Chat Time Sam!
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Exploratory Sheets for the Math Circles Events
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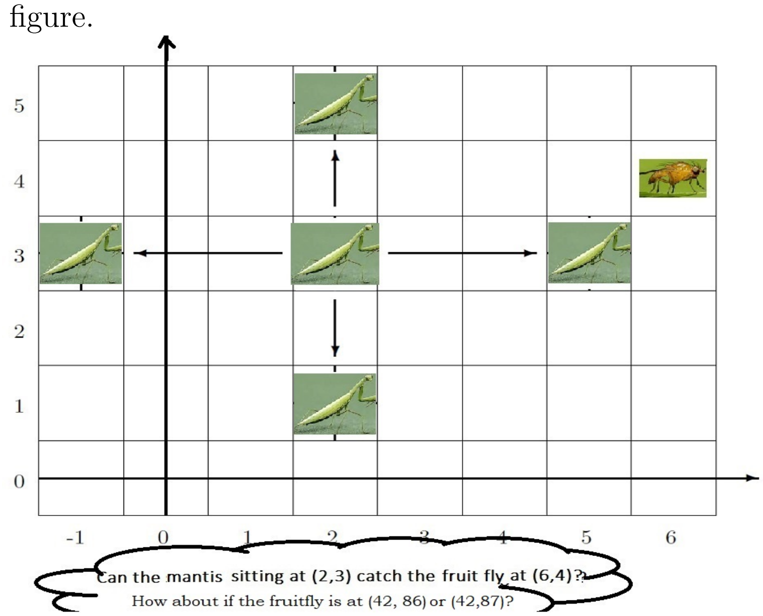
Have Fun!

Catch-me-if-you-can

Consider the following predator-prey puzzle where a praying mantis which is on the horizontal plane, wishes to catch a fruit fly, but our mantis is a lattice mantis - insists on travelling only along certain lattice points.
In the following figure, the centres of those squares are lattice points and the mantis always sits at the centre of the square it occupies.

The movement of the mantis is restricted; from a lattice point (u, v) it can move to one of the four points $(u+v, v), (u-v, v), (u, v+u), (u, v-u)$ - see the figure.



Some natural questions are:
Which lattice points can be reached by the mantis starting from a point other than the origin?

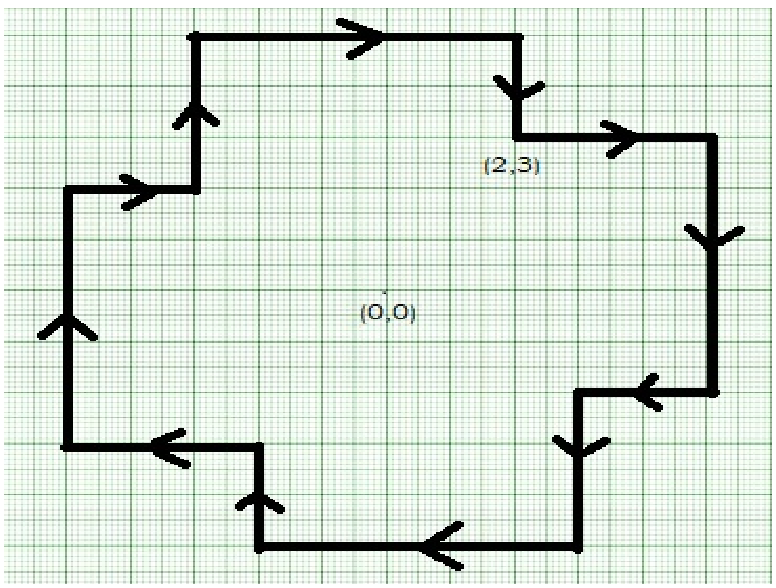
Does the mantis ever return to a point after finitely many steps without simply reversing any of its steps?

The point $(2,3)$ can return:

$$(2,3) \mapsto (5,3) \mapsto (5,-2) \mapsto (3,-2) \mapsto (3,-5)$$

$$\mapsto (-2,-5) \mapsto (-2,-3) \mapsto (-5,-3) \mapsto (-5,2)$$

$$\mapsto (-3,2) \mapsto (-3,5) \mapsto (2,5) \mapsto (2,3).$$



Tiling rectangles by rectangles

Let us now discuss tiling integer-sided rectangles by copies of a fixed integer-sided rectangle.

Question 0. Can we tile a rectangle of size 28×17 by copies of a 7×7 rectangle?

Question 1. Can we tile a rectangle of size 28×17 by copies of a rectangle of size 4×7?

Question 2. Can a 10×15 rectangle be tiled with copies of a 1×6 rectangle?

Problem. Find necessary and sufficient conditions for tiling an $m \times n$ rectangle by copies of an $a \times b$ rectangle. Give an algorithm to tile when tiling exists. Finally, generalize this to higher dimensional parallelopipeds.

