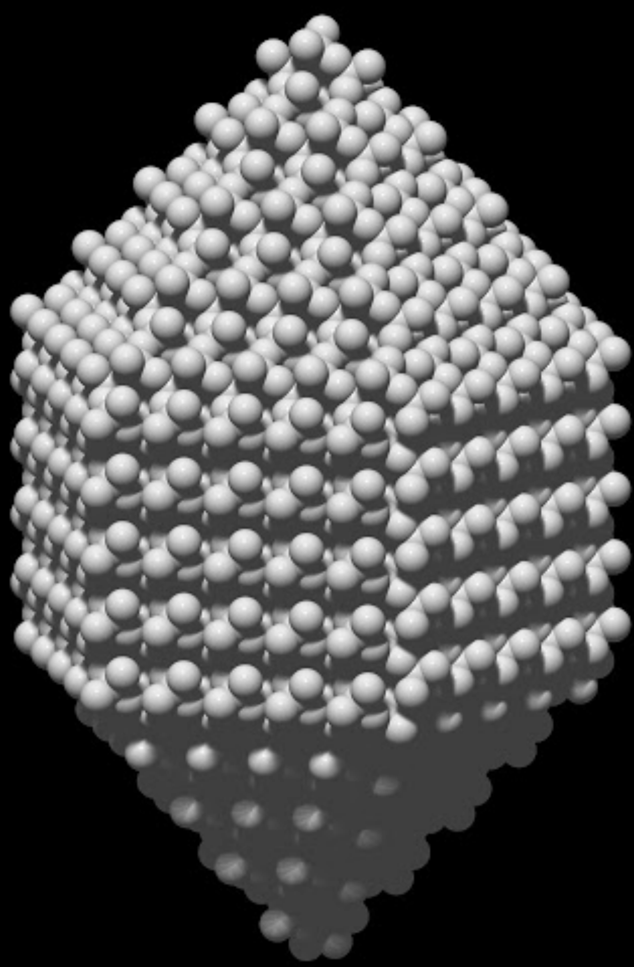
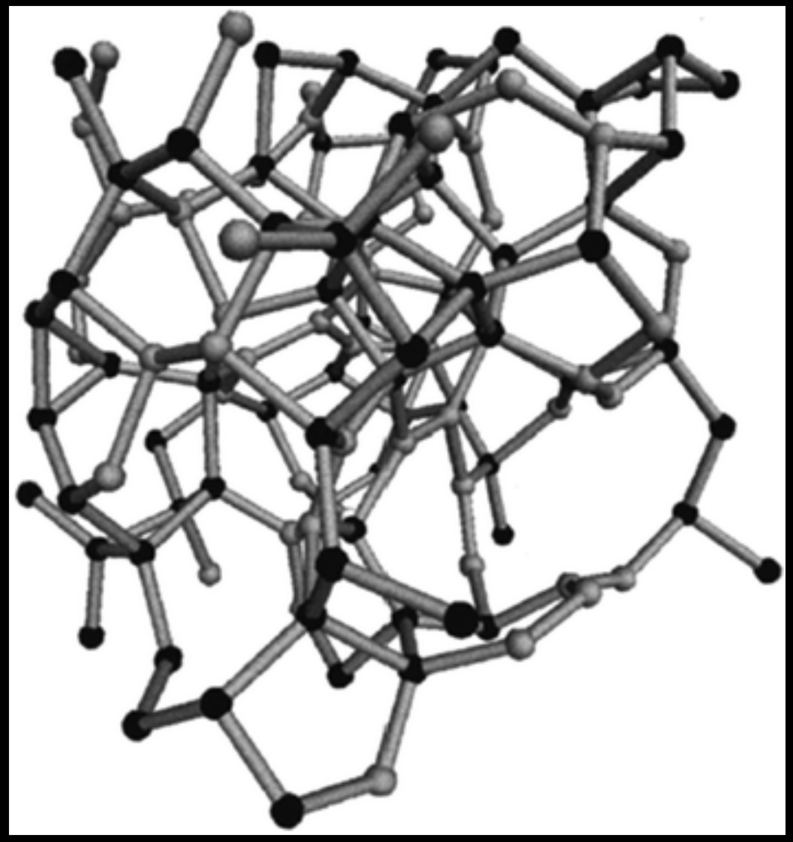
Emergence of elasticity in amorphous solids





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Athermal, Disordered Solids: Jammed in the broadest sense of the word

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Once created, what is their mechanical response ?

Does a field-theoretic description exist and if so, what does it look like?

Emergence of elasticity in amorphous solids

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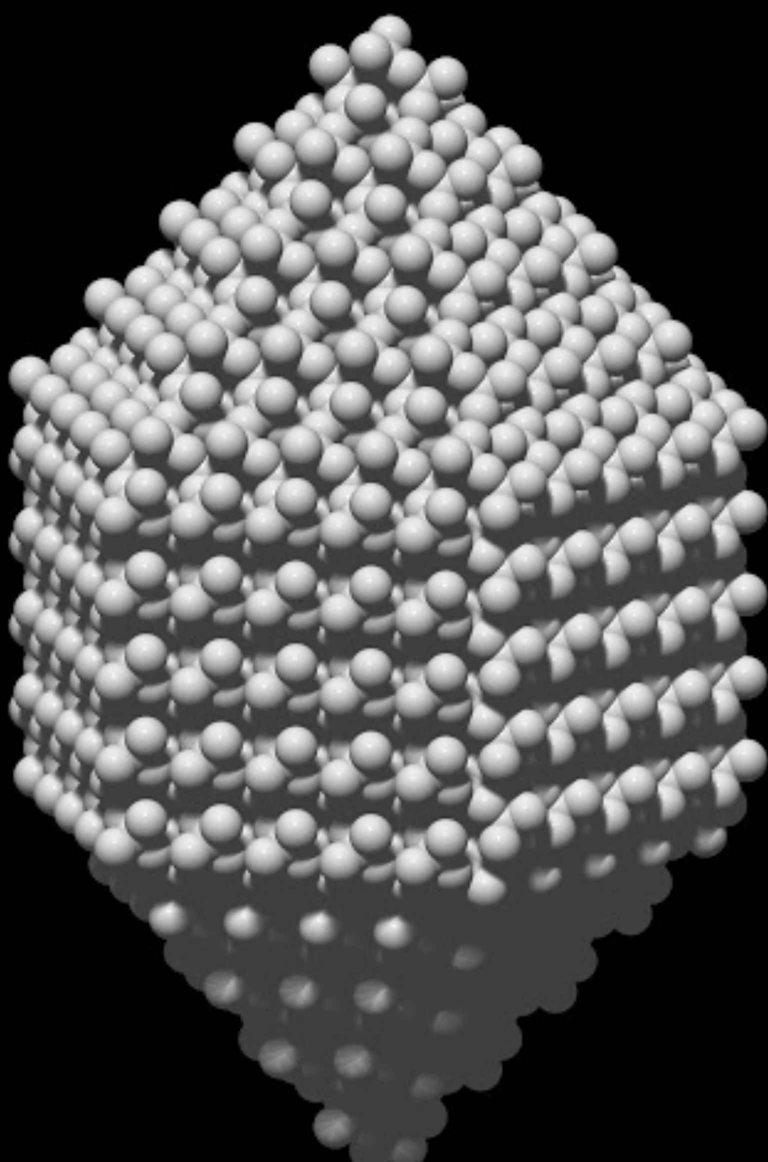
Collaborators: Theory

Silke Henkes, Dapeng Bi, Sumantra Sarkar, Kabir Ramola, Jishnu Nampoothiri, Desh Bedi, Michael D'Eon, & Subhro Bhattacharya

Collaborators: Experiments

The Behringer Group, Duke University
Jie Zhang, Yiqiao Wang, SJTU

Foundations of Elasticity Theory



- Long-range order distinguishes crystalline solids from liquid: "reference structure"
- Displacements from this reference structure, coarse-grained defines a symmetric tensor: the strain tensor
- Stress is defined as the derivative of the free-energy with respect to strain
- Broken Symmetry: Goldstone modes, gapless (phonons)

Jammed Solids

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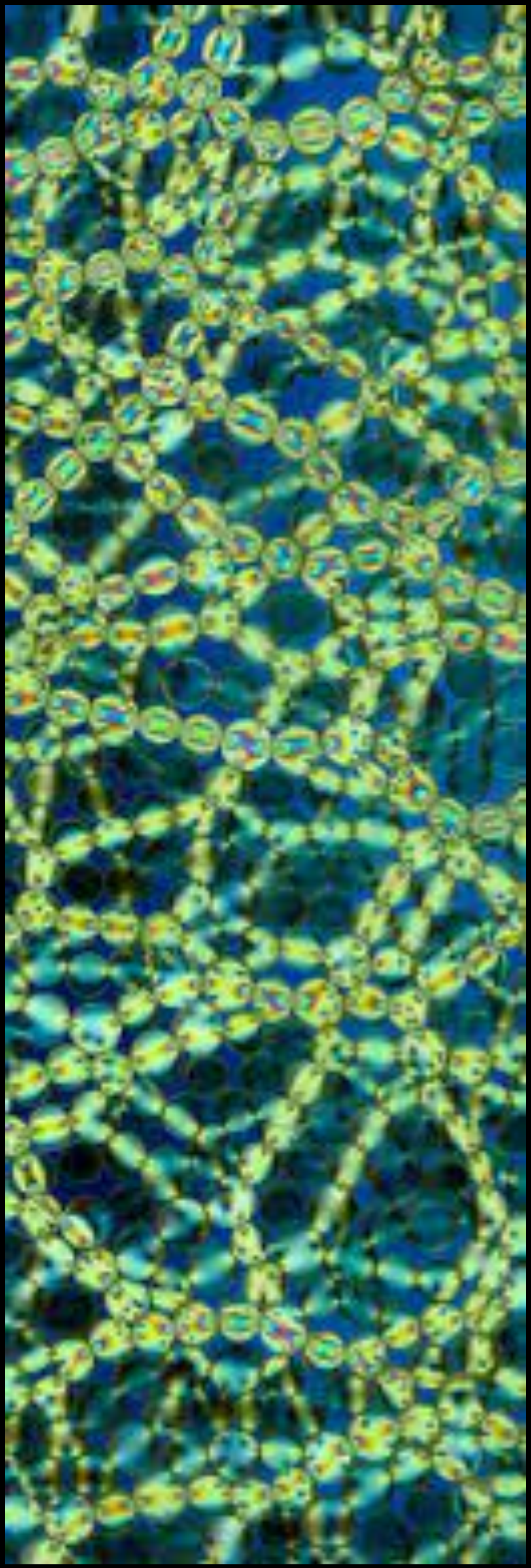

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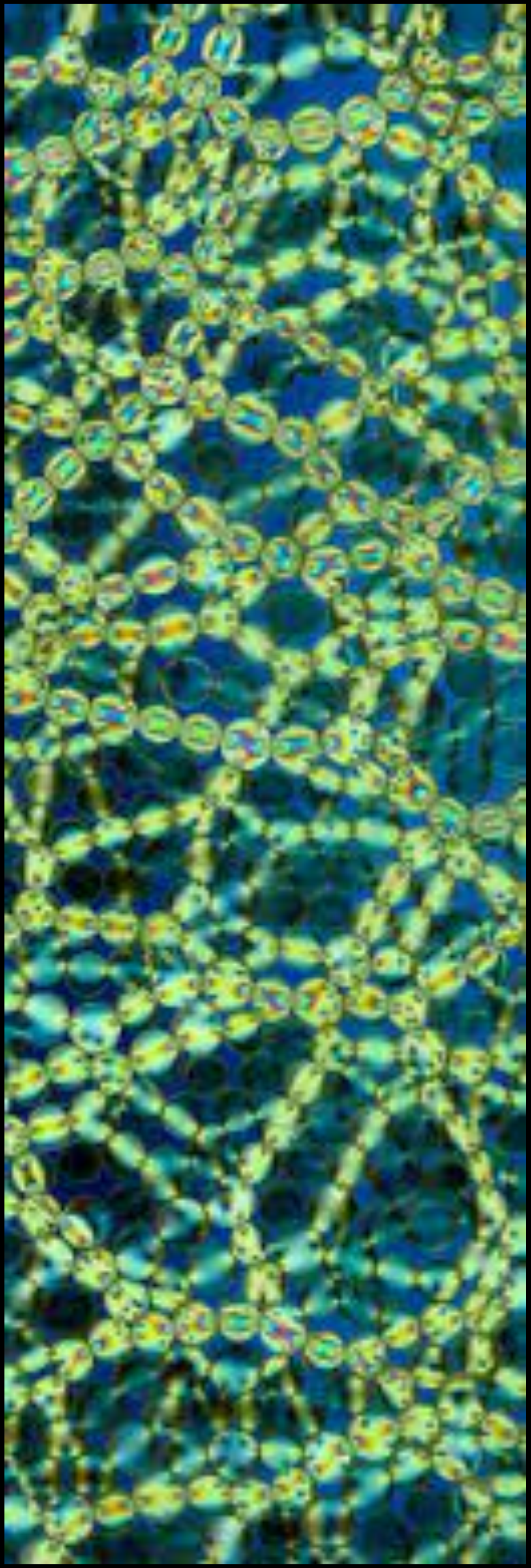

