-$$

- [Haldane '82]: Ishimori chain (integrable classical spin chain) arises as the classical limit of the integrable quantum XXX_s spin chain as $s \rightarrow \infty$
(thanks to M. Koulkarni):

$$\psi(x) \sim \log(x) \quad \text{as } x \rightarrow \infty$$

(II.4) LDP Proof

We prove the results for the partial exclusion process.

- The same strategy can be adapted to the other processes.

The key steps are

- LLN for the tilted process (by Grönwall's inequality and Dinkyn martingale).
- LD upper bound (by Markov inequality and exponential martingale).
- LD lower bound (by Jensen inequality and relative entropy).

LLN for the tilted process

Tilted process

- Define the tilted process $\mathbf{X}^{s,h} = \{X^{s,h}(t) : t \geq 0\}$, taking values in Ω^s , with generator

$$\begin{aligned} L^{s,h}f(x) &= s \sum_{i \in \mathbb{T}_N} x_i(1 - x_{i+1}) e^{h_{i+1}(t) - h_i(t)} (f(x - \delta_i + \delta_{i+1}) - f(x)) \\ &+ s \sum_{i \in \mathbb{T}_N} x_{i+1}(1 - x_i) e^{h_i(t) - h_{i+1}(t)} (f(x + \delta_i - \delta_{i+1}) - f(x)). \end{aligned}$$

Here $h : \mathbb{R} \rightarrow \mathbb{R}^N$ is a continuous function which is used to tune the tilting of the path space measure.

LLN for the tilted process

- ▶ Let $\mathbf{X}^h = \{X^h(t) : t \geq 0\}$ the solution of the equations

$$\frac{dX^h(t)}{dt} = F^h(X(t))$$

with $F^h : [0, 1]^N \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} F_i^h(x) &= x_{i-1}(1 - x_i)e^{h_i(t) - h_{i-1}(t)} - x_i(1 - x_{i-1})e^{h_{i-1}(t) - h_i(t)} \\ &\quad - x_i(1 - x_{i+1})e^{h_{i+1}(t) - h_i(t)} + x_{i+1}(1 - x_i)e^{h_i(t) - h_{i+1}(t)}. \end{aligned}$$

- ▶ If $X^h(0) = \theta \in (0, 1)^N$ and $\mathbf{X}^h$ is started from ν^s , then for $T > 0$ and $\epsilon > 0$ we have

$$\lim_{s \rightarrow \infty} \mathbb{P}_\theta \left(\sup_{t \in [0, T]} |X^{s,h}(t) - X^h(t)| > \epsilon \right) = 0.$$

- ▶ Rem.: If $h = 0$ then the discrete Laplacian is recovered.

Proof of LLN for the tilted process

- Defining $f_i(x) = x_i$ for $i \in \mathbb{T}_N$, we have that the vector $\mathbf{M}^{s,h}$ with components

$$M_i^{s,h}(t) = X_i^{s,h}(t) - X_i^{s,h}(0) - \int_0^t L^{s,h} f_i(X^{s,h}(u)) du$$

is a martingale. Therefore we can write

$$X_i^{s,h}(t) = X_i^{s,h}(0) + \int_0^t L^{s,h} f_i(X^{s,h}(u)) du + M_i^{s,h}(t).$$

Similarly, for the flow $\mathbf{X}^h$ we can write

$$X_i^h(t) = X_i^h(0) + \int_0^t L^h f_i(X^h(u)) du.$$

From the definition of the generators $L^{s,h}$ and L^h we have that

$$\begin{aligned} L^{s,h} f_i(x) &= F_i^h(x), \\ L^h f_i(x) &= F_i^h(x). \end{aligned}$$

Triangular inequality implies

$$|X^{s,h}(t) - X^h(t)| \leq |X^{s,h}(0) - X^h(0)| + \int_0^t |F^h(X^{s,h}(u)) - F^h(X^h(u))| du + |M^{s,h}(t)|.$$

Proof of LLN for the tilted process (cont'd)

- $F^h : \Omega \rightarrow \mathbb{R}^N$ is a **Lipschitz** function, i.e. for any time $t \in [0, T]$ and for any $x, y \in \Omega$ there exists a positive real constant $K(t)$ s.t.

$$|F^{h(t)}(x) - F^{h(t)}(y)| \leq K(t)|x - y|.$$

As a consequence, defining $K = \sup_{t \in [0, T]} K(t)$,

$$|X^{s,h}(t) - X^h(t)| \leq |X^{s,h}(0) - X^h(0)| + K \int_0^t |X^{s,h}(u) - X^s(u)| du + |M^{s,h}(t)|.$$

- **Grönwall's** inequality: if one has a bounded function $y : [0, T] \rightarrow \mathbb{R}$ such that for all $t \in [0, T]$

$$y(t) \leq A + B \int_0^t y(u) du$$

where A and B are some positive constants (which might depend on T), then for all $t \in [0, T]$

$$y(t) \leq Ae^{BT}.$$

Therefore, if $y(t) > Ae^{BT}$ for some $t \in [0, T]$, then $y(t) > A + B \int_0^t y(u) du$.