Domino Tilings

Suppose, one tiles a rectangular $m \times n$ grid using dominos. For a $2 \times n$ grid, it is very easy to determine the number c_n of domino tilings.

Question 0. Find c_n.

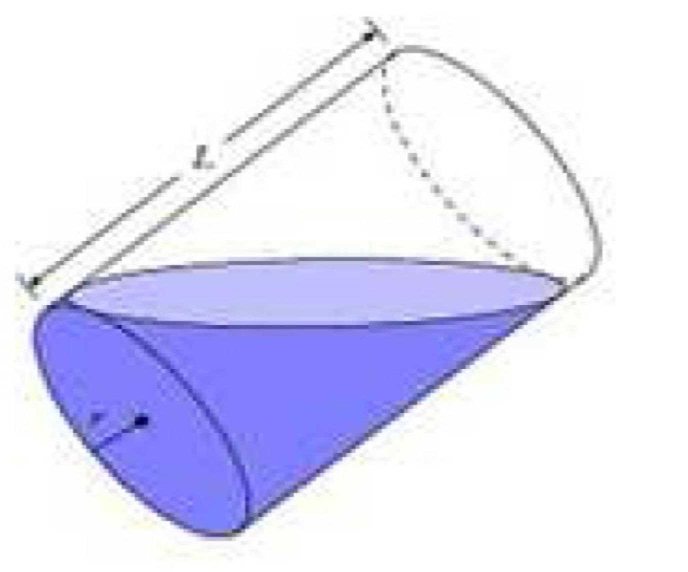
Question 1. For a $3 \times 2n$ grid, can you find a recursive formula in terms of n for the number of domino tilings?

Question 2. What about an $4 \times n$ grid, for general n?

Question 3. Finally, what about $m \times n$ grid for general m, n with mn even?

Filling Glasses

If we have a perfectly cylindrical glass vessel with some amount of liquid but no measuring aid, how do we know if the glass is more than half filled or not? That is easy:

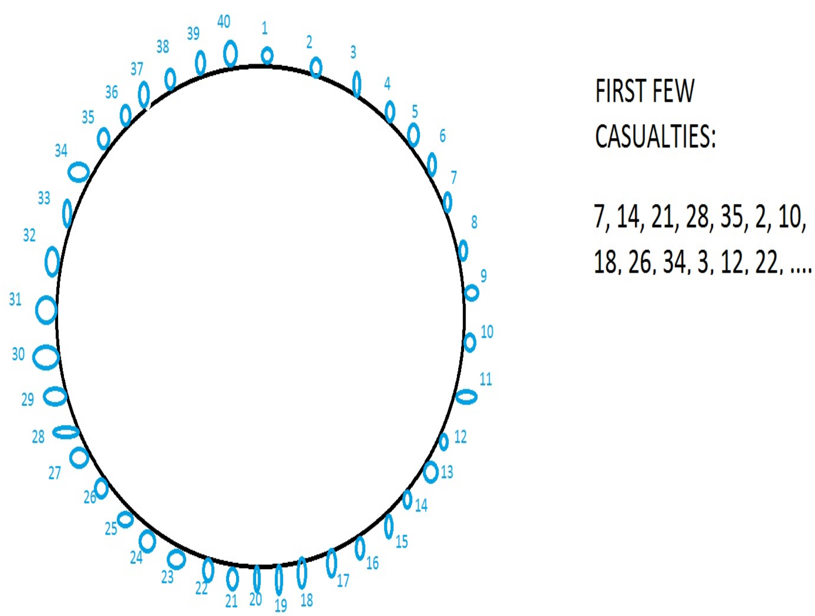


Consider now the following problem where we have two glass vessels of equal size. Suppose, we can always check if the two glasses have equal amounts of liquids by placing them side by side and checking the levels. Given an unlimited quantity of water, what proportions of glass vessels can be measured just using these two glasses? Can you also provide an algorithm that carries out the process of filling the exact amounts which are possible to measure and fill precisely?

If you can't beat them, join them

Flavius Josephus and 39 of his comrades were surrounded when holding a revolt against the Romans during the 1st century A.D. Rather than become slaves, they decided to kill themselves by their own comrades. They arranged themselves along a circle. Starting somewhere, they went clockwise around the circle and every 7th person was eliminated. This continued with the 7th among the surviving ones being killed at each step. Apparently, Josephus was a clever mathematician and arranged himself in such a position that he would be the last survivor. The story goes that he did not kill himself but came and joined the Romans!

We need to find out Josephus's position. The general problem is of n people, designated by $1, 2, \cdots, n$ in clockwise order, say, and every d-th person is eliminated going around in the clockwise direction. Can we describe somehow the position of the last survivor?



When $d = 2$, there is a beautiful formula to find the position of the last person alive; can you find it?