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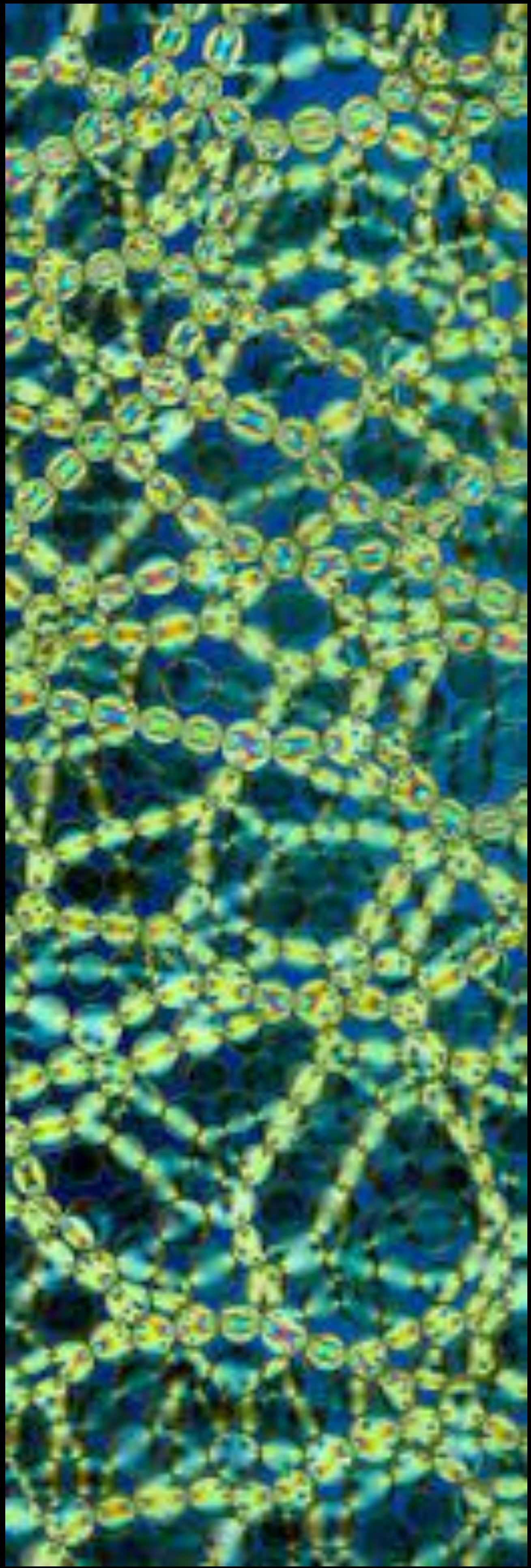

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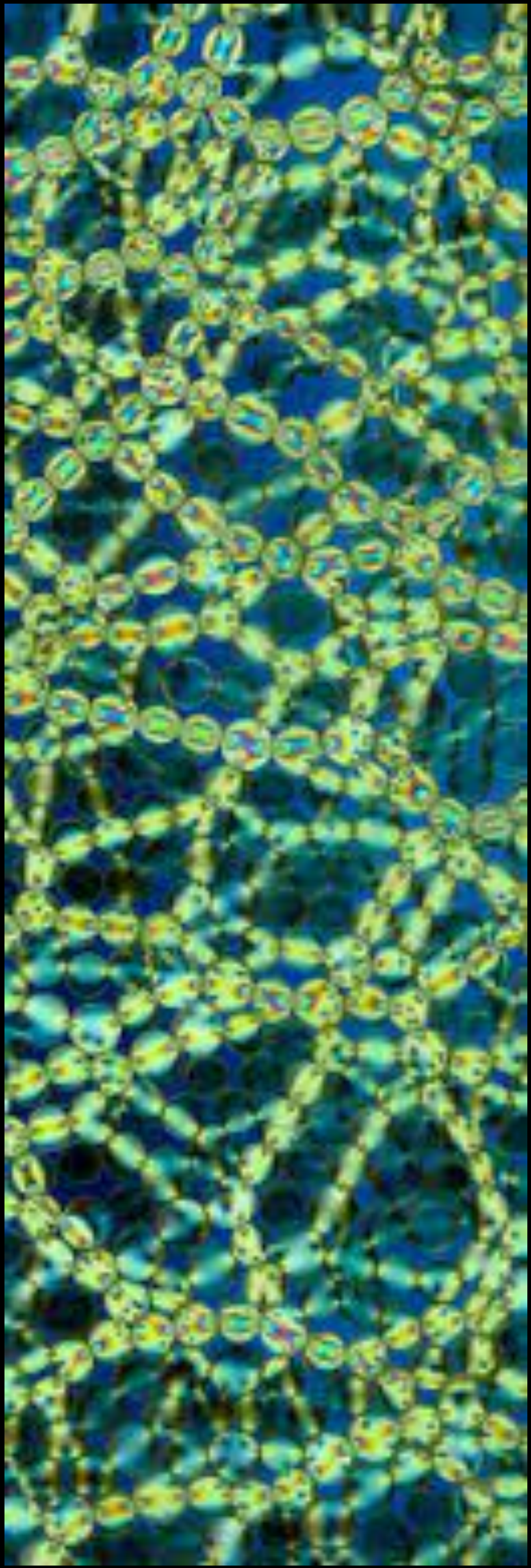

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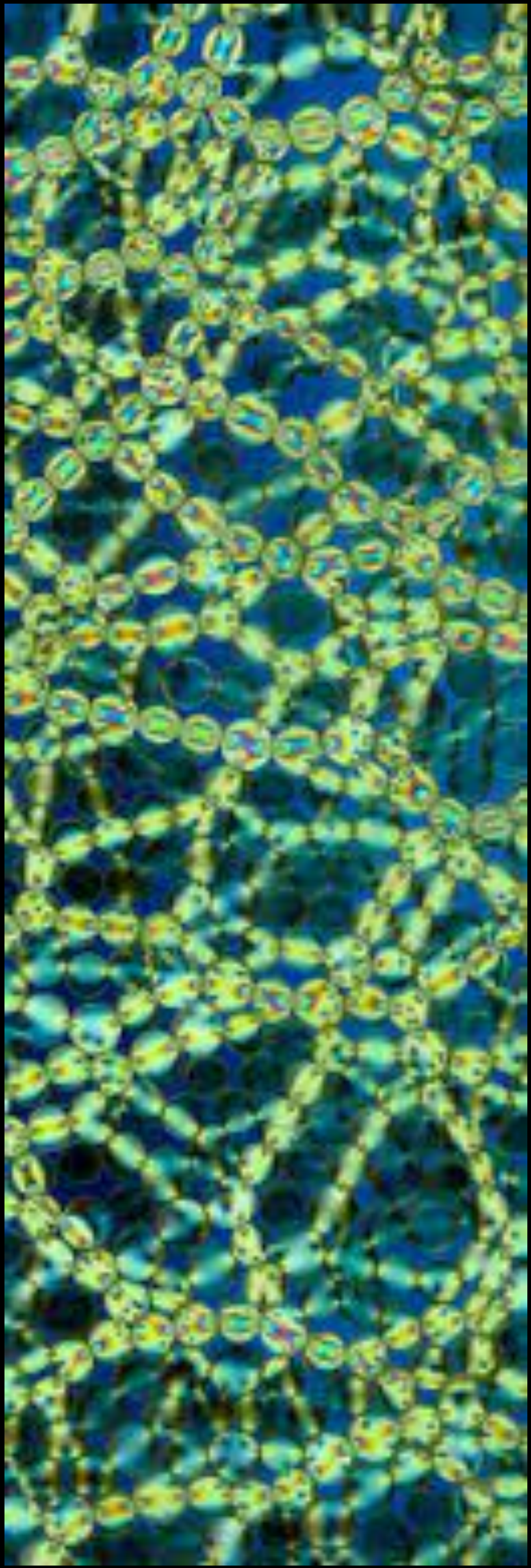

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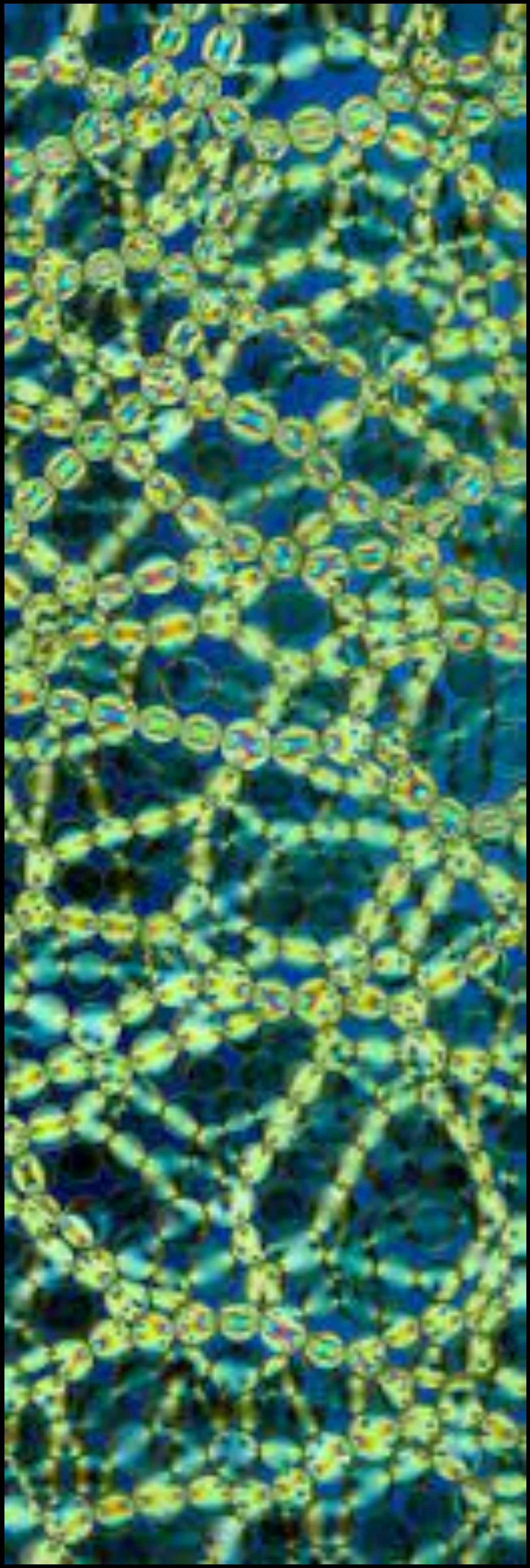

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Continuum Elasticity

$\partial_i \sigma_{ij} = 0$; $\sigma_{ij} = \sigma_{ji}$

force balance   torque balance

Not enough equations

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Closure: constitutive Equation

$$\sigma_{ij} = \Lambda_{ijkl} \Gamma_{kl} \; ;$$ linear elasticity

Strain Tensor

$$\Gamma_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)$$

$\vec{u}$: displacement from reference

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Note: $\epsilon_{iab}\epsilon_{jcd}\partial_a\partial_c \Gamma_{bd} = 0$

"double curl" of strain field is zero: provides enough equations

Jammed Solids

Lack of reference structure invalidates the foundational role of the strain tensor in elasticity theory

Jammed Solids at $T = 0$

$$\partial_i \sigma_{ij} = \boxed{f_j^{ext}} \qquad \sigma_{ij} = \sigma_{ji}$$

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Hunt for a "stress-only" formalism: active field in early 2000's :
J.-P. Bouchaud Les Houches Lectures

$$\partial_i \sigma_{ij} = \boxed{f_j^{ext}} .$$

Where have we seen this before ?

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Generates Gauge Transformations

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<u>Electrostatics</u>

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Formulated in the context of "Tensorial Spin Liquids": M. Pretko (2017)

Analog of Faraday's Law ?

$$\partial_i \sigma_{ij} = f_j^{ext} - \partial_t \Pi_j$$

$\vec{\Pi}$: momentum density

Define $\partial_i B_{ij} = \Pi_j$

σ & B : symmetric tensors

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Gives us the statics: stress-only description of the mechanical response of jammed solids

