Weak universality, quantum many-body scars and anomalous autocorrelations in a one-dimensional spin model with duality

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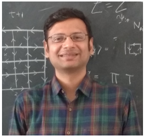


Udupa et al, Phys. Rev. B 108, 214430 (2023)

Outline

- Motivation: transverse field Ising model
- Model with three-spin Ising interaction and transverse field: duality, symmetries and energy spectrum
- Critical exponents and universality class
- Distribution of energy level spacing
- Zero energy states and scar states
- Autocorrelation functions near one end of the system

General motivation

A well-known one-dimensional model with duality is the transverse field Ising model

This model is integrable and has a duality.
The self-dual point corresponds to a quantum critical point, separating an ordered phase from a disordered phase

We want to generalize this to other models with duality but which have a different symmetry and may not be integrable

The nature of the self-dual point is then interesting to study

General motivation

A family of such models was introduced long ago and some of the models were studied using mean-field theory and series expansions

The models have Ising interactions involving p neighboring sites and a transverse field at all sites

Turban, J. Phys. C 15, L65 (1982)

Penson, Jullien and Pfeuty, Phys. Rev. B 26, 6334 (1982)

Igloi, Kapor, Skrinjar and Sólyom, J. Phys. A 19, 1189 (1986)

The transverse field Ising model corresponds to the case $p = 2$, summarized in the next two slides

We will then present a detailed study of the model with $p = 3$. Similar models can be realized in triangular configurations of optical lattices with two atomic species

Recall: model with two-spin Ising interaction

Using X, Y, Z to denote the Pauli matrices $\sigma^x, \sigma^y, \sigma^z$, the Hamiltonian for the transverse field Ising model is

$$H_2 = -\sum_{n=1}^{L} Z_n Z_{n+1} - h \sum_{n=1}^{L} X_n$$

for a L-site system with periodic boundary conditions

This has a duality: define a dual lattice with coordinates at the mid-points of the original lattice, and dual spin variables

$$\tilde{X}_{n+1/2} = Z_n Z_{n+1}$$

This gives $X_n = \tilde{Z}_{n-1/2} \tilde{Z}_{n+1/2}$. Hence the dual Hamiltonian is

$$\tilde{H}_2 = -\sum_{n=1}^{L} \tilde{X}_{n+1/2} - h \sum_{n=1}^{L} \tilde{Z}_{n-1/2} \tilde{Z}_{n+1/2}$$

Model with two-spin Ising interaction $\cdots$

Thus the Hamiltonian changes from

$$H_2 \;=\; -\sum_{n=1}^{L} Z_n\, Z_{n+1} \;-\; h \sum_{n=1}^{L} X_n$$

to

$$\tilde{H}_2 \;=\; -\sum_{n=1}^{L} \tilde{X}_{n+1/2} \;-\; h \sum_{n=1}^{L} \tilde{Z}_{n-1/2}\, \tilde{Z}_{n+1/2}$$

Effectively, we have mapped $h \;\to\; 1/h$

There is a subtlety that, with periodic boundary conditions, the mapping only works if both $\prod_n \tilde{X}_{n+1/2} = 1$ and $\prod_n X_n = 1$

Model with three-spin Ising interaction

The Hamiltonian is

$$H_3 \;=\; -\sum_{n=1}^{L} Z_n\, Z_{n+1}\, Z_{n+2} \;-\; h \sum_{n=1}^{L} X_n$$

This also has a duality. The sites of the dual lattice coincide with the sites of the original lattice

We define

$$\tilde{X}_n \;=\; Z_{n-1}\, Z_n\, Z_{n+1}$$

This gives $X_n = \tilde{Z}_{n-1}\, \tilde{Z}_n\, \tilde{Z}_{n+1}$. Hence the dual Hamiltonian is

$$\tilde{H}_3 \;=\; -\sum_{n=1}^{L} \tilde{X}_n \;-\; h \sum_{n=1}^{L} \tilde{Z}_{n-1}\, \tilde{Z}_n\, \tilde{Z}_{n+1}$$