Proof of LLN for the tilted process (cont'd)

- Applying Grönwall's inequality to

$$Y(t) = |X^{s,h}(t) - X^h(t)|$$

with $A = \epsilon e^{-KT}$ and $B = K$ we obtain

$$\begin{aligned} & \mathbb{P}\left(\sup_{t \in [0, T]} |X^{s,h}(t) - X^h(t)| > \epsilon\right) \\ & \leq \mathbb{P}\left(\sup_{t \in [0, T]} |X^{s,h}(t) - X^h(t)| - K \int_0^t |X^{s,h}(u) - X^h(u)| du > \epsilon e^{-KT}\right). \end{aligned}$$

- Using the Lipschitz bound we arrive to

$$\begin{aligned} & \mathbb{P}\left(\sup_{t \in [0, T]} |X^{s,h}(t) - X^h(t)| > \epsilon\right) \\ & \leq \mathbb{P}(|X^{s,h}(0) - X^h(0)| + \sup_{t \in [0, T]} |M^{s,h}(t)| > \epsilon e^{-KT}). \end{aligned}$$

Since $\lim_{s \rightarrow \infty} \mathbb{P}(|X^{s,h}(0) - X^h(0)| > \epsilon) = 0$, to prove LLN it is enough to show that

$$\lim_{s \rightarrow \infty} \mathbb{P}\left(\sup_{t \in [0, T]} |M^{s,h}(t)| > \epsilon e^{-KT}\right) = 0.$$

Proof of LLN for the tilted process (cont'd)

- We compute the quadratic covariation defined as

$$\langle M_j^{s,h}, M_k^{s,h} \rangle(t) = \int_0^t \Gamma(f_j, f_k)(X^{s,h}(u)) du$$

where

$$\begin{aligned} \Gamma(f_j, f_k)(x) &= s \sum_{i \in \mathbb{T}_N} x_i (1 - x_{i+1}) e^{h_{i+1}(t) - h_i(t)} \left(f_j(x^{i,i+1}) - f_j(x) \right) \left(f_k(x^{i,i+1}) - f_k(x) \right) \\ &+ s \sum_{i \in \mathbb{T}_N} x_{i+1} (1 - x_i) e^{h_i(t) - h_{i+1}(t)} \left(f_j(x^{i+1,i}) - f_j(x) \right) \left(f_k(x^{i+1,i}) - f_k(x) \right). \end{aligned}$$

Since this is clearly $O(\frac{1}{s})$, we get $\lim_{s \rightarrow \infty} \langle M_i^{s,h}, M_j^{s,h} \rangle(t) = 0$.

- Finally, applying Doob's inequality and Ito's isometry we conclude that

$$\begin{aligned} \lim_{s \rightarrow \infty} \mathbb{P} \left(\sup_{t \in [0, T]} |M_i^{s,h}(t)| > \epsilon e^{-KT} \right) &\leq \lim_{s \rightarrow \infty} \frac{\mathbb{E}[(M_i^{s,h}(T))^2]}{\epsilon^2 e^{-2KT}} \\ &= \lim_{s \rightarrow \infty} \frac{\mathbb{E}[\langle M_i^{s,h}, M_i^{s,h} \rangle(T)]}{\epsilon^2 e^{-2KT}} = 0. \end{aligned}$$

LDP Upper bound

Proof of Upper bound

- Given $f : \mathbb{R}_+ \times \Omega^S \rightarrow \mathbb{R}$, the following is a martingale:

$$M^S(t) = \frac{f(t, X(t))}{f(t, X(0))} \exp \left\{ - \int_0^t \frac{(\partial_u + L^S)f(u, X(u))}{f(u, X(u))} du \right\}.$$

- Choosing $f(t, x) = \exp \left\{ s \sum_{i \in \mathbb{T}_N} h_i(t) x_i \right\}$, for any vector function h in $C^1([0, T])$, we get the **exponential martingale**

$$\mathbf{Z}^{s,h} = \{Z^{s,h}(t) : 0 \leq t \leq T\} = \exp \{s l_{\text{dyn}}^h(\mathbf{X}^S)\}$$

where

$$l_{\text{dyn}}^h(\mathbf{x}) = h(t) \cdot x(t) - h(0) \cdot x(0) - \int_0^t \left[\dot{h}(u) \cdot x(u) - \mathcal{H}(x(u), h(u)) \right] du$$

and

$$\mathcal{H}(x, h) = \sum_{i \in \mathbb{T}_N} \left(x_i(1 - x_{i+1})(e^{h_{i+1} - h_i} - 1) + (1 - x_i)x_{i+1}(e^{-(h_{i+1} - h_i)} - 1) \right)$$

Proof of Upper bound (cont'd)

- We now use Lemma 3.3 in Appendix 2 of [KL89], which allows to obtain LD upper bound for compact sets $\mathcal{K}$ from the LD upper bound for open sets $\mathcal{O}$ by replacing $\inf \sup$ with $\sup \inf$. Therefore we get

$$\begin{aligned} \limsup_{s \rightarrow \infty} \frac{1}{s} \log \mathbb{P}(\mathbf{X}^s \in \mathcal{K}) &\leq \sup_{\mathbf{x} \in \mathcal{K}} \inf_{\substack{h \in C^1 \\ a \in \mathbb{R}^N}} \left[-l_{\text{dyn}}^h(\mathbf{x}) - a \cdot x(0) + \sum_{i \in \mathbb{T}_N} \log(1 - \theta_i + \theta_i e^{a_i}) \right] \\ &= - \inf_{\mathbf{x} \in \mathcal{K}} \sup_{\substack{h \in C^1 \\ a \in \mathbb{R}^N}} \left[l_{\text{dyn}}^h(\mathbf{x}) + a \cdot x(0) - \sum_{i \in \mathbb{T}_N} \log(1 - \theta_i + \theta_i e^{a_i}) \right]. \end{aligned}$$

- The supremum separates into the dynamical and static parts:

$$\begin{aligned} \sup_h l_{\text{dyn}}^h(\mathbf{x}) &= l_{\text{dyn}}(\mathbf{x}). \\ \sup_{a \in \mathbb{R}^N} \left\{ a \cdot x(0) - \sum_{i \in \mathbb{T}_N} \log(1 - \theta_i + \theta_i e^{a_i}) \right\} &= l_0(x(0)). \end{aligned}$$