Compare to the "classical" theory of elasticity: crystals

Static limit

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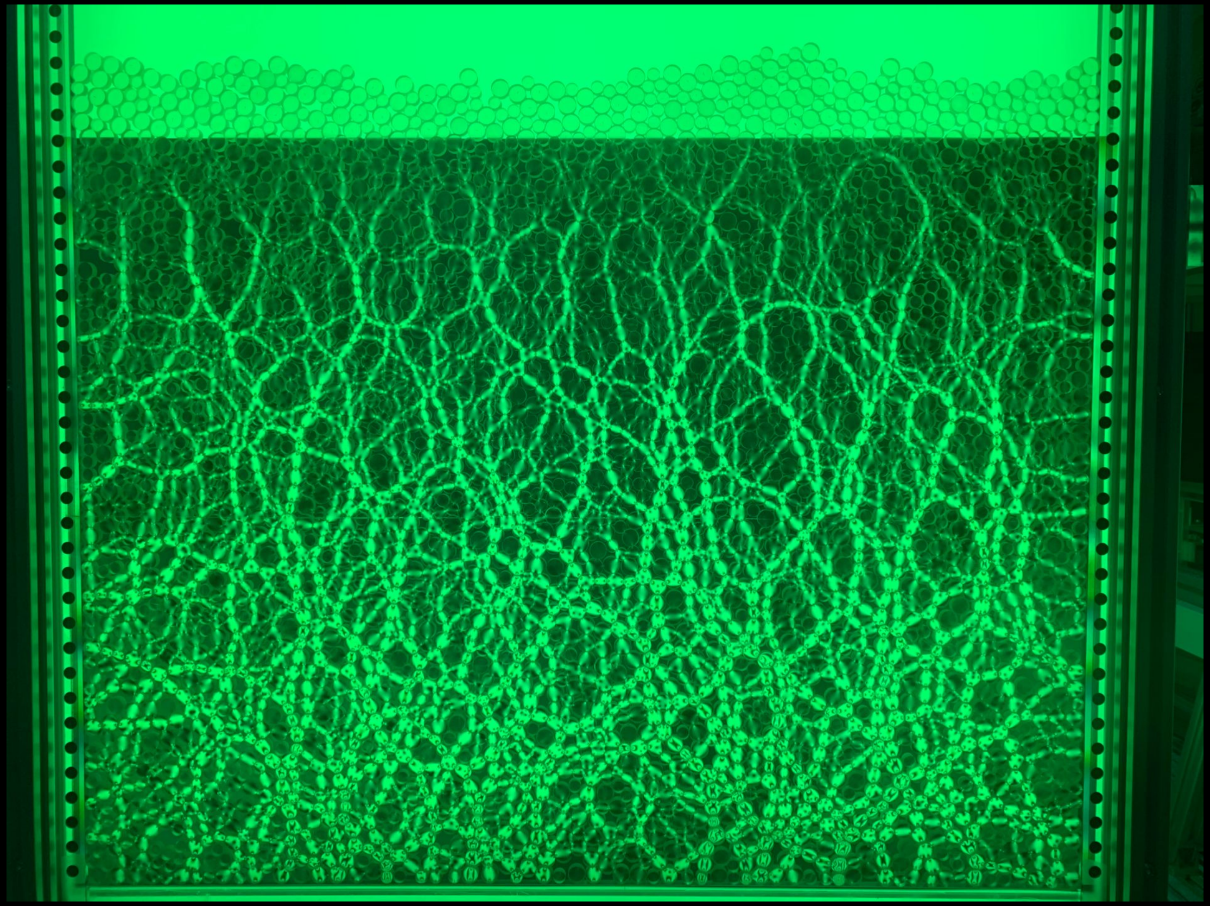
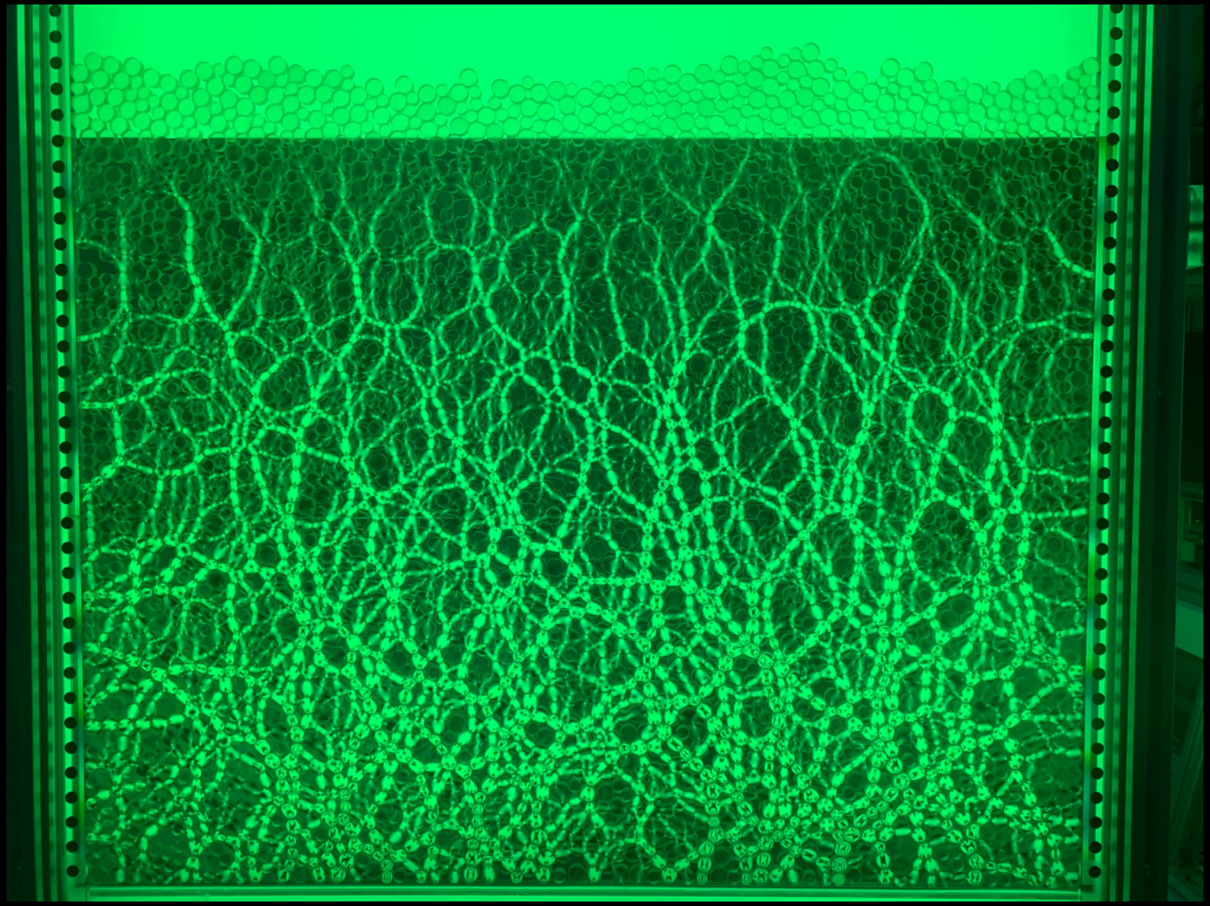
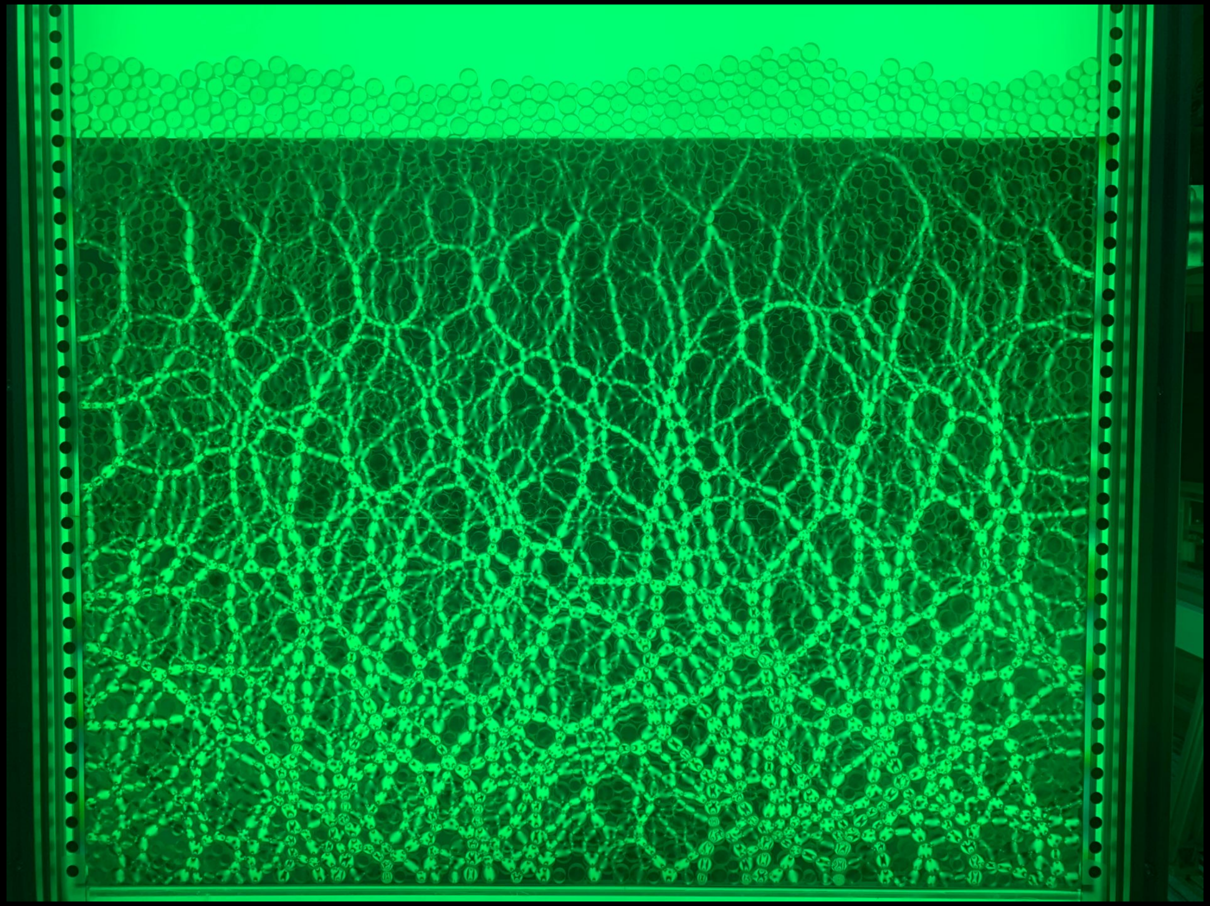
Yes ! Consider amorphous solids as a "polarizable" medium: a dielectric





This solid has been created by forces exerted at the boundary



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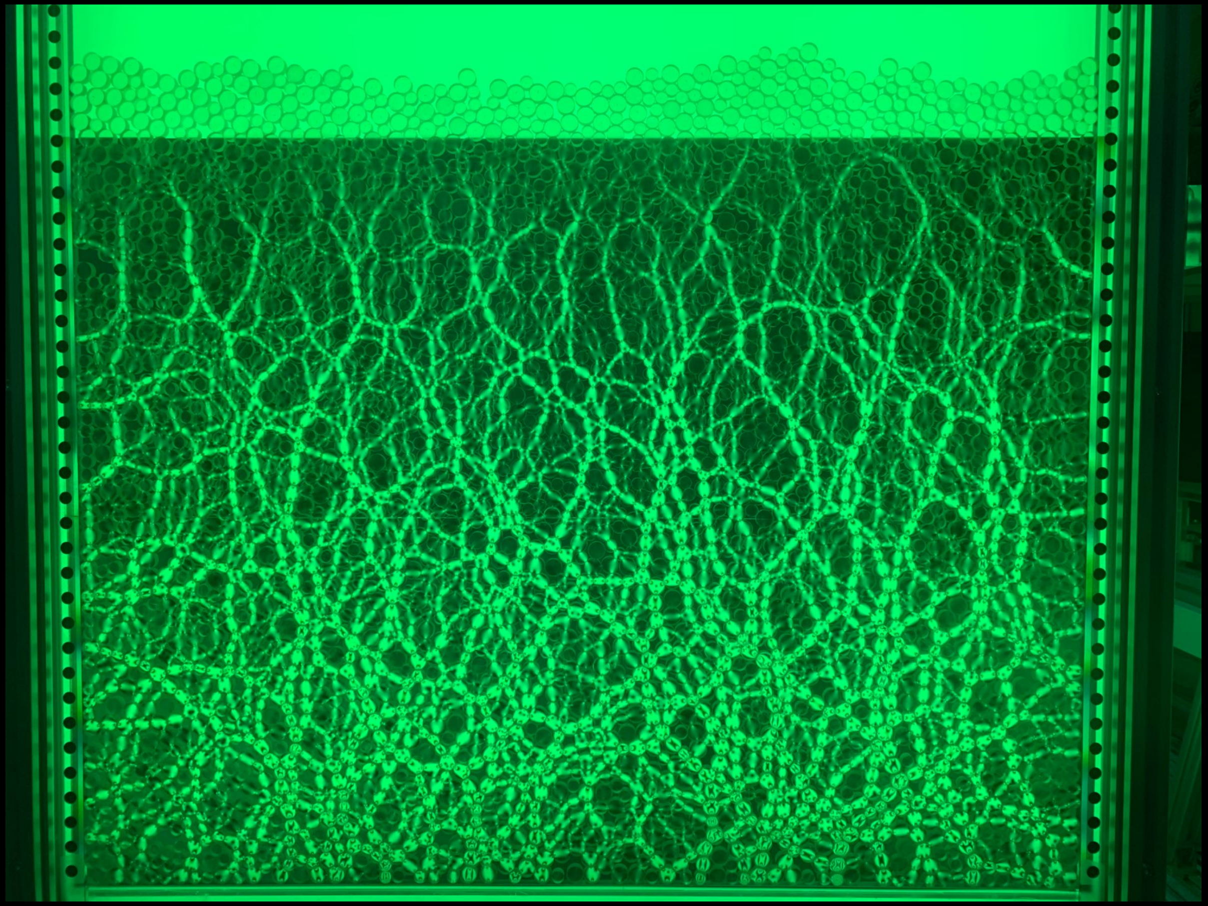
Electric field created by charges: no E field if there are no charges



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Bound charges (dipoles)
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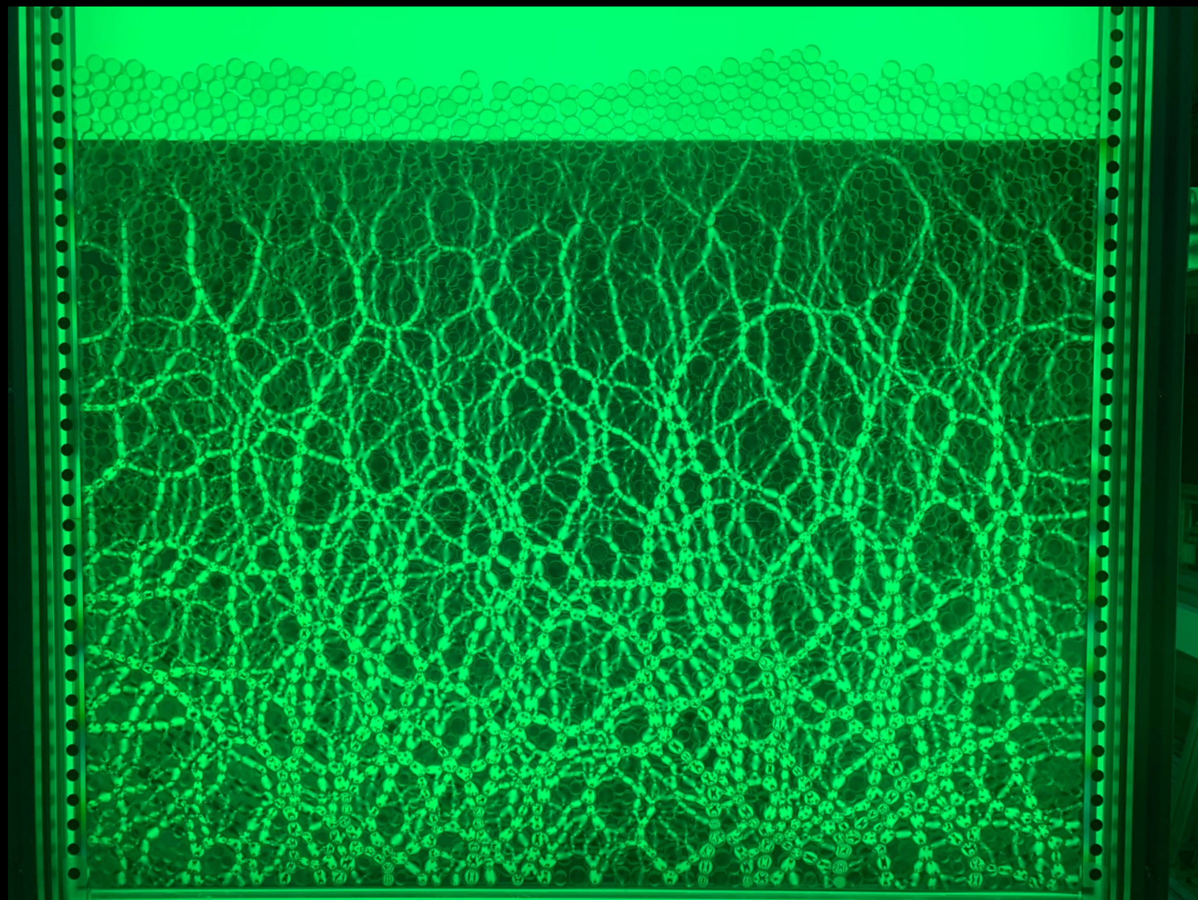

