Calculus: Exploration Sheet 2

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1 Introduction

In the last session, we explored differentiation and discussed methods of finding the derivatives of functions. In this session, we will explore the inverse problem, where we seek to determine one or more functions given their derivatives. Such equations are called "differential equations." The simplest differential equations can be solved by a method called "integration".

Differential equations are of great importance in science. For example, Newton's second law — perhaps the most basic law of physics — relates the force on an object to its acceleration. But what we can observe directly is the position of the object. Since the acceleration is the derivative of the velocity, which is the derivative of the position, it is often necessary to solve differential equations to extract observable information from physics.

In the previous exploration, we introduced differentiation in terms of dual numbers. But in this exploration, you are required to recall the physical insight behind that definition: dual numbers are a trick for keeping track of small quantities. They help us keep track of first-order terms, while dropping higher-order terms. You will need to do this below.

2 Integration

The simplest differential equations are those where we need to find an unknown function, $y(x)$, when we are given its derivative explicitly as a function of x.

$$\frac{dy(x)}{dx} = g(x). \tag{1}$$

P1 Show that (1) does not have a unique solution. If $y(x)$ is a solution, then $y(x)+c$, where c is any constant, is also a solution. However if the solution exists, then for any two values x_1, x_2

$$y(x_2) - y(x_1)$$

is uniquely fixed. Alternately, one could say that if one is given an "initial value", $y(x_1) = y_1$, to supplement (1) then the solution is uniquely fixed.

The solution to the equation (1) is written as

$$y(x_2) - y(x_1) = \int_{x_1}^{x_2} g(x)dx \tag{2}$$

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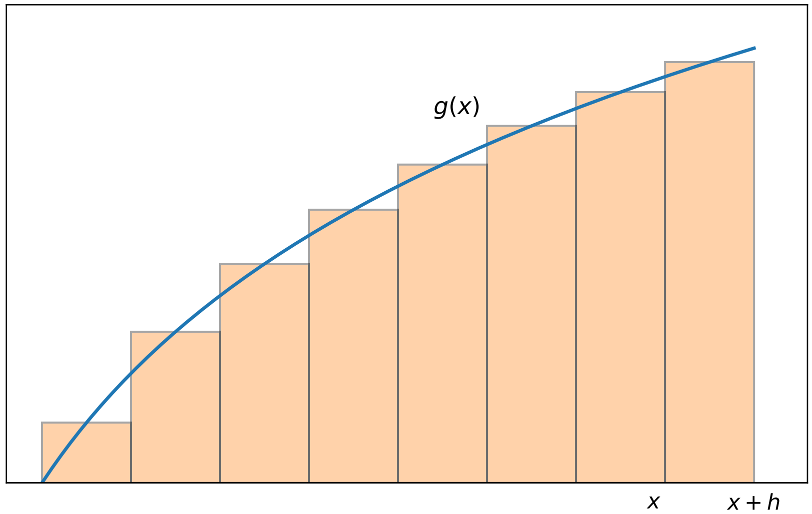


Figure 1: *The figure shows how the area under the curve of a function can be approximated by drawing many thin rectangles. This also allows us to argue that the area under the curve between two points is the integral of the function.*

pronounced as "integral from x_1 to x_2 of g(x) d x".

This integral can be interpreted as the area under the curve of the function $g(x)$ between the points x_1 and x_2. To see this, let us approximate the area between x_1 to x, denoted by $A(x)$, by means of many small rectangles as shown in Figure 1. The figure shows that $A(x+h) = A(x) + h * g(x)$ for small h. Keeping track of first order terms in h and dropping higher order terms, we reach the conclusion that

$$\frac{dA}{dx} = g(x). \tag{3}$$

The initial condition is $A(x_1) = 0$, which uniquely fixes the solution.

P2 The rules that you derived for differentiation in the last section have analogues for integration. Show the following results.