Consequently,

$$\limsup_{s \rightarrow \infty} \frac{1}{s} \log \mathbb{P}(\mathbf{X}^s \in \mathcal{K}) \leq - \inf_{\mathbf{x} \in \mathcal{K}} [l_{\text{dyn}}(\mathbf{x}) + l_0(x(0))] = - \inf_{\mathbf{x} \in \mathcal{K}} l(\mathbf{x}).$$

Proof of Upper bound (cont'd)

- To extend the large deviation upper bound from compact sets to closed sets we show **exponential tightness**: for any $n \in \mathbb{N}$ there exists a compact set $\mathcal{K}_n \subset D([0, T], \Omega)$ such that

$$\limsup_{s \rightarrow \infty} \frac{1}{s} \ln \mathbb{P}(\mathbf{X}^s \notin \mathcal{K}_n) \leq -n$$

- We adapt the approach in [KL89]. Namely, we proceed in two steps: in the first step it is proved that for all vectors $h \in \mathbb{R}^N$ and for all $\epsilon > 0$

$$\lim_{\delta \rightarrow 0} \limsup_{s \rightarrow \infty} \frac{1}{s} \ln \mathbb{P}\left(\sup_{|t-t'| \leq \delta} |X^s(t) \cdot h - X^s(t') \cdot h| > \epsilon\right) = -\infty.$$

In the second step, the compact set $\mathcal{K}_n$ is constructed explicitly.

- All in all, we get that for every closed set $\mathcal{C} \in D([0, T], \Omega)$

$$\limsup_{s \rightarrow \infty} \frac{1}{s} \log \mathbb{P}(\mathbf{X}^s \in \mathcal{C}) \leq -\inf_{\mathbf{x} \in \mathcal{C}} \{I(\mathbf{x})\}$$

LDP lower bound

Proof of lower bound

- ▶ To prove the lower bound we first provide an alternative representation of the rate function in terms of **relative entropy**.
- ▶ The rate function $I : D([0, T], \Omega) \rightarrow [0, \infty]$ can be written as

$$I(\mathbf{x}) = \begin{cases} \inf_{h \in A(\mathbf{x})} \left\{ \liminf_{s \rightarrow \infty} \frac{1}{s} \mathcal{H}(\mathbb{P}^h \parallel \mathbb{P}) \right\} & \text{if } A(\mathbf{x}) \neq \emptyset \\ +\infty & \text{otherwise.} \end{cases}$$

where $\mathcal{H}(\mathbb{P}^h \parallel \mathbb{P})$ is the relative entropy between the two path space measures and

$$A(\mathbf{x}) = \{h \in C^0 \mid \dot{\mathbf{x}} = F^h(\mathbf{x})\}.$$

Proof of lower bound (cont'd)

► Relative entropy:

$$\mathcal{H}(\mathbb{P}^h \parallel \mathbb{P}) = \mathbb{E}^h \left[\log \frac{d\mathbb{P}^h}{d\mathbb{P}} \right].$$

We compute the Radon-Nikodym derivative using Girsanov formula. Denoting

$$c_{i,j}(x) = x_i(1 - x_j) \qquad c_{i,j}^h(x) = x_i(1 - x_j)e^{h_j(t) - h_i(t)}$$

we have

$$\begin{aligned} & \log \left(\frac{d\mathbb{P}^h}{d\mathbb{P}} \right) \\ &= \sum_{i \in \mathbb{T}_N} \left\{ \int_0^T \log \left(\frac{c_{i,i+1}^h(X(t-))}{c_{i,i+1}(X(t-))} \right) dJ_{i,i+1}(t) + \int_0^T s \left(c_{i,i+1}(X(t-)) - c_{i,i+1}^h(X(t-)) \right) dt \right. \\ & \quad \left. + \int_0^T \log \left(\frac{c_{i+1,i}^h(X(t-))}{c_{i+1,i}(X(t-))} \right) dJ_{i+1,i}(t) + \int_0^T s \left(c_{i+1,i}(X(t-)) - c_{i+1,i}^h(X(t-)) \right) dt \right\}. \end{aligned}$$

Here $X(t-)$ denotes the left limit and $J_{i,j}(t)$ is the number of jumps from site i to site j up to time t .

Proof of lower bound (cont'd)

- Taking expectation under $\mathbb{P}^h$ and using

$$\mathbb{E}^h[dJ_{i,j}(t) \mid \mathcal{F}(t-)] = s c_{i,j}^h(X(t-))dt$$

we get

$$\begin{aligned} & \mathcal{H}(\mathbb{P}^h \parallel \mathbb{P}) \\ &= \sum_{i \in \mathbb{T}_N} \left\{ \int_0^T \left(c_{i,i+1}^h(X(t-)) \log \left(\frac{c_{i,i+1}^h(X(t-))}{c_{i,i+1}(X(t-))} \right) + c_{i,i+1}(X(t-)) - c_{i,i+1}^h(X(t-)) \right) dt \right. \\ & \quad \left. + \int_0^T \left(c_{i+1,i}^h(X(t-)) \log \left(\frac{c_{i+1,i}^h(X(t-))}{c_{i+1,i}(X(t-))} \right) + c_{i+1,i}(X(t-)) - c_{i+1,i}^h(X(t-)) \right) dt \right\}. \end{aligned}$$

Proof of lower bound (cont'd)

- The expression of the rate function in terms of relative entropy follows from

$$\sup_h \mathcal{I}^h(x) = \inf_{h: \dot{x} = F^h(x)} \mathcal{J}^h(x)$$

where

$$\mathcal{I}^h(x) = \sum_{i \in \mathbb{T}_N} \left[h_i \dot{x}_i - x_i(1 - x_{i+1})(e^{h_{i+1} - h_i} - 1) - x_{i+1}(1 - x_i)(e^{h_i - h_{i+1}} - 1) \right]$$

and

$$\begin{aligned} \mathcal{J}^h(x) = \sum_{i \in \mathbb{T}_N} & \left\{ c_{i,i+1}^h(x) \log \left(\frac{c_{i,i+1}^h(x)}{c_{i,i+1}(x)} \right) + \left(c_{i,i+1}(x) - c_{i,i+1}^h(x) \right) \right. \\ & \left. + c_{i+1,i}^h(x) \log \left(\frac{c_{i+1,i}^h(x)}{c_{i+1,i}(x)} \right) + \left(c_{i+1,i}(x) - c_{i+1,i}^h(x) \right) \right\}. \end{aligned}$$