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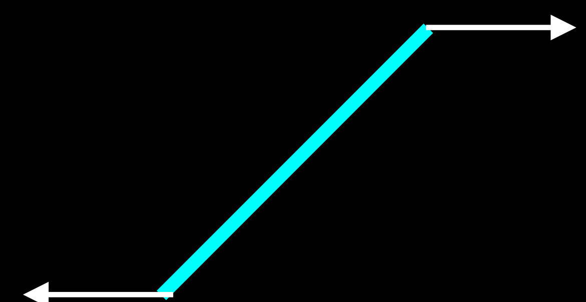

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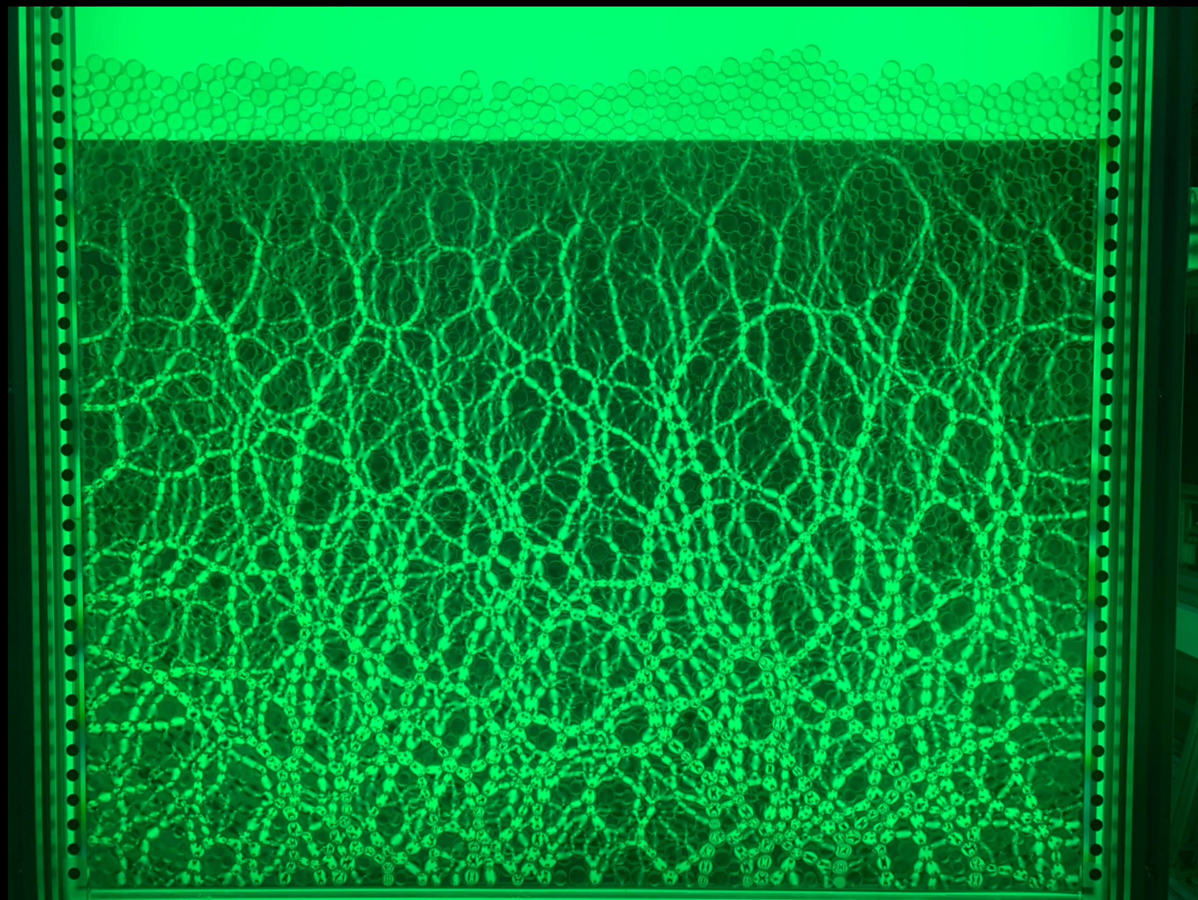

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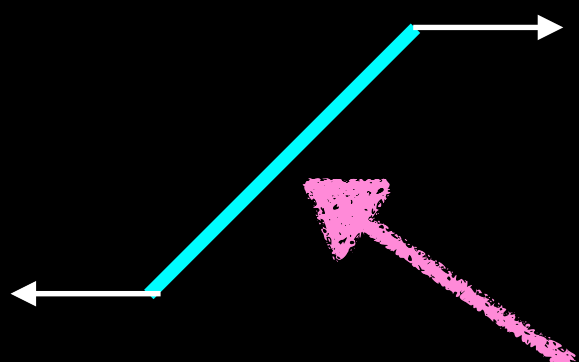
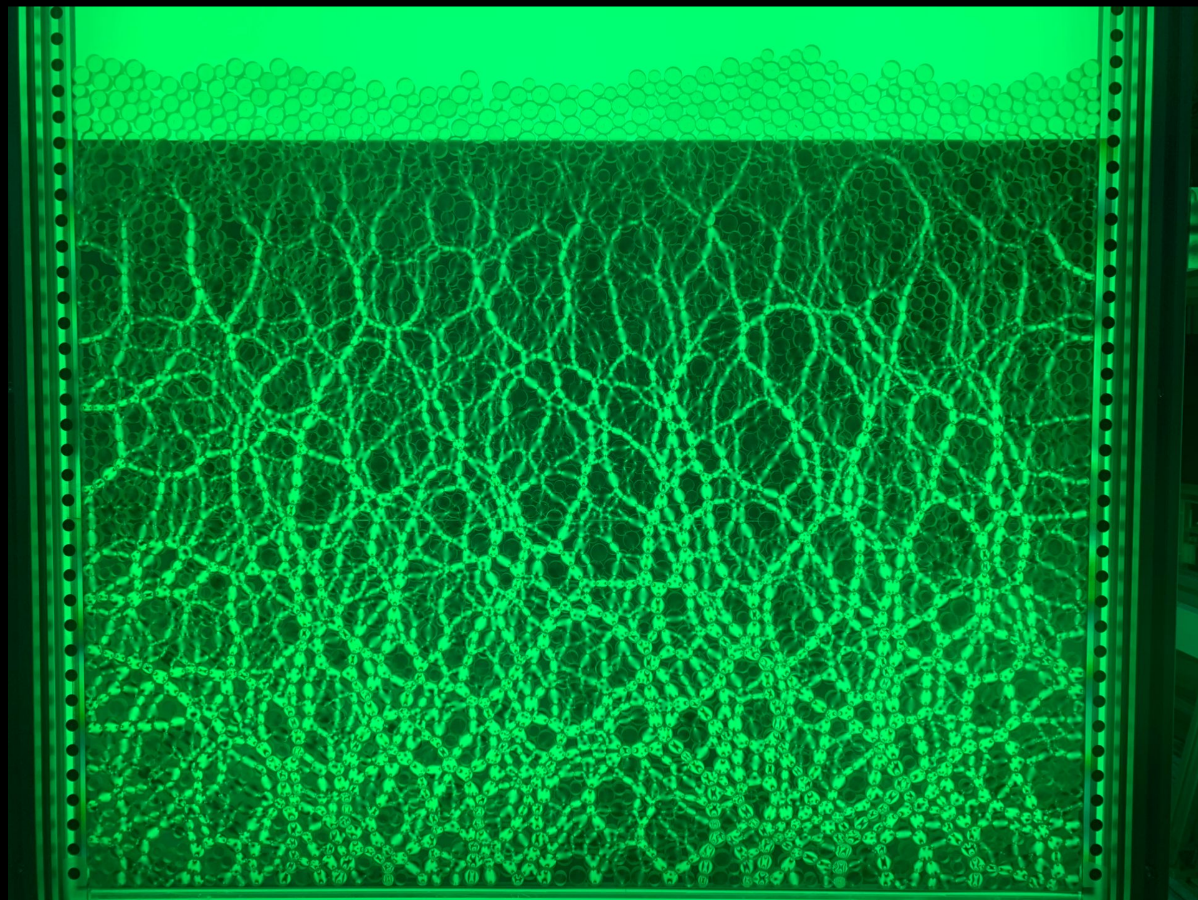


Dipole tensor: contact vector + contact force



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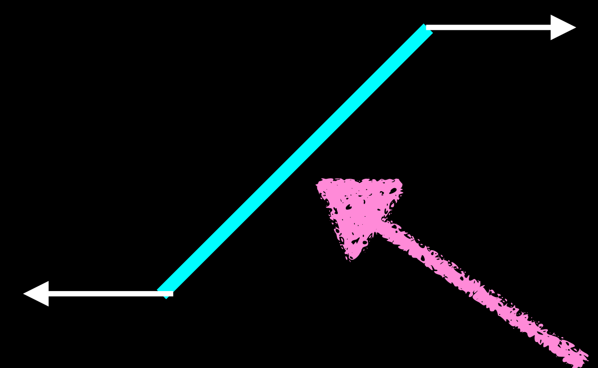

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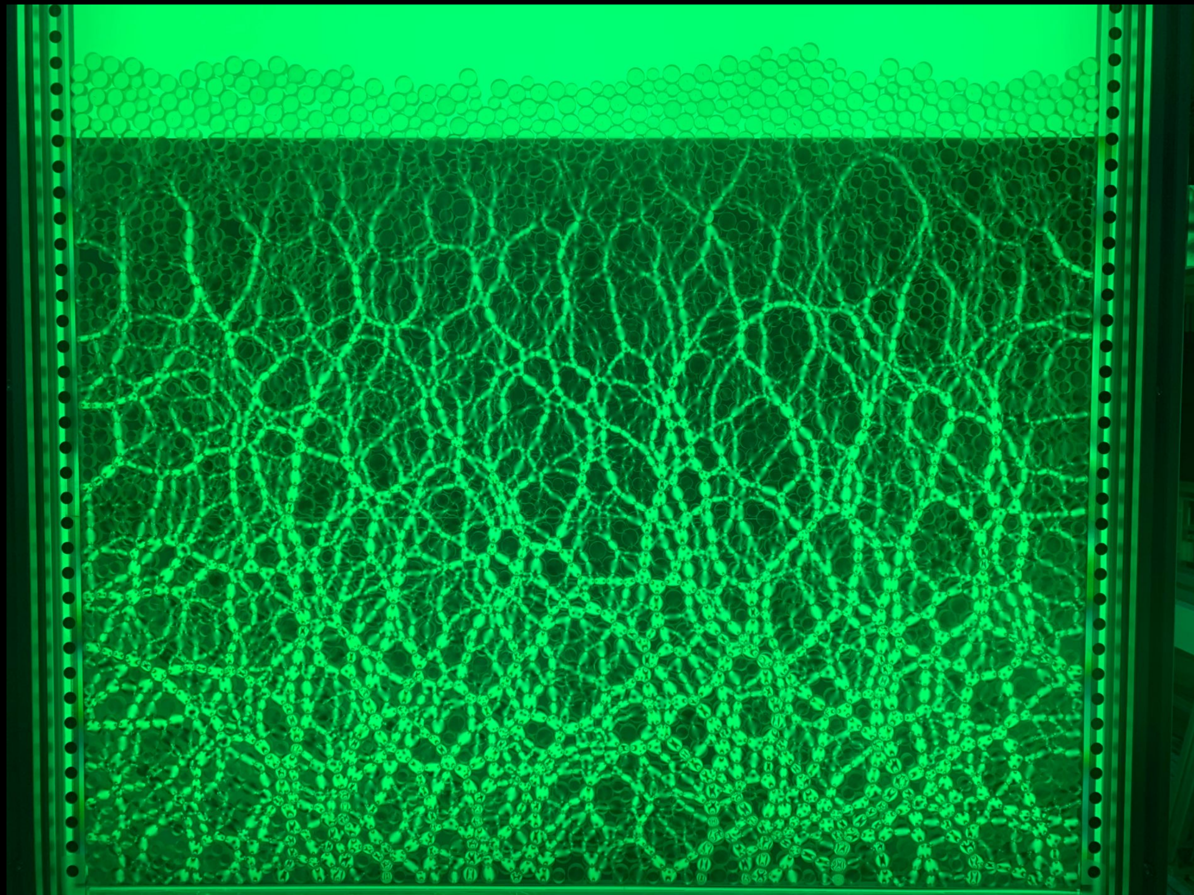
Jishnu N. Nampoothiri, Yinqiao Wang, Kabir Ramola, Jie Zhang, Subhro Bhattacharjee, and Bulbul Chakraborty
Phys. Rev. Lett. 125, 118002