which effectively maps $h \to 1/h$

Model with three-spin Ising interaction

$$H_3 = - \sum_{n=1}^{L} Z_n Z_{n+1} Z_{n+2} - h \sum_{n=1}^{L} X_n$$

This commutes with three Z_2-valued operators. Assuming periodic boundary conditions and L to be a multiple of $3,$ the operators are

$$D_1 = \prod_{j=1}^{L/3} X_{3j-2} X_{3j-1}$$

$$D_2 = \prod_{j=1}^{L/3} X_{3j-1} X_{3j}$$

$$D_3 = \prod_{j=1}^{L/3} X_{3j-2} X_{3j}$$

Only two of these are independent operators since $D_1 D_2 D_3 = I$. Hence the system has a $Z_2 \times Z_2$ symmetry

Symmetries

Given the two commuting and conserved Z_2 operators, the states of the system lie in 4 sectors in which states have the following combinations of eigenvalues of the operators (D_1, D_2, D_3)

$(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$ and $(1, 1, 1)$

One-fourth of all the states lie in each of these sectors.

Next, translation by one site, T, commutes with H.
We can show that

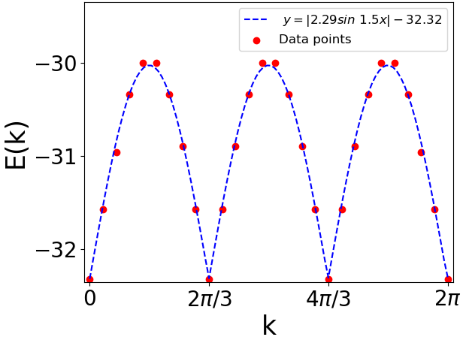
$$T\, D_1 T^{-1} = D_2 \,, \quad T\, D_2 T^{-1} = D_3 \,, \quad T\, D_3 T^{-1} = D_1$$

We can then prove that states $|\psi\rangle$ with momentum k, satisfying $T|\psi\rangle = e^{ik}|\psi\rangle$ will come with a 3-fold degeneracy with momenta k, $k + 2\pi/3$ and $k + 4\pi/3$ if they lie in the first three sectors, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$,

and will not be degenerate if they lie in the sector $(1, 1, 1)$

Energy-momentum dispersion at $h = 1$

We first used exact diagonalization to study this model with periodic boundary conditions



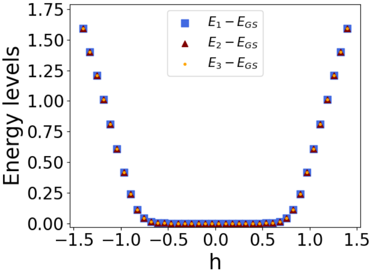
The system size $L = 27$. The dispersion is gapless and linear near $k = 0,\ 2\pi/3$ and $4\pi/3$. The velocity in those regions is $v = 3.44$

Four lowest energy levels

For $|h| > 1$, the ground state is unique and is separated by a gap from the next three energy levels (which are degenerate)

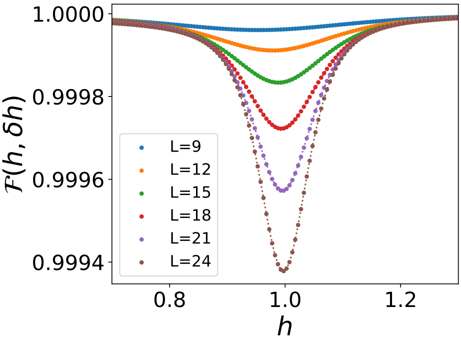
For $|h| < 1$, the ground state is separated from the next three energy levels by a gap which goes to zero exponentially as L increases. So there is a four-fold degeneracy as $L \to \infty$

This implies a symmetry breaking phase transition at $h = \pm 1$



$L = 15$

Ground state fidelity



The fidelity is defined as $\mathcal{F}(h) = \langle\psi_0(h-\delta h/2)|\psi_0(h+\delta h/2)\rangle$

We take $\delta h = 0.005$. There is a clear dip at $h = 1$

Critical exponents

Near a second-order quantum phase transition, various quantities scale as powers of $h - h_c$ as $L \to \infty$ or powers of L at $h = h_c$