- Generalization of the simple identity:

$$\sup_x [ax - \lambda(e^x - 1)] = a \log\left(\frac{a}{\lambda}\right) - a + \lambda$$

Proof of lower bound (cont'd)

- Consider the tilted process $\mathbf{X}^{s,h}$ with generator $L^{s,h}$ and denote $\mathbb{P}^h$ the corresponding probability measure. We have

$$\begin{aligned}\mathbb{P}(\mathbf{X}^s \in B(\mathbf{x}, \delta)) &= \mathbb{E}[\mathbf{1}_{\{\mathbf{X}^s \in B(\mathbf{x}, \delta)\}}] \\ &= \mathbb{E}^h[\mathbf{1}_{\{\mathbf{X}^{s,h} \in B(\mathbf{x}, \delta)\}} \frac{d\mathbb{P}}{d\mathbb{P}^h}],\end{aligned}$$

where $\mathbb{E}^h$ denotes expectation with respect to $\mathbb{P}^h$.

- Applying Jensen inequality, one gets

$$\log(\mathbb{P}(\mathbf{X}^s \in B(\mathbf{x}, \delta))) \geq \log(\mathbb{P}^h(\mathbf{X}^{s,h} \in B(\mathbf{x}, \delta))) + \frac{\mathbb{E}^h[\mathbf{1}_{\{\mathbf{X}^{s,h} \in B(\mathbf{x}, \delta)\}} \ln(\frac{d\mathbb{P}}{d\mathbb{P}^h})]}{\mathbb{P}^h(\mathbf{X}^{s,h} \in B(\mathbf{x}, \delta))}.$$

Proof of lower bound (cont'd)

- Now, for a given path $\mathbf{x} \in D([0, T], \Omega)$, choose $h = \{h(t) : t \geq 0\}$ so that the typical value of $\mathbf{X}^{s,h}$ process is $\mathbf{x}$. In view of LLN for the tilted process this amounts to requiring that

$$\dot{\mathbf{x}} = F^h(\mathbf{x})$$

- If

$$\lim_{s \rightarrow \infty} \mathbb{P}^h(\mathbf{X}^{s,h} \in B(\mathbf{x}, \delta)) = 1 \quad \text{for all } \delta > 0$$

then we get

$$\liminf_{s \rightarrow \infty} \frac{1}{s} \log (\mathbb{P}(\mathbf{X}^s \in B(\mathbf{x}, \delta))) \geq \liminf_{s \rightarrow \infty} \frac{1}{s} \mathbb{E}^h \left[\ln \left(\frac{d\mathbb{P}}{d\mathbb{P}^h} \right) \right].$$

- This holds for every $\mathbb{P}^h$ provided the constraint $\dot{\mathbf{x}} = F^h(\mathbf{x})$ is satisfied. To get the best lower bound we optimize:

$$\begin{aligned} \liminf_{s \rightarrow \infty} \frac{1}{s} \log (\mathbb{P}(\mathbf{X}^s \in B(\mathbf{x}, \delta))) &\geq \sup_{h: \dot{\mathbf{x}}=F^h(\mathbf{x})} \left\{ \liminf_{s \rightarrow \infty} \frac{1}{s} \mathbb{E}^h \left[\ln \left(\frac{d\mathbb{P}}{d\mathbb{P}^h} \right) \right] \right\} \\ &= - \inf_{h: \dot{\mathbf{x}}=F^h(\mathbf{x})} \left\{ \liminf_{s \rightarrow \infty} \frac{1}{s} \mathcal{H}(\mathbb{P}^h \parallel \mathbb{P}) \right\} \\ &= -I(\mathbf{x}). \end{aligned}$$

Harmonic process: LLN for the tilted process

The forward and backward packet clocks on $(i, i + 1)$ have tilted intensities

$$s\varphi_{s/2}(k, sx_i)e^{k(h_{i+1}-h_i)}, \quad s\varphi_{s/2}(k, sx_{i+1})e^{-k(h_{i+1}-h_i)}$$

For $s \rightarrow \infty$,

$$\varphi_{s/2}(k, sx_i) \longrightarrow c_k(x_i) := \frac{1}{k} \left(\frac{x_i}{1+x_i} \right)^k.$$

If the response field stays strictly inside

$$\frac{x_i}{1+x_i}e^{\alpha_i} < 1, \quad \frac{x_{i+1}}{1+x_{i+1}}e^{-\alpha_i} < 1, \quad (\star)$$

then the limiting drift and current are

$$\begin{aligned} \dot{x}_i &= F_i^h(x) = J_{i-1}^h(x) - J_i^h(x) \\ J_i^h(x) &= \frac{x_i e^{\alpha_i}}{1+x_i-x_i e^{\alpha_i}} - \frac{x_{i+1} e^{-\alpha_i}}{1+x_{i+1}-x_{i+1} e^{-\alpha_i}} \end{aligned}$$

On a fixed-mass set, condition $(\star)$ gives

$$\varphi_{s/2}(k, sx_i)e^{\pm k(h_{i+1}-h_i)} \leq \frac{C\rho^k}{k} \quad (\rho < 1), \quad \mathbb{E}\langle M_i^{s,h} \rangle_T \leq \frac{C_T}{s} \sum_{k \geq 1} k\rho^k = O\left(\frac{1}{s}\right).$$

Mass conservation gives compact containment. The bound above gives uniform convergence, local Lipschitz continuity of F^h , and the tilted LLN by the same Doob–Grönwall argument.