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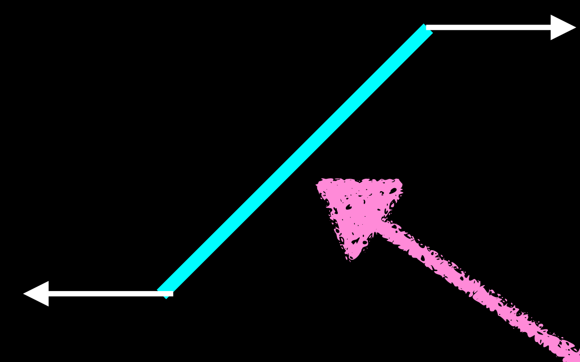

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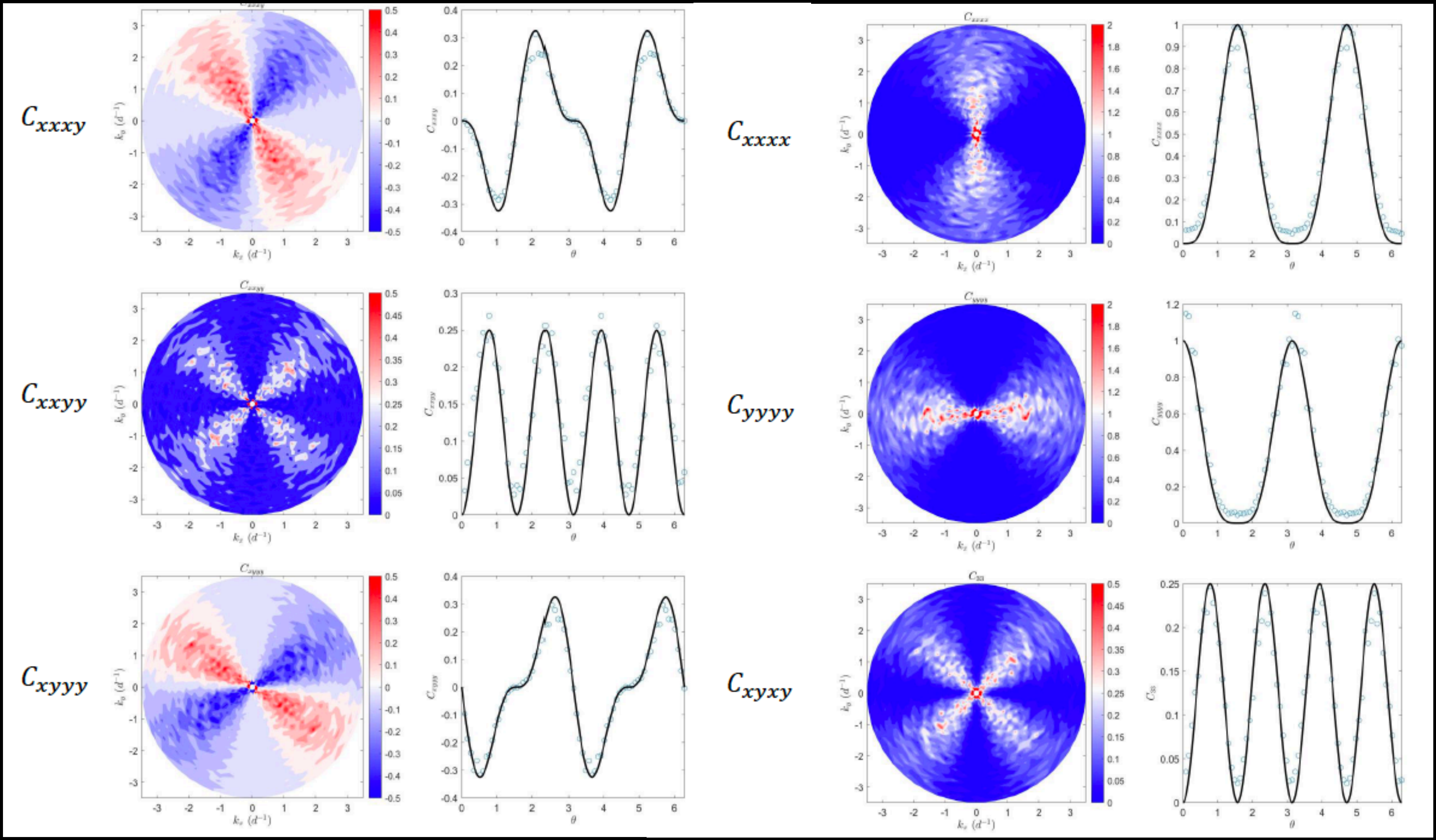
Polarizability maps to elastic modulus tensor: can be measured through stress-stress spatial correlations

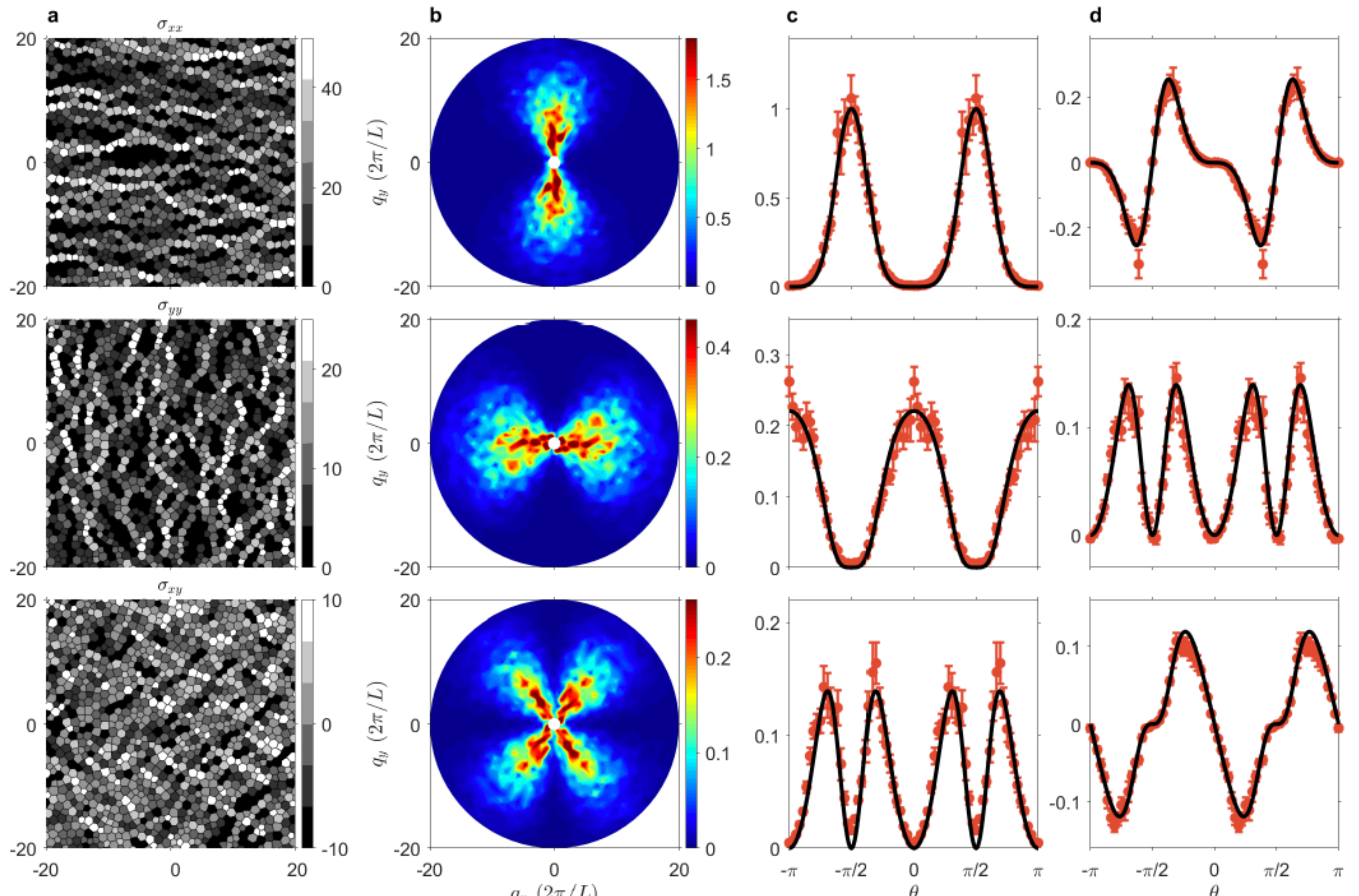
Experiments (Jie Z and Yinqiao W)

Stress-correlations (isotropic compression)

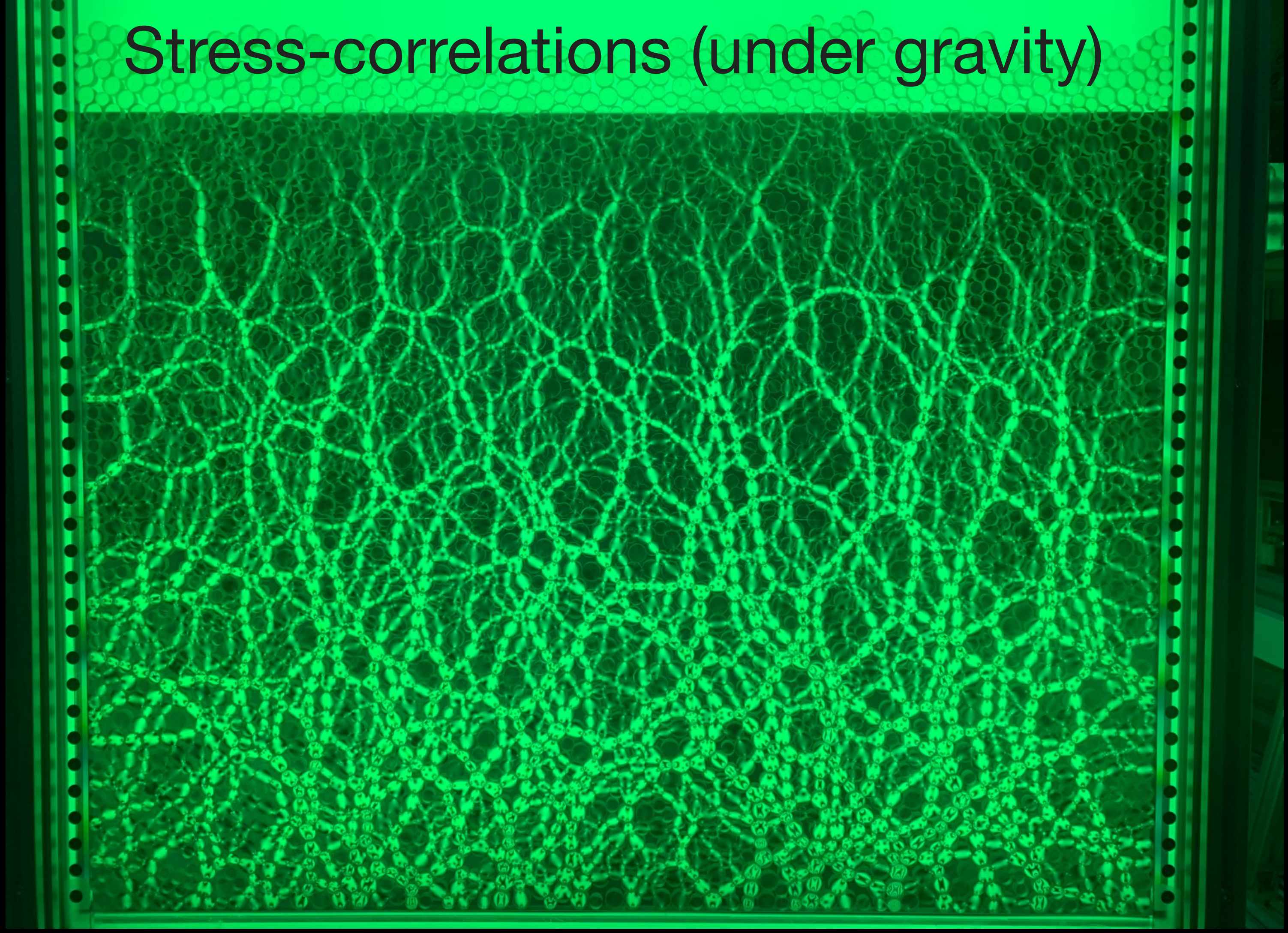


Quadrupolar structure (monopoles and dipoles are conserved in this “E&M”)

Stress-correlations (Sheared)



Stress-correlations (under gravity)



