Chat Time Sam!

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Exploratory Sheets for the Math Circles Session

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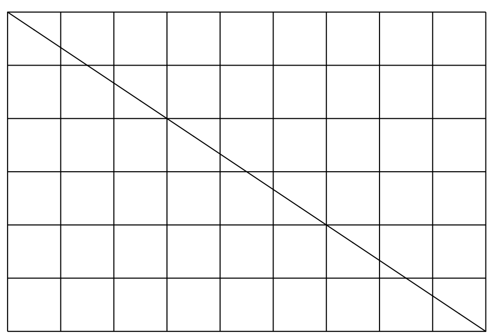
Have Fun!

We go from very easy problems to increasingly difficult ones.

Rectangular grid

Draw a rectangular grid made up of unit squares; for example, a rectangle of length 9 units and height 6 units.
Draw a diagonal - say, from the left upper corner to the lower right corner.
How many of the small unit squares does this diagonal pass through?



What about a 42×18 rectangle? Or a general $m \times n$ rectangle?!

Can you prove the claim you make?

Sun-Tsu

For relatively prime integers $m, n > 1$ look at an $m \times n$ grill made up of unit squares. How does one fill the entries by the numbers from 1 to mn on the unit squares such that starting from the top left corner, when we come down by i squares and move j squares to the right, the entry in that unit square gives a remainder i while divided by m and remainder j when divided by n?

Jarasandha

Write down any n-digit integer. Divide it into a right part of r digits and a left part of n-r digits; to the left part add a number $0 < L < 10$ and to the right part add some $0 < R < 10$. The addition is done modulo 10 and the "carry-over" is ignored. Transfer the left part to the right of the right part and we again get an n-digit number - if there are any 0s, don't erase them. Repeat this process with the new number. Iterating this several times, we can ask if we always get the original number back eventually.

Here is an example.

Example: We take $n = 8, r = 2, L = 4, R = 2$.
Starting with the number 56240317, the iteration gives:

56240317	26051556	07175426
19562407	58260519	28071758
09195628	11582609	50280711
20091950	01115820	13502801
52200913	22011152	03135022
15522003	54220115	24031354
05155224	17542205	56240317

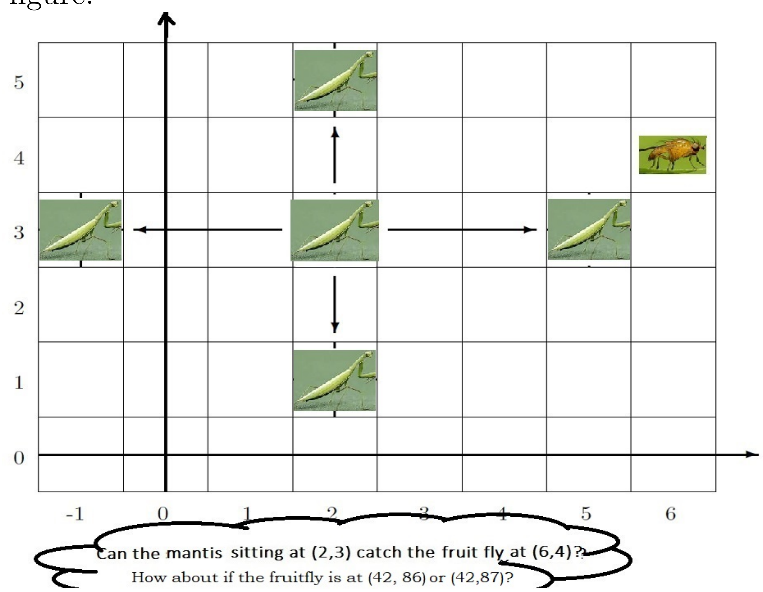
which gives back the original number at the 20-th step.

Question. If every number must eventually reappear, what is the least number N of steps required that works for every n-digit number? Do this for any base in place of base 10.

Catch-me-if-you-can

Consider the following predator-prey puzzle where a praying mantis which is on the horizontal plane, wishes to catch a stationary fruit fly, but our mantis is a lattice mantis - insists on travelling only along certain lattice points. ˘ In the following figure, the centres of those squares are lattice points and the mantis always sits at the centre of the square it occupies.

The movement of the mantis is restricted; from a lattice point (u, v) it can move to one of the four points $(u + v, v), (u - v, v), (u, v + u), (u, v - u)$ - see the figure.



Some natural questions are:
Which lattice points can be reached by the mantis starting from a point other than the origin?
Does the mantis ever return to a point after finitely many steps without simply reversing any of its steps?

Tiling rectangles by rectangles

Let us now discuss tiling integer-sided rectangles by copies of a fixed integer-sided rectangle.

Question 0. Can we tile a rectangle of size 28×17 by copies of a 7×7 rectangle?

Question 1. Can we tile a rectangle of size 28×17 by copies of a rectangle of size 4×7?

Question 2. Can a 10×15 rectangle be tiled with copies of a 1×6 rectangle?

Problem. Find necessary and sufficient conditions for tiling an $m \times n$ rectangle by copies of an $a \times b$ rectangle. Give an algorithm to tile when tiling exists. Finally, generalize this to higher dimensional parallelopipeds.

Domino Tilings

Suppose, one tiles a rectangular $m \times n$ grid using dominos. For a $2 \times n$ grid, it is very easy to determine the number c_n of domino tilings.

Question 0. Find c_n.

Question 1. For a $3 \times 2n$ grid, can you find a recursive formula in terms of n for the number of domino tilings?

Question 2. What about an $4 \times n$ grid, for general n?

Question 3. Finally, what about $m \times n$ grid for general m, n with mn even?