Harmonic process: LDP upper bound

- For $f(t, x) = e^{sh(t) \cdot x}$,

$$Z_T^{s,h} = \exp \left\{ s \left[h(T) \cdot X^s(T) - h(0) \cdot X^s(0) - \int_0^T \left(\dot{h} \cdot X^s + \mathcal{H}_s(X^s, h) \right) dt \right] \right\}.$$

The geometric majorant permits passage to the limit in the full packet sum:

$$\mathcal{H}_{\text{harm}}(x, h) = - \sum_j \log([1 - x_i(e^{\alpha_j} - 1)][1 - x_{i+1}(e^{-\alpha_j} - 1)])$$

- For the negative-binomial initial law,

$$\Lambda_0(a) := \lim_{s \rightarrow \infty} \frac{1}{s} \log \mathbb{E} e^{sa \cdot X^s(0)} = - \sum_i \log[1 - \theta_i(e^{a_i} - 1)],$$

for $a_i < \log(1 + 1/\theta_i)$, and $l_0 = \Lambda_0^*$. The martingale and minimax proof for compact sets is unchanged.

- For closed sets, first use the initial exponential moments and conservation:

$$\lim_{A \rightarrow \infty} \limsup_{s \rightarrow \infty} \frac{1}{s} \log \mathbb{P} \left(\sum_i X_i^s(0) > A \right) = -\infty,$$

On $\{\sum_j X_j^s \leq A\}$, the geometric tail also gives

$$s \sum_{k \geq \varepsilon s} \varphi_{s/2}(k, \mathbf{s}x_i) \leq e^{-sC_{A,\varepsilon}}, \quad \mathbb{P}(N_\varepsilon^s(T) > m) \leq e^{-smC_{A,\varepsilon} + o(s)},$$

where N_ε^s counts packets $k \geq \varepsilon s$. This controls the number of macroscopic jumps; the small-jump martingale gives the KOV modulus estimate. Hence exponential tightness holds in $D([0, T], \Omega)$.

Harmonic process: LDP lower bound

- Tilt every packet clock, not only the total bond current:

$$c_{i,k}^{s,h,+} = \varphi_{s/2}(k, sx_i) e^{k(h_{i+1}-h_i)}, \quad c_{i,k}^{s,h,-} = \varphi_{s/2}(k, sx_i) e^{-k(h_{i+1}-h_i)}.$$

With $d(u \mid v) = u \log(u/v) - u + v$, the dynamical entropy is

$$\frac{1}{s} \mathcal{H}_{\text{dyn}}(\mathbb{P}^{s,h} \mid \mathbb{P}^s) = \mathbb{E}^{s,h} \int_0^T dt \sum_{i,k} [d(c_{i,k}^{s,h,+} \mid c_{i,k}^s) + d(c_{i,k}^{s,h,-} \mid c_{i+1,k}^s)].$$

- For a regular path x and a field h strictly inside $(\star)$ such that

$$\dot{x} = F^h(x) = \partial_h H(x, h),$$

the tilted LLN and dominated convergence yield

$$\lim_{s \rightarrow \infty} \frac{1}{s} \mathcal{H}_{\text{dyn}}(\mathbb{P}^{s,h} \mid \mathbb{P}^s) = \int_0^T [h \cdot \dot{x} - \mathcal{H}_{\text{harm}}(x, h)] dt.$$

The initial negative-binomial tilt adds $l_0(x(0))$. Jensen's inequality gives

$$\lim_{\delta \downarrow 0} \liminf_{s \rightarrow \infty} \frac{1}{s} \log \mathbb{P}(X^s \in B(x, \delta)) \geq -l(x).$$

(II.5) Comments on dMFT

(i) Two successive limits

- ▶ At fixed lattice size N , the large-spin limit $s \rightarrow \infty$ gives the discrete-MFT path LDP

$$\mathbb{P}[(X^{(s)}, J^{(s)}) \simeq (x, j)] \asymp \exp\{-sI_N(x, j)\}.$$

Its Hamiltonian keeps the full exponential structure of the bond currents.

- ▶ We then let $N \rightarrow \infty$ with diffusive scaling

$$u = \frac{i}{N}, \quad \tau = N^2 t, \quad x_i(N^2 t) \simeq \rho(t, u).$$

For smooth paths, I_N is of order N and

$$\frac{1}{N} I_N \longrightarrow \mathcal{I}_{\text{MFT}}.$$

Thus the combined exponential speed is sN .

- ▶ The calculation below identifies the continuum functional. It is a **formal continuum limit on smooth trajectories**, not by itself a proof of the full hydrodynamic path-space LDP.

A general conservative bond Hamiltonian

- Write the discrete-MFT Hamiltonian as

$$\mathcal{H}_N(x, h) = \sum_{i \in \mathbb{T}_N} \Psi(x_i, x_{i+1}, h_{i+1} - h_i), \quad \Psi(x, y, 0) = 0.$$

- The variable $z = h_{i+1} - h_i$ counts the oriented current from i to $i + 1$.
Consequently

$$J(x, y) := \partial_z \Psi(x, y, 0), \quad K(x, y) := \partial_z^2 \Psi(x, y, 0).$$

are the first two cumulants mean of the oriented bond current.

- Assume, for $y = r + \delta$,

$$J(r, r + \delta) = -D(r) \delta + o(|\delta|), \quad K(r, r) = \sigma(r).$$

Only these two local current cumulants survive the diffusive limit.