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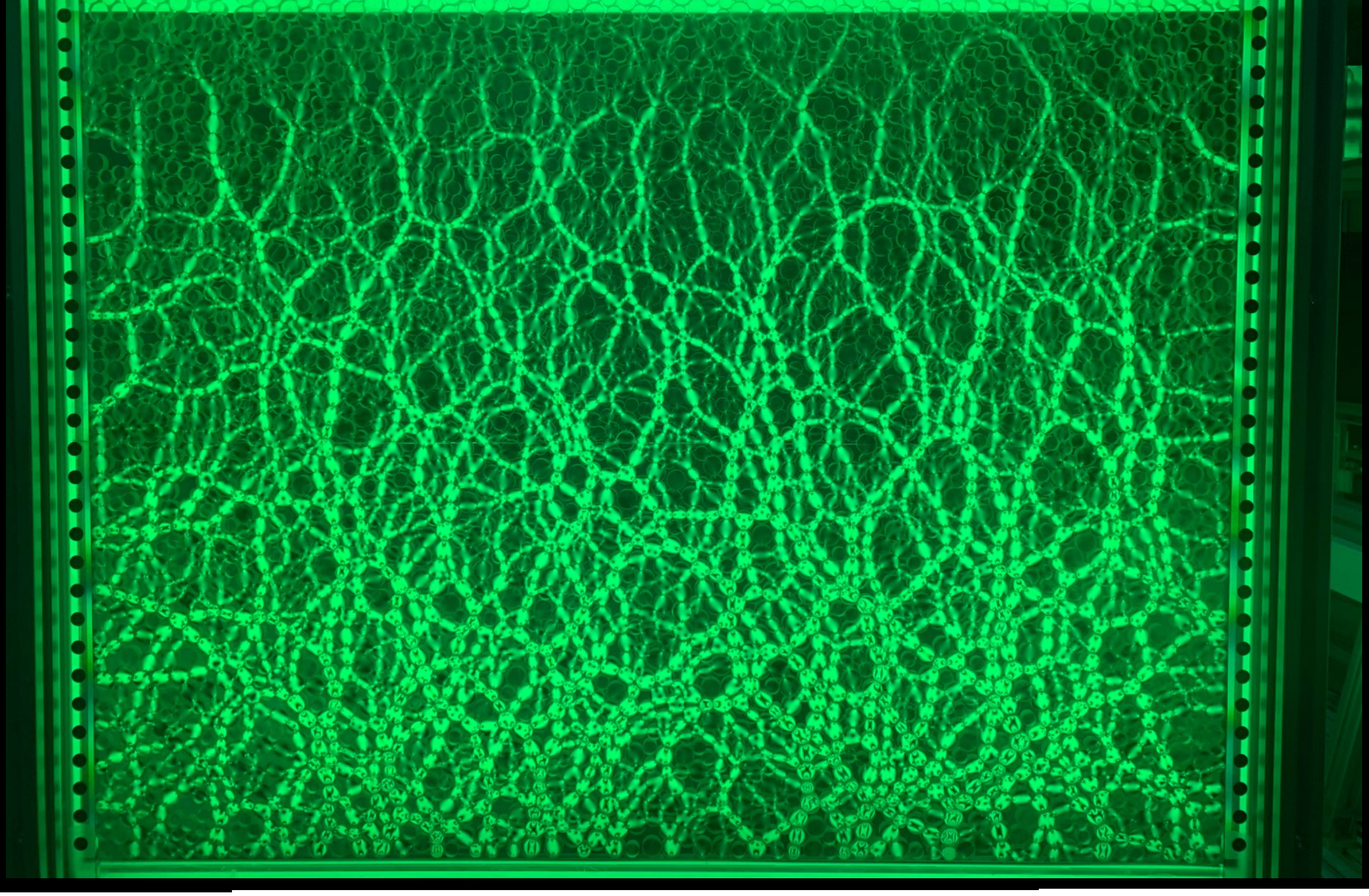
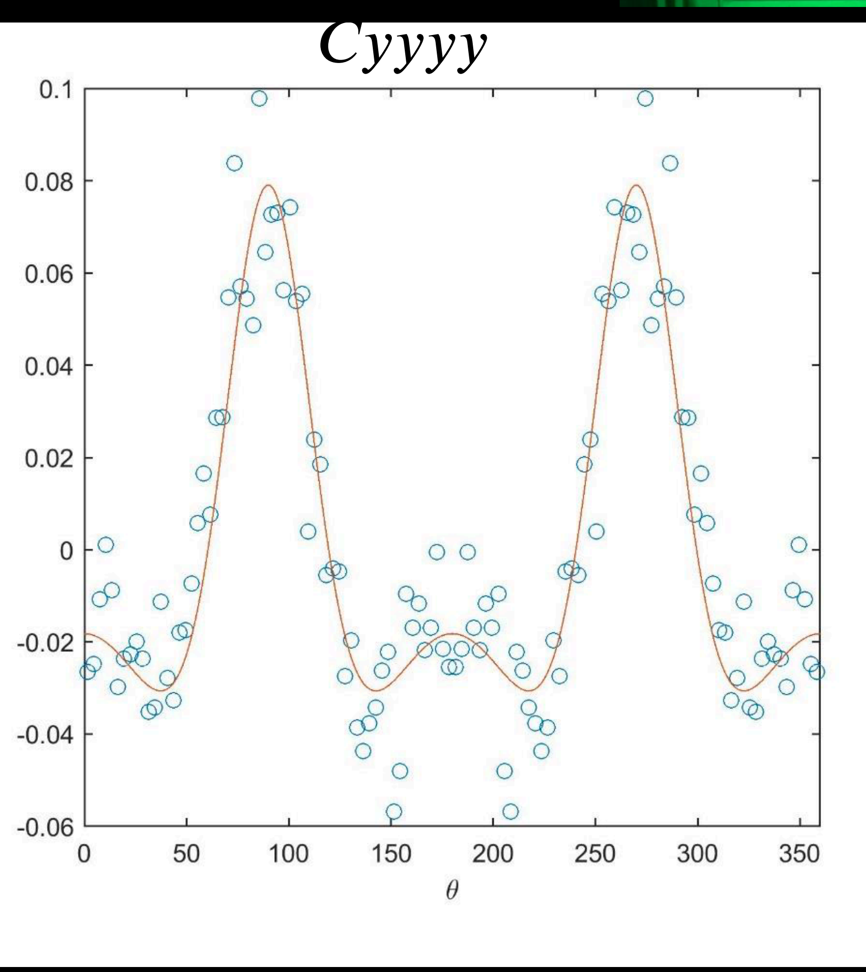
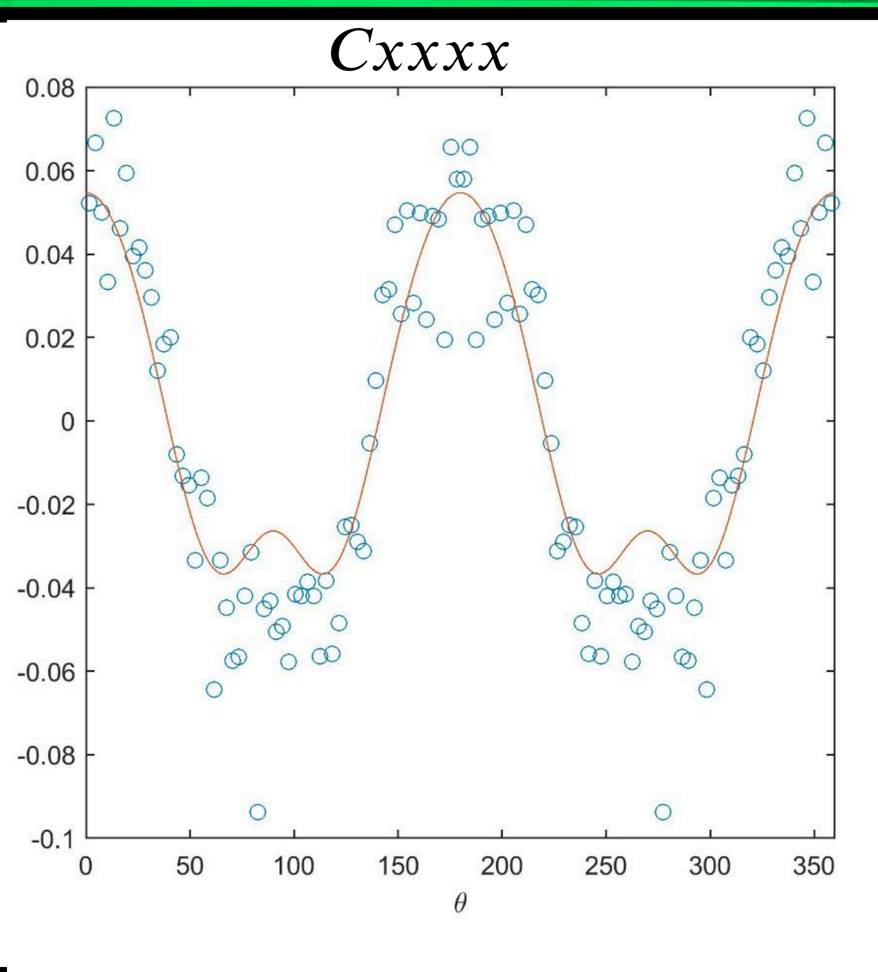
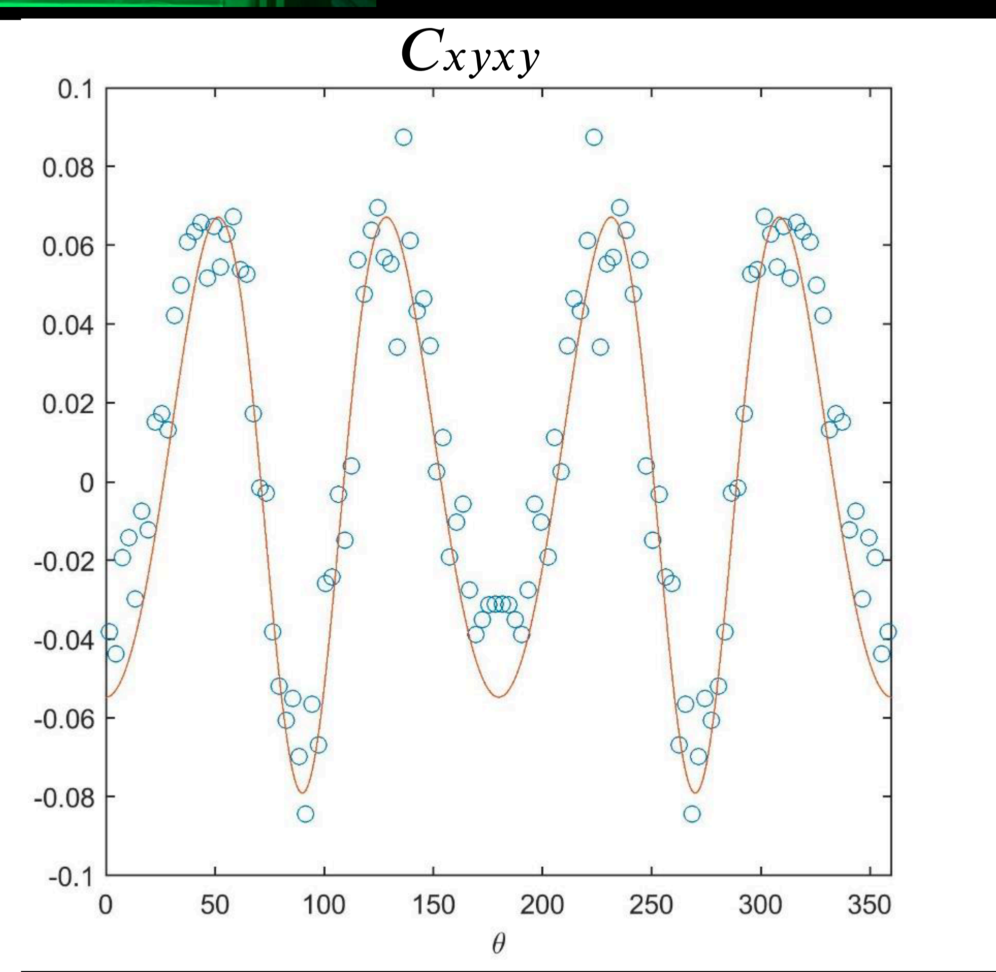
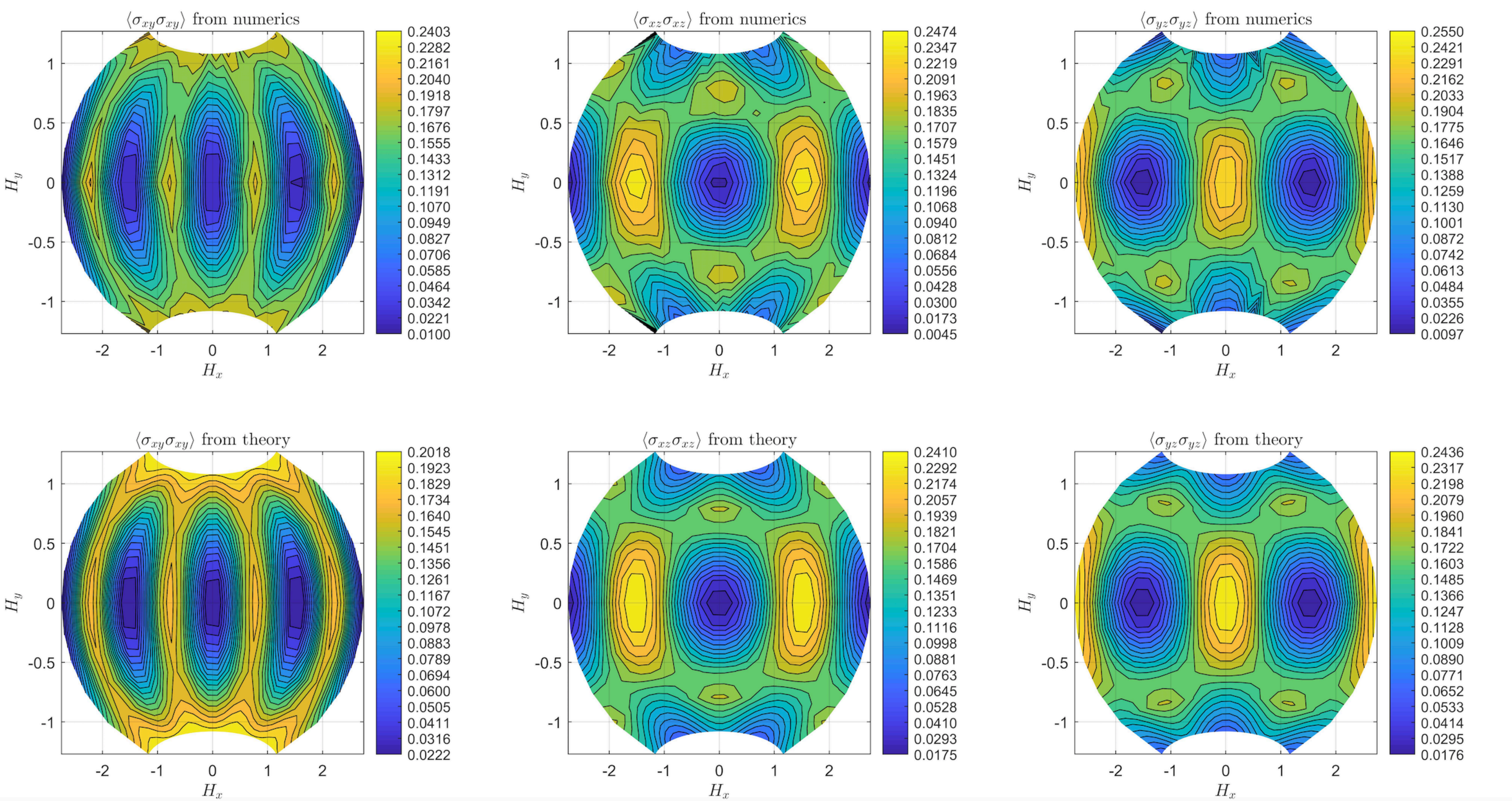