1. The sum rule for differentiation implies that if $h(x) = f(x) + g(x)$

$$\int_{x_1}^{x_2} h(x)dx = \int_{x_1}^{x_2} f(x)dx + \int_{x_1}^{x_2} g(x)dx. \tag{4}$$

2. The product rule for differentiation implies that

$$\int_{x_1}^{x_2} f(x)g'(x)dx = f(x_2)g(x_2) - f(x_1)g(x_1) - \int_{x_1}^{x_2} f'(x)g(x)dx. \tag{5}$$

3. The chain rule for differentiation implies that

$$\int_{x_1}^{x_2} f(g(x))g'(x)dx = \int_{g(x_1)}^{g(x_2)} f(x)dx. \tag{6}$$

Note the change in the limits of integration on the right hand side. You should also recall that that the "x" inside the integral sign is just a dummy variable.

3 Projectile motion

Lets use our newfound understanding to derive the formulas for projectile motion that many of you might have already seen in school. Let $y(t)$ be the height of the projectile at time t, and $x(t)$ be its displacement in the horizontal direction. Then the velocity of the projectile in the x and y directions is given by

$$v_y(t) = \frac{dy(t)}{dt}; \qquad v_x(t) = \frac{dx(t)}{dt}. \tag{7}$$

Neglecting air resistance, a projectile thrown into the air only feels a downward force. This leads to the equations

$$\frac{dv_y(t)}{dt} = -g; \qquad \frac{dv_x(t)}{dt} = 0, \tag{8}$$

where g is the acceleration due to gravity. Lets also choose the initial conditions

$$y(0) = 0; \quad x(0) = 0; \quad v_y(0) = u_y; \quad v_x(0) = u_x. \tag{9}$$

P3 Solve the equations above to find $x(t)$ and $y(t)$. Then eliminate t from these solutions and find $y(x)$ — the height as a function of the horizontal position.

When you plot it you should find a graph like Fig. 2. This is a real application of Calculus to everyday life!

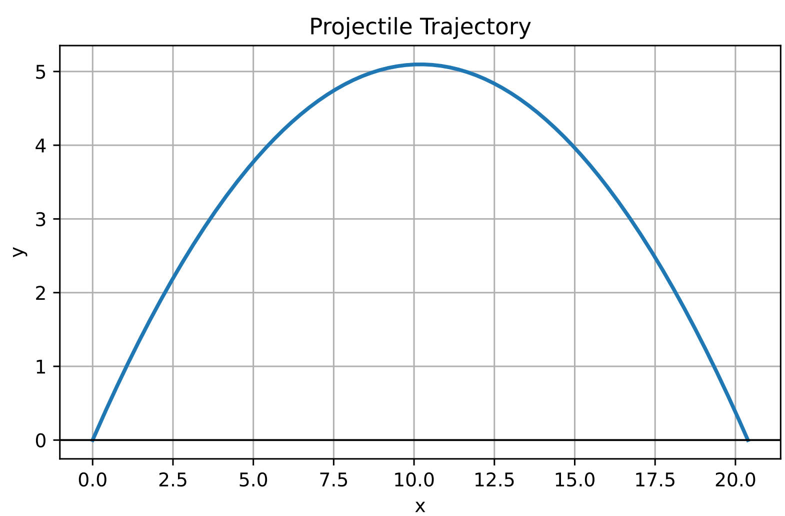


Figure 2: *The predicted trajectory of a projectile launched with $u_x = u_y = 10m/s$ in the absence of air resistance.*

4 Trigonometric functions

Last time, there were many questions about the trigonometric functions. A good way to study the trigonometric functions and exponential functions is through differential equations, as we do below.