Diffusive limit of the Hamiltonian

- Set $u_i = i/N$, $x_i = \rho(u_i)$ and $h_i = H(u_i)$. Then

$$x_{i+1} - x_i = \frac{1}{N} \partial_u \rho + O(N^{-2}), \quad h_{i+1} - h_i = \frac{1}{N} \partial_u H + O(N^{-2}).$$

- Taylor expansion of one bond gives

$$\begin{aligned} \Psi(x_i, x_{i+1}, h_{i+1} - h_i) &= J(x_i, x_i + 1)(h_{i+1} - h_i) + \frac{1}{2} K(x_i, x_{i+1})(h_{i+1} - h_i)^2 + O((h_{i+1} - h_i)^3) \\ &= \frac{1}{N^2} \left[-D(\rho) \partial_u \rho \partial_u H + \frac{1}{2} \sigma(\rho) (\partial_u H)^2 \right] + O(N^{-3}). \end{aligned}$$

The diffusive time acceleration contributes N^2 , while the spatial large-deviation speed contributes N . Hence the Hamiltonian per unit macroscopic length is $N\mathcal{H}_N$, and

$$\lim_{N \rightarrow \infty} N\mathcal{H}_N(\rho, H) = \int_0^1 du \left[-D(\rho) \partial_u \rho \partial_u H + \frac{1}{2} \sigma(\rho) (\partial_u H)^2 \right].$$

This is the MFT Hamiltonian.

(ii) Different discrete theories can have the same MFT limit

For $x, y \geq 0$, let

$$a = x(1 + y), \quad b = y(1 + x), \quad \Phi_{x,y}(z) = a(e^z - 1) + b(e^{-z} - 1).$$

► Inclusion process:

$$\Psi_{\text{SIP}}(x, y, z) = \Phi_{x,y}(z).$$

► Harmonic process:

$$\Psi_{\text{Harm}}(x, y, z) = -\log(1 - \Phi_{x,y}(z)).$$

Although the two functions are different,

$$\partial_z \Psi(r, r, 0) = 0, \quad \partial_z^2 \Psi(r, r, 0) = 2r(1 + r),$$

and, for $y = x + \delta$,

$$\partial_z \Psi(x, y, 0) = -\delta + o(|\delta|).$$

Therefore both give

$$D(\rho) = 1, \quad \sigma(\rho) = 2\rho(1 + \rho).$$

(iii) Discrete joint large deviations of density and current

- Let $j_i(t)$ be the oriented current through the bond $(i, i+1)$. Then

$$\dot{x}_i(t) = j_{i-1}(t) - j_i(t).$$

The joint discrete-MFT action is

$$I_N(x, j) = I_{0,N}(x(0)) + \int_0^T dt \sum_{i \in \mathbb{T}_N} \ell(x_i, x_{i+1}, j_i),$$

where

$$\ell(x, y, j) = \sup_{z \in \mathbb{R}} \{zj - \Psi(x, y, z)\}.$$

- **Relative entropy interpretation:** for independent forward and backward Poisson currents with intensities $a(x, y)$ and $b(x, y)$,

$$\ell(x, y, j) = \inf_{u^+ - u^- = j} \{ \varphi(u^+ | a) + \varphi(u^- | b) \},$$

$$\varphi(u | c) = u \log \frac{u}{c} - u + c.$$

Introduce a multiplier z for $u^+ - u^- = j$, stationarity gives $u_*^+ = ae^z$, $u_*^- = be^{-z}$. Then $\ell(x, y, j) = \sup_z \{zj - a(e^z - 1) - b(e^{-z} - 1)\}$

The MFT fundamental formula from the discrete action

- Under diffusive scaling, impose a macroscopic current $j(t, u)$ through

$$j_i^{\text{disc}}(N^2 t) = \frac{1}{N} j(t, u_i).$$

Since

$$J(x_i, x_{i+1}) = -\frac{1}{N} D(\rho) \partial_u \rho + o(N^{-1}), \quad K(x_i, x_{i+1}) = \sigma(\rho) + o(1),$$

the local Legendre transform satisfies

$$\ell\left(x_i, x_{i+1}, \frac{j}{N}\right) = \frac{1}{2N^2} \frac{[j + D(\rho) \partial_u \rho]^2}{\sigma(\rho)} + o(N^{-2}).$$

- Therefore

$$\boxed{\frac{1}{N} I_N \longrightarrow \mathcal{F}(\rho_0) + \frac{1}{2} \int_0^T dt \int_0^1 du \frac{[j + D(\rho) \partial_u \rho]^2}{\sigma(\rho)}}$$

subject to

$$\partial_t \rho + \partial_u j = 0.$$

This is the fundamental density-current formula of MFT.

(iv) Fluctuation relation in discrete MFT

- For a bond with forward and backward intensities a and b , define the affinity

$$\mathcal{A}(x, y) = \log \frac{a(x, y)}{b(x, y)}.$$

The bond Hamiltonian satisfies the Gallavotti–Cohen symmetry

$$\Psi(x, y, z) = \Psi(x, y, -\mathcal{A}(x, y) - z).$$

Consequently,

$$\ell(x, y, j) - \ell(x, y, -j) = -\mathcal{A}(x, y)j.$$

- Summing over bonds, if

$$x_i^\dagger(t) = x_i(T - t), \quad j_i^\dagger(t) = -j_i(T - t),$$

then

$$I_{N,\text{dyn}}(x, j) - I_{N,\text{dyn}}(x^\dagger, j^\dagger) = - \int_0^T dt \sum_i \mathcal{A}(x_i, x_{i+1}) j_i.$$

The right-hand side is minus the entropy flux.