The correlation length scales as

$$\xi \sim |h - h_c|^{-\nu}$$

The scaling of a quantity $\mathcal{O}$ is typically given by

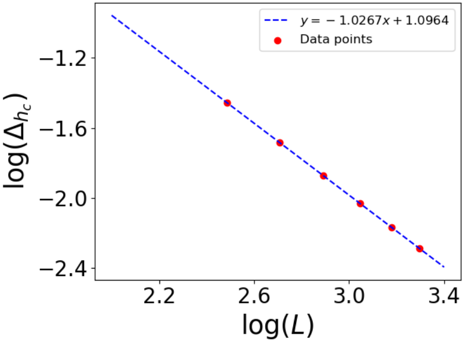
$$\mathcal{O} \sim |h - h_c|^{-\theta} \sim \xi^{\theta/\nu} \quad \text{for} \quad L \gg \xi$$

$$\sim L^{\theta/\nu} \quad \text{for} \quad h = h_c$$

Dynamical critical exponent z

Taking $h = 1$ as the critical field, the dynamical critical exponent z gives the scaling with the system size L of the energy gap $\Delta E = E_1 - E_0$ between the ground state and the first excited state

$$\Delta E \quad \sim \quad L^{-z} \quad \text{for} \quad h = h_c$$



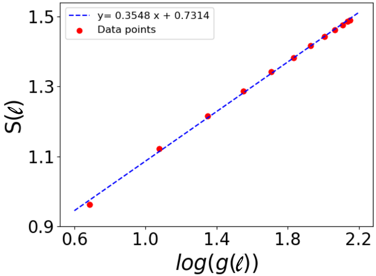
Using data for $L = 12,\ 15,\ 21,\ 24,\ 27$ we find $z \simeq 1.027$

Central charge

The fact that $z \simeq 1$ suggests that the low-energy sector at the critical point is Lorentz invariant and is described by a conformal field theory characterized by a central charge c.
The value of c can be extracted in two ways

The entanglement entropy between subsystems of lengths $l,\ L-l$ is

$$S(l) = \frac{c}{3} \ln \left[\frac{L}{\pi} \sin\left(\frac{\pi l}{L}\right) \right] + c'$$



This gives $c \simeq 1.064$

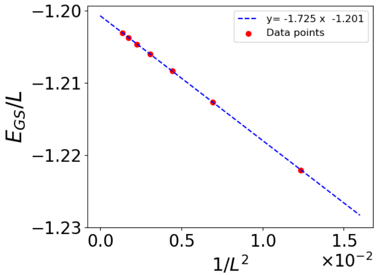
Central charge

At $h = h_c$, the first two terms in the ground state energy go as

$$E_0 = \alpha L - \frac{\pi v c}{6L}$$

where α is a non-universal constant, and $v = 3.44$ is the velocity near the gapless points where the dispersion is linear

Cardy, J. Phys. A 19, L1093 (1986)



This gives $c \simeq 0.959$. We see that $c = 1$

Magnetization

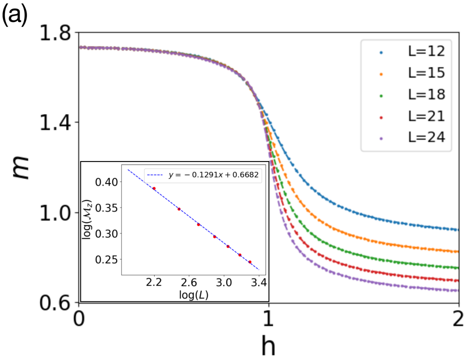
Taking the three-sublattice structure into account, we define sublattice order parameters

$$m_A = \frac{3}{L}\sum_{j=1}^{L/3} Z_{3j-2}, \quad m_B = \frac{3}{L}\sum_{j=1}^{L/3} Z_{3j-1}, \quad m_C = \frac{3}{L}\sum_{j=1}^{L/3} Z_{3j}$$