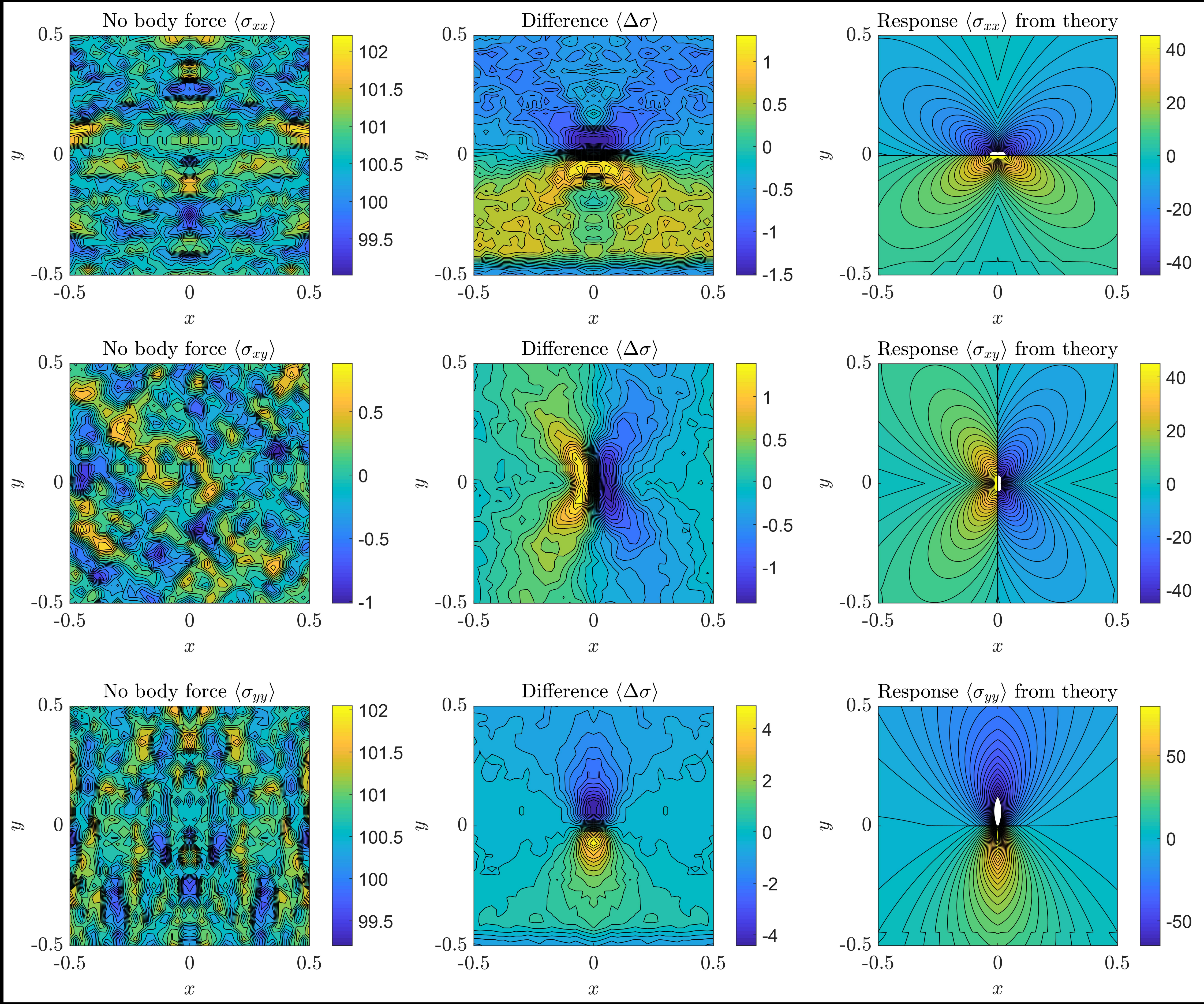




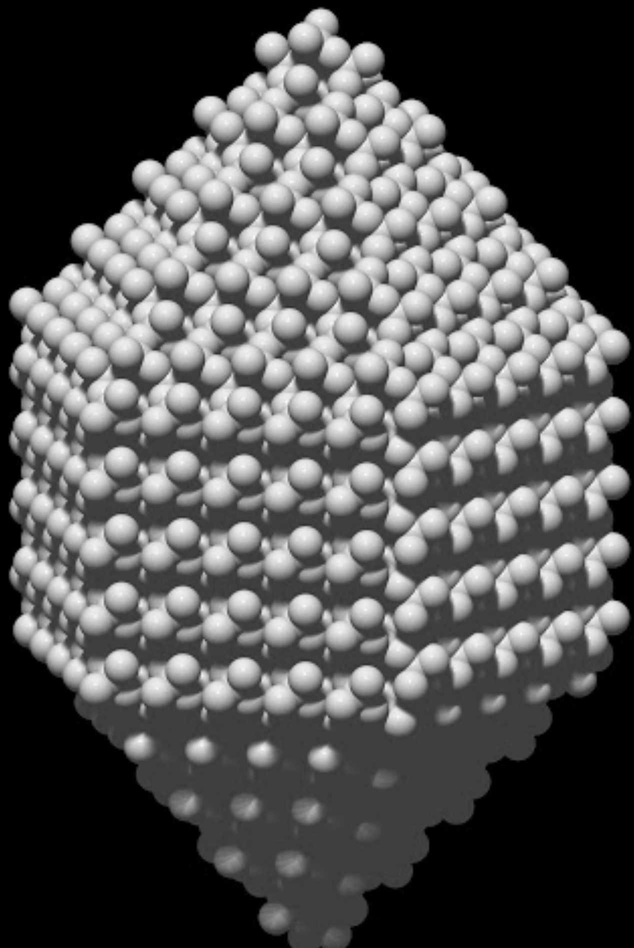
3D Simulations (frictionless): Stress-correlations (isotropic compression)



2D Simulations (frictionless)
Response to point force (isotropic compression)

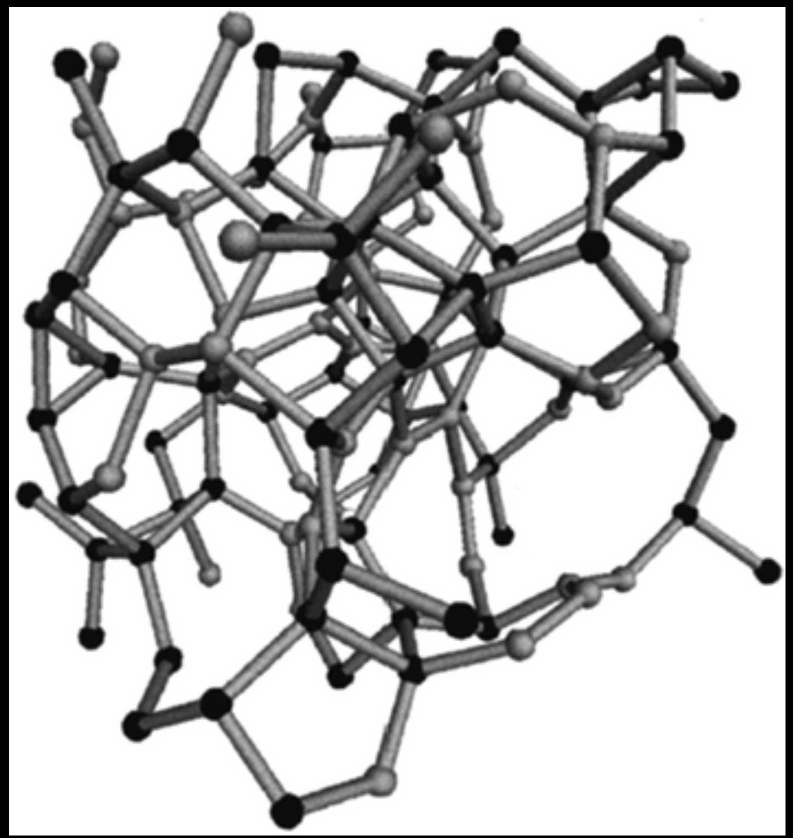


Different Paradigms



Elasticity from broken symmetry

Elasticity from constraints and
an emergent gauge theory



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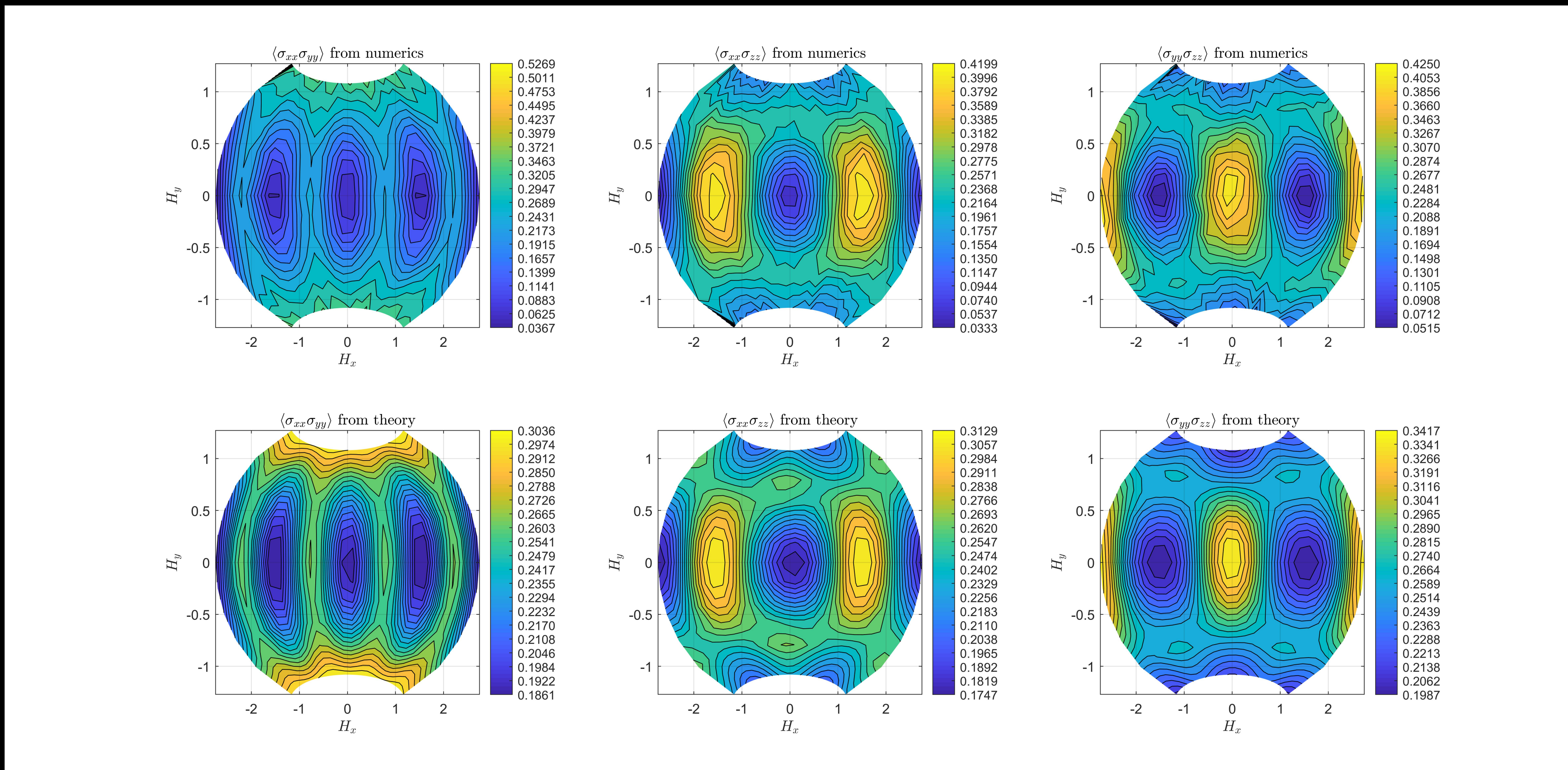