Fluctuation relation in MFT

- Assume local detailed balance and write, on the torus,

$$\mathcal{A}_i^N = f'(x_i) - f'(x_{i+1}) + \frac{1}{N}\mathcal{E}(u_i) + o(N^{-1}), \quad f''(\rho) = \frac{2D(\rho)}{\sigma(\rho)}.$$

Then

$$N\mathcal{A}_i^N \longrightarrow -\partial_u f'(\rho) + \mathcal{E},$$

and the typical macroscopic current is

$$\bar{j}(\rho) = -D(\rho)\partial_u \rho + \frac{1}{2}\sigma(\rho)\mathcal{E}.$$

- For

$$\mathcal{L}(\rho, j) = \frac{[j - \bar{j}(\rho)]^2}{2\sigma(\rho)},$$

one obtains

$$\mathcal{L}(\rho, j) - \mathcal{L}(\rho, -j) = j\partial_u f'(\rho) - \mathcal{E}j.$$

After integration,

$$\mathcal{I}_{\text{dyn}}(\rho, j) - \mathcal{I}_{\text{dyn}}(\rho^\dagger, j^\dagger) = \mathcal{F}(\rho_T) - \mathcal{F}(\rho_0) - \int_0^T dt \int_0^1 du \mathcal{E}j.$$

(vi) MFT, spin chains, integrability

► Universality

$$\mathcal{H}_{MFT}(\rho, H) = \int dx \left[-D(\rho) \partial_x \rho \partial_x H + \frac{1}{2} \sigma(\rho) (\partial_x H)^2 \right]$$

$$\{\rho(x), H(y)\} = \delta(x - y)$$

► Integrability

$$S^+ = e^H \quad S^- = \sigma(\rho) e^{-H} \quad S^z = \frac{1}{2} \sigma'(\rho)$$

$$\{S^+, S^-\} = -2S^z \quad \{S^z, S^\pm\} = \pm \frac{\sigma''(\rho)}{2} S^\pm$$

$D(\rho)$ constant, $\sigma(\rho)$ quadratic

$$\begin{array}{ccccc} \mathcal{H}_{LLE}(\mathbf{S}) & \longleftrightarrow & \mathcal{H}_{MFT}(\rho, H) & \longleftrightarrow & \mathcal{H}_{AKNS}(u, v) \\ \int dx (\partial_x \mathbf{S} \cdot \partial_x \mathbf{S}) & & & & \int dx (\partial_x u \partial_x v + u^2 v^2) \end{array}$$

Discrete-MFT, spin chains, integrability

► Model dependent

$$\mathcal{H}_{\text{dMFT}}^{\text{SIP}}(x, h) = - \sum_i -x_i(1+x_{i+1})(e^{-(h_i-h_{i+1})} - 1) - (1+x_i)x_{i+1}(e^{-(h_{i+1}-h_i)} - 1)$$

$$\mathcal{H}_{\text{dMFT}}^{\text{Harm}}(x, h) = - \sum_i \log \left[1 - x_i(1+x_{i+1})(e^{-(h_i-h_{i+1})} - 1) - (1+x_i)x_{i+1}(e^{-(h_{i+1}-h_i)} - 1) \right]$$

► Integrability

$$\begin{aligned} \mathcal{S}_i^+ &= (1+x_i)e^{h_i} & \mathcal{S}_i^- &= x_i e^{-h_i} & \mathcal{S}_i^z &= x_i + \frac{1}{2} \\ \{\mathcal{S}_i^+, \mathcal{S}_i^-\} &= -2\mathcal{S}_i^z & \{\mathcal{S}_i^z, \mathcal{S}_i^+\} &= \mathcal{S}_i^+ & \{\mathcal{S}_i^z, \mathcal{S}_i^-\} &= -\mathcal{S}_i^- \end{aligned}$$

$$\begin{aligned} \mathcal{H}_{\text{XXX}}(\mathbf{S}) &\longleftrightarrow \mathcal{H}_{\text{dMFT}}^{\text{SIP}}(\rho, H) \\ - \sum_i (\mathbf{S}_i \cdot \mathbf{S}_{i+1}) &\text{not integrable} \end{aligned}$$

$$\begin{aligned} \mathcal{H}_{\text{Ishimori}}(\mathbf{S}) &\longleftrightarrow \mathcal{H}_{\text{dMFT}}^{\text{Harm}}(\rho, H) &\longleftrightarrow \mathcal{H}_{\text{Ablowitz-Ladick}}(Q, R) \\ - \sum_i \log \left(\frac{1}{4} - \mathbf{S}_i \cdot \mathbf{S}_{i+1} \right) &\text{integrable!} & \sum_i [Q_i R_{i+1} + R_i Q_{i+1} + 2 \log(1 - Q_i R_i)] \end{aligned}$$

THANK YOU FOR
YOUR ATTENTION

Some references on the harmonic process

- ▶ R. Frassek, **CG**, J. Kurchan
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Duality and hidden equilibrium in transport models
SciPost Phys. 9, 054 (2020)
- ▶ R. Frassek, **CG**
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J. Math. Phys. 63, 103301 (2022)
- ▶ C. Franceschini, R. Frassek, **CG**
Integrable heat conduction model
J. Math. Phys. 64, 043304 (2023)
- ▶ G. Carinci, C. Franceschini, D. Gabrielli, **CG**, D. Tsagkarogiannis
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- ▶ G. Carinci, C. Franceschini, R. Frassek, **CG**, F. Redig
Large deviations and additivity principle for the open harmonic process
Comm. Math. Phys. 406, 103 (2025)
- ▶ **CG**, F. Redig, B. van Tol
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J. Stat. Phys. 192, 21 (2025)

MFT

current large deviations

Current large deviations (step initial condition):

► Macroscopic Fluctuation Theory (MFT)

$$\mathcal{H}(\rho, H) = \frac{1}{2}\sigma(\rho)(\partial_x H)^2 - D(\rho)(\partial_x \rho)(\partial_x H)$$