P4 Argue the the trigonometric functions obey the differential equations

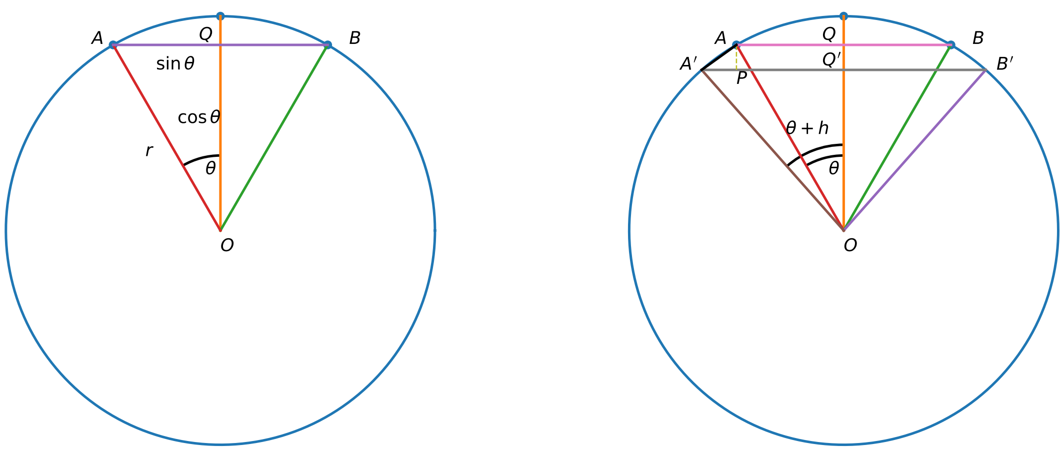
$$\frac{d\sin\theta}{d\theta} = \cos\theta, \tag{10}$$

and

$$\frac{d\cos\theta}{d\theta} = -\sin\theta, \tag{11}$$

with the initial conditions $\sin(0) = 0; \quad \cos(0) = 1$.

You can use any method that you like. But I find it useful to think of these functions in terms of triangles inscribed in a unit circle as shown in Figure 3a. To compute the derivative, we think of a slightly different configuration shown in Figure 3b. Here AP is perpendicular to



(a) (b)

Figure 3: *The first figure shows how to define the trigonometric functions within a unit circle. The length of the half-chord AQ is $\sin\theta$ and OQ is $\cos\theta$. The second figure shows how to relate their derivatives to each other.*

the chord $A'B'$. The key point is to analyze the triangle $AA'P$. Also, recall the definition of the angle: the length of the curve along the unit circle for an angle h, measured in radians, is h. Using this, and keeping track of *first order* terms in h, see if you can determine the lengths of AP and $A'P$. This will immediately give you the derivatives.

P5 Check that the following infinite series solve the equations above.

$$\begin{aligned} \sin\theta &= \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k+1}}{(2k+1)!} \\ \cos\theta &= \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k}}{(2k)!} \end{aligned} \tag{12}$$

You should not be intimidated by the fact that the series involves an infinite number of terms. The key point is that, for any value of x, the size of the terms starts declining rapidly once k becomes larger than $\frac{x}{2}$, and successive terms become less-and-less important. So we can always obtain any desired accuracy by keeping track of a finite number of terms.

5 Exponential function

Now, we will turn to the exponential function although we will introduce it in a slightly unusual manner. Let us define the function $y(x)$ through the equation

$$\frac{dy(x)}{dx} = y(x), \tag{13}$$

with the initial condition $y(0) = 1$.

P6 1. Check that the following infinite series solves the equation above

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \tag{14}$$

The comments that we made about the infinite series above, for the sine and cosine functions, also apply here.

2. Using this series, and the binomial theorem, check that $y(x)$ obeys the property

$$y(x_1 + x_2) = y(x_1)y(x_2). \tag{15}$$

The second property above tells us that $y(x)$ can be interpreted as an exponential. So we can set $y(x) = e^x$, where e is some number. This is the same number, e, that is called the second-most famous irrational number after π. You can even evaluate e to a few decimal places by summing some terms of the series above.

The series (12) and (14) look similar. The precise way to relate them is to examine $e^{i\theta}$ where i is an imaginary number with the property that $i^2 = -1$.

P7 Show that

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{16}$$

From here, you can pair e with its even-more illustrious counterpart through $e^{i\pi} = -1$.