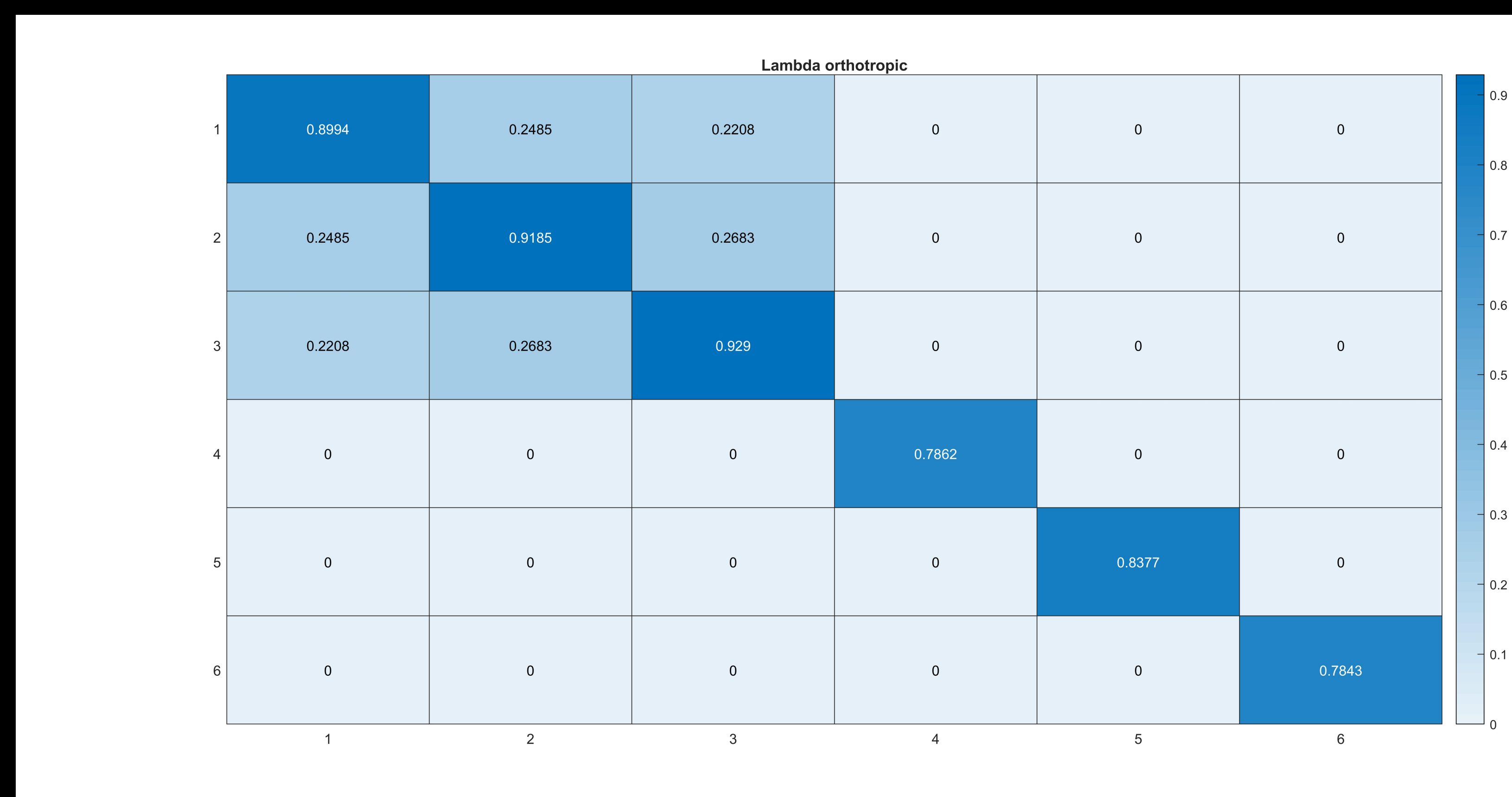

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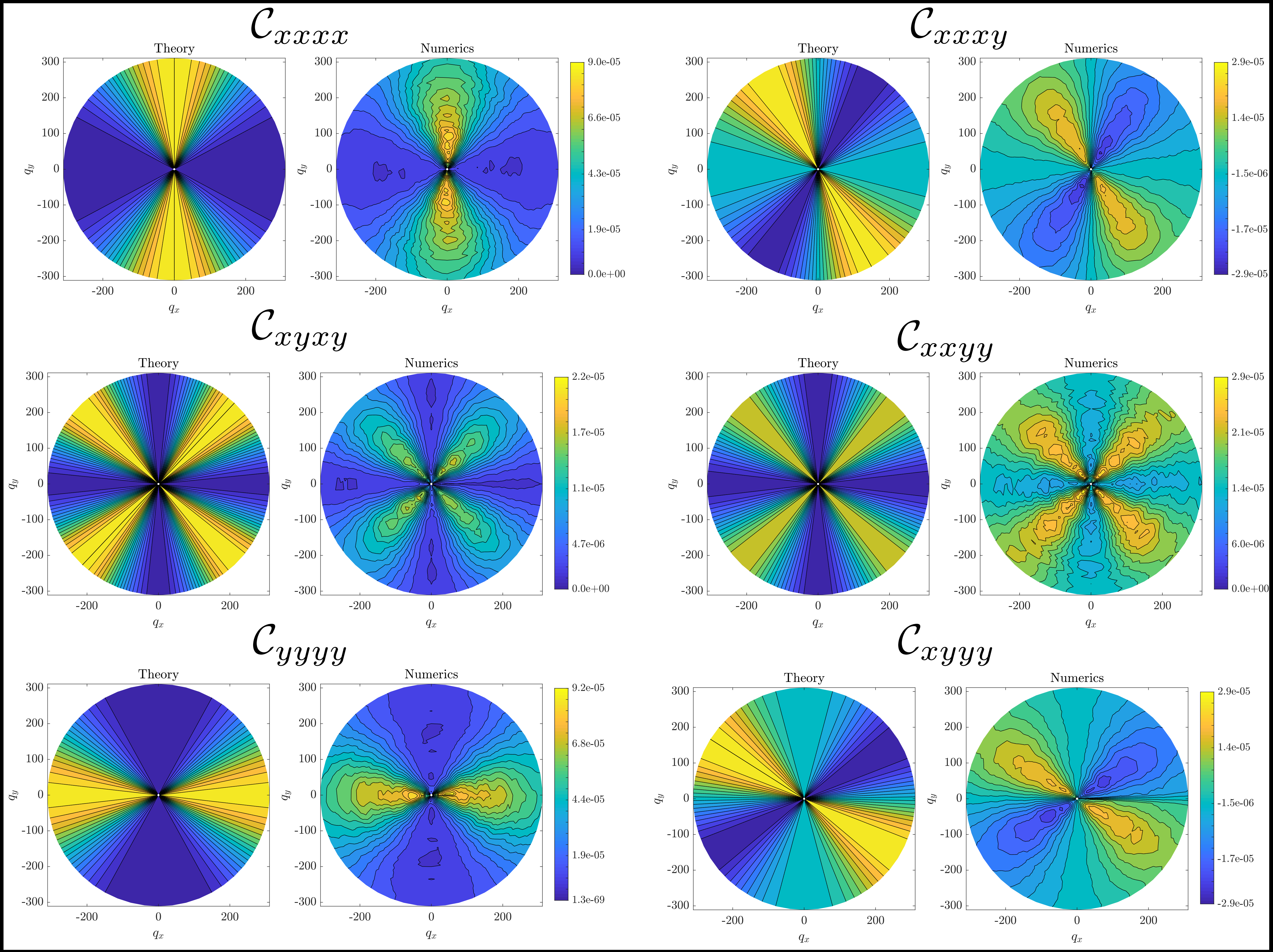
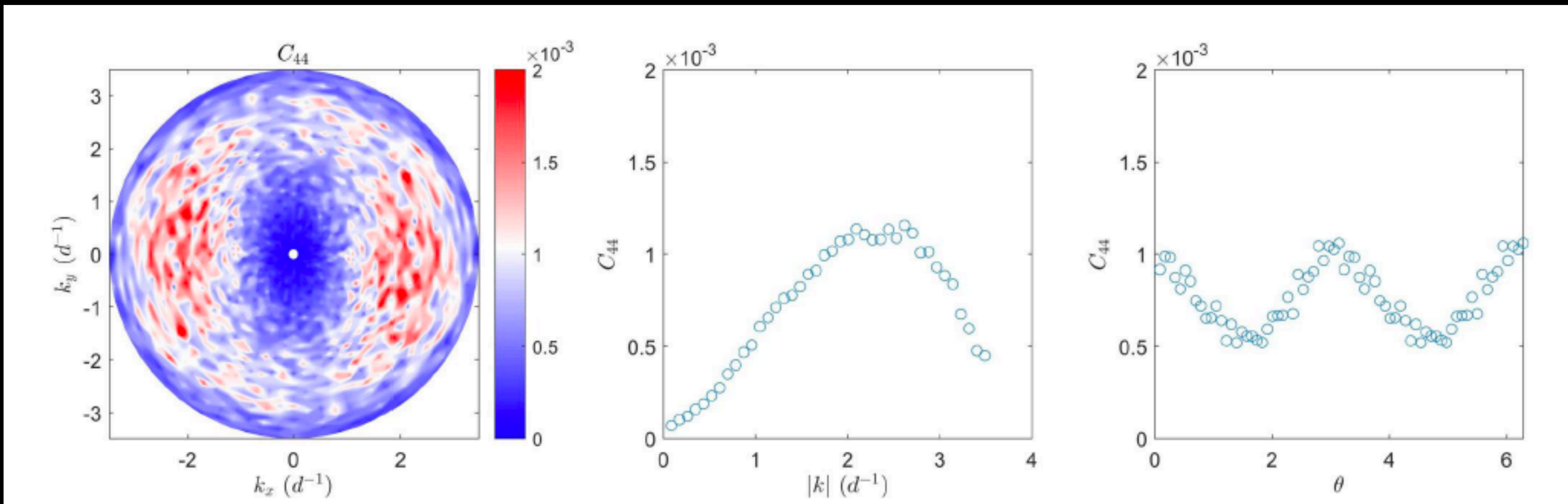
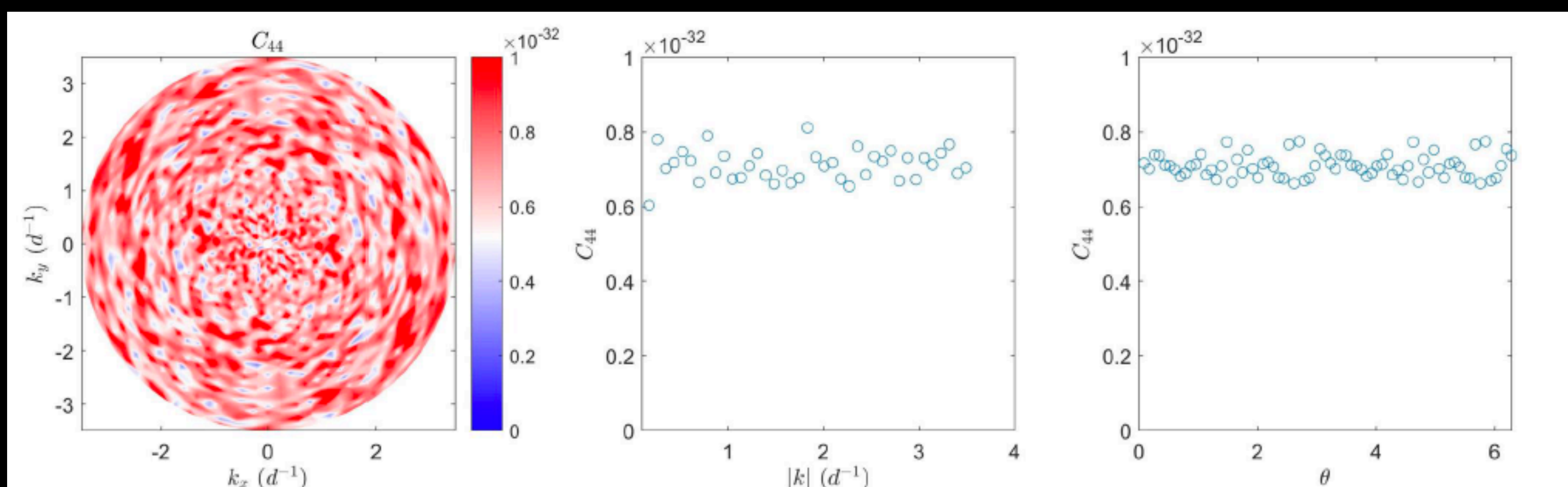


Torque-correlations

Contact-stresses



Particle-stresses



Dielectric formalism

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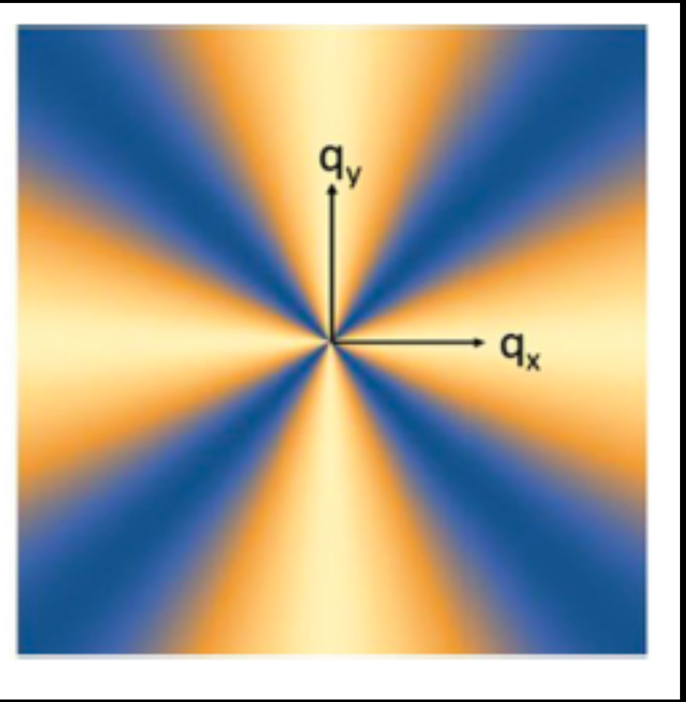
Like displacements BUT these are gauge potentials NOT physically observable fields



q_y
q_x