► MFT equations

$$\partial_t \rho = \partial_x [D(\rho) \partial_x \rho] - \partial_x [\sigma(\rho) \partial_x H]$$

$$\partial_t H = D(\rho) \partial_{xx} H - \frac{1}{2} \sigma'(\rho) (\partial_x H)^2$$

with

$$H(x, T) = \lambda 1_{\{x > 0\}}$$

$$H(x, 0) = \lambda 1_{\{x > 0\}} + f'(\rho(x, 0)) - f'(\bar{\rho}(x))$$

$$f'(\rho) = \ln\left(\frac{\rho}{1-\rho}\right)$$

$$\bar{\rho}(x) = \rho_L 1_{\{x \leq 0\}} + \rho_R 1_{\{x > 0\}}$$

Cumulant generating function of current (step initial condition):

- ▶ For D constant and σ quadratic, [Mallick, Moriya, Sasamoto '22] introduced a generalized Cole-Hopf transformation to new coordinates $(\rho, H) \mapsto (u, v)$.
- ▶ The MFT equations become AKNS system

$$\partial_t u = \partial_{xx} u - 2u^2 v$$

$$\partial_t v = -\partial_{xx} v + 2uv^2$$

with

$$u(x, 0) = \omega \delta(x)$$

$$v(x, T) = \delta(x)$$

- ▶ AKNS system is solvable by Inverse Scattering Method. [MMS 22] could reproduce cgf of SEP using classical integrability.

Discrete MFT

current large deviations

Cumulant generating function (step initial condition)

► Current:

$$\mu(\lambda, T) := \lim_{s \rightarrow \infty} \frac{1}{2s} \log \mathbb{E} [\exp (\lambda C(2sT))]$$

$$C(t) := \sum_{i \in \mathbb{N}} (\eta_i(t) - \eta_i(0))$$

► Height:

$$\mu_x(\lambda, T) := \lim_{s \rightarrow \infty} \frac{1}{2s} \log \mathbb{E} [\exp (\lambda C_x(2sT))], \quad x \in \mathbb{Z}$$

$$C_x(t) := \begin{cases} C(t) - \sum_{i=1}^x \eta_i(t) & x \geq 1, \\ C(t) + \sum_{i=x+1}^0 \eta_i(t) & x \leq -1 \end{cases}$$

Current large deviations for harmonic process (step initial condition)

► Lattice Hamiltonian

$$\mathcal{H}(q, p) = - \sum_i \ln \left(1 - q_i(1 + q_{i+1})(e^{-(p_i - p_{i+1})} - 1) - (1 + q_i)q_{i+1}(e^{-(p_{i+1} - p_i)} - 1) \right)$$

► Discrete-MFT equations

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial q_i} \qquad \dot{p}_i = - \frac{\partial \mathcal{H}}{\partial p_i}$$

with

$$p_i(T) = \lambda 1_{\{i > 0\}}$$

$$p_i(0) = \lambda 1_{\{i > 0\}} + f'(q_i(0)) - f'(\bar{q}_i)$$

$$f'(q_i) = \ln\left(\frac{q_i}{1 + q_i}\right)$$

$$\bar{q}_i = q_L 1_{\{i \leq 0\}} + q_R 1_{\{i > 0\}}$$

Current large deviations for harmonic process (step initial condition):
large s

- ▶ In [G, Mallick, Sasamoto, Suda, (2026+)] we introduce a discrete generalized Cole-Hopf transformation to new coordinates $(q, p) \mapsto (Q, R)$.
- ▶ The discrete MFT equations become Ablowitz-Ladik system

$$\dot{Q}_i = -Q_{i+1} - Q_{i-1} + 2Q_i + Q_i R_i (Q_{i+1} + Q_{i-1})$$

$$\dot{R}_i = R_{i+1} + R_{i-1} - 2R_i - Q_i R_i (R_{i+1} + R_{i-1})$$

with

$$R_i(0) = R_0(0) \delta_{i,0}$$

$$Q_i(T) = Q_0(T) \delta_{i,0}$$

- ▶ AL system is solvable by Inverse Scattering Method and therefore we obtain the cumulants generating function using classical integrability.

Current large deviations for harmonic process (step initial condition):
large s

- Cumulant generating function [G, Mallick, Sasamoto, Suda, (2026+)]

$$\begin{aligned}\mu_x(\lambda, T) &= T \sum_{n=1}^{\infty} \frac{(-\omega e^{-2T})^n}{n} (2I_{nx}(2nT) + I_{nx+1}(2nT) + I_{nx-1}(2nT)) \\ &+ \frac{|x|}{2} \sum_{n=1}^{\infty} \frac{(-\omega e^{-2T})^n}{n} \left(I_{n|x|}(2nT) + \sum_{m=-n|x|+1}^{n|x|-1} I_m(2nT) \right) \\ &- \frac{x}{2} \log \left(\frac{1 - (e^{-\lambda} - 1) q_R}{1 - (e^{\lambda} - 1) q_L} \right)\end{aligned}$$

where

$$\omega := \left(1 - (e^{-\lambda} - 1) q_R\right) \left(1 - (e^{\lambda} - 1) q_L\right) - 1$$

$I_m(z)$ modified Bessel function

Comments

- The large s large deviation provide a lattice version of MFT: if $\frac{x}{2\sqrt{T}} = \xi$ then

$$\begin{aligned}\mu_\xi(\lambda) &:= \lim_{T \rightarrow \infty} \frac{\mu_{\lfloor 2\xi\sqrt{T} \rfloor}(\lambda, T)}{\sqrt{T}} \\ &= \sum_{n=1}^{\infty} \frac{(-\omega)^n}{n^{3/2}} \left(\frac{e^{-n\xi^2}}{\sqrt{\pi}} + \sqrt{n}|\xi| \operatorname{erf}(\sqrt{n}\xi) \right) - \xi \log \left(\frac{1 - (e^{-\lambda} - 1) q_R}{1 - (e^\lambda - 1) q_L} \right)\end{aligned}$$

- For the harmonic process, the following scheme emerges:

continuum :

LLE

MFT

AKNS

discrete :

Ishimori

discrete-MFT

Ablowitz-Ladik