We then define a combined order parameter

$$m = [\,\langle m_A^2 + m_B^2 + m_C^2 \rangle\,]^{1/2}$$

Magnetization exponent β



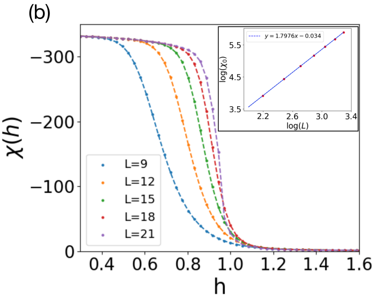
From $m|_{h_c} \sim L^{-\beta/\nu}$, we find that $\beta/\nu \simeq 0.129$

Susceptibility exponent γ

We add a longitudinal field h_z in the Hamiltonian

$$H_3 = -\sum_{n=1}^{L} Z_n Z_{n+1} Z_{n+2} - h \sum_{n=1}^{L} X_n - h_z \sum_{n=1}^{L} Z_n$$

At the critical point $h = h_c,$ the magnetic susceptibility $\chi = (\partial m / \partial h_z)_{h_z \to 0}$ scales as $L^{\gamma/\nu}$



We find that $\gamma/\nu \simeq 1.788$

Weak universality

The facts that $z = 1$, $c = 1$, and β/ν and γ/ν are close to $1/8$ and $7/4$ suggest that the critical point of this model lies on the Ashkin-Teller (AT) line of critical models which exhibit weak universality

Conformal field theories with $c = 1$ have a marginal operator which can give rise to a line of critical points on which ν, β, γ can vary continuously but the ratios $\beta/\nu, \gamma/\nu$ do not change along the line

Weak universality

The Ashkin-Teller line is described by a Hamiltonian of the form

$$H_{AT} \;=\; \sum_n \big(\, \sigma_n^x \,+\, \tau_n^x \,+\, \lambda\, \sigma_n^x \tau_n^x \\ \quad + \, \sigma_n^z \sigma_{n+1}^z \,+\, \tau_n^z \tau_{n+1}^z \,+\, \lambda\, \sigma_n^z \tau_n^z \sigma_{n+1}^z \tau_{n+1}^z \,\big)$$

where $\sigma_n^a, \; \tau_n^a$ describe two spin-1/2 objects at each site, and λ is a parameter which varies along the line

Two special points on the AT line correspond to two copies of the transverse field Ising model $(\lambda = 0, \; \nu = 1)$ and the four-state Potts model $(\lambda = 1, \; \nu = 2/3).$ It is known that

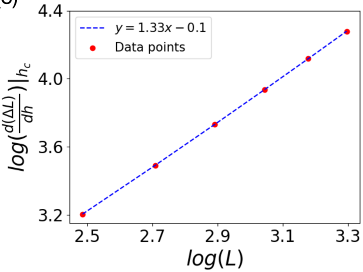
$$\nu \;=\; \frac{1}{2 \,-\, \frac{\pi}{2}\, [\arccos(-\lambda)]^{-1}}$$

Correlation length exponent ν

For our model, we can extract ν from the energy gap Δ at $h = h_c$ for different system sizes

$$\frac{d(\Delta L^z)}{dh}\Big|_{h_c} \sim L^{1/\nu}$$

(c)



We find that $\nu \simeq 0.754$

Correlation length exponent ν

Given the Ashkin-Teller line of Hamiltonians

$$H_{AT} = \sum_n \left(\sigma_n^x + \tau_n^x + \lambda\, \sigma_n^x \tau_n^x \right.$$
$$\left. + \sigma_n^z \sigma_{n+1}^z + \tau_n^z \tau_{n+1}^z + \lambda\, \sigma_n^z \tau_n^z \sigma_{n+1}^z \tau_{n+1}^z \right)$$

and the expression

$$\nu = \frac{1}{2 - \frac{\pi}{2}\, [\arccos(-\lambda)]^{-1}}$$

the value $\nu \simeq 0.754$ implies that $\lambda \simeq 0.827$

However this disagrees with results that we get from the Binder cumulant

Binder cumulant U_2

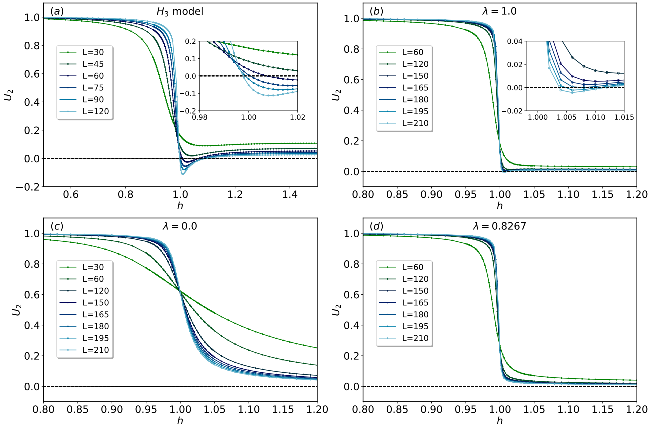
In our model, the Binder cumulant is defined as

$$U_2 = 2 - \frac{\langle m^4 \rangle}{\langle m^2 \rangle^2}$$

$$m^2 = \frac{1}{L^2} \left[(\sum_n \sigma_n^z)^2 + (\sum_n \tau_n^z)^2 \right]$$

U_2 is defined in such a way that it approaches 1 and 0 as h approaches 0 and ∞ respectively

Binder cumulant U_2



We see that the Binder cumulant is non-monotonic both for our model and the four-state Potts model, but is monotonic for the transverse field Ising model $(\lambda = 0)$ and $\lambda \simeq 0.827$

Binder cumulant U_2

The Binder cumulant suggests that our model lies close to the four-state Potts model $(\lambda = 1)$ rather than near $\lambda \simeq 0.827$ on the Ashkin-Teller line

It is known that there are significant log corrections at the critical point of the four-state Potts model. These may make estimates of critical exponents from exact diagonalization of small systems unreliable

We therefore studied the system using the density-matrix renormalization group (DMRG) method which can go up to much large system sizes, but with open boundary conditions

DMRG studies

Including log corrections, we use the expressions

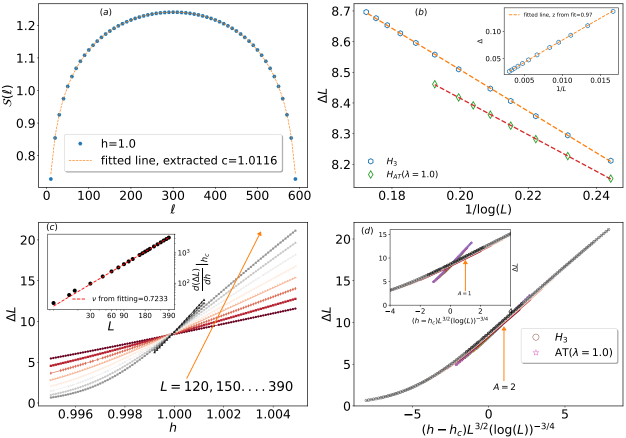
$$\frac{\Delta|_{h_c} L}{|h_c|} = a^* + \frac{b}{\ln(L)} + \cdots$$

$$\Delta(h)\, L = \mathcal{F}(A(h-h_c)L^{3/2}(\ln L)^{-3/4})$$

as where $\mathcal{F}$ is a universal function, to fit the DMRG data

We find that the DMRG data for our model is consistent with the four-state Potts model with $\lambda = 1$ and $A = 2$

DMRG results



(a) Entanglement entropy showing $c = 1.012,$ (b) ΔL at $h = h_c$ showing agreement between our model and the four-state Potts model. (c-d) $\Delta(h)\, L$ versus h

Why does the four-state Potts model appear?

The Hamiltonian on the Ashkin-Teller line

$$
\begin{aligned}
H_{AT} \;=\; \sum_n \Big(& \sigma_n^x + \tau_n^x + \lambda\, \sigma_n^x \tau_n^x \\
& + \sigma_n^z \sigma_{n+1}^z + \tau_n^z \tau_{n+1}^z + \lambda\, \sigma_n^z \tau_n^z \sigma_{n+1}^z \tau_{n+1}^z \Big)
\end{aligned}
$$

has a $Z_2 \times Z_2$ symmetry for all $\lambda.$ But for the four-state Potts model with $\lambda = 1,$ the low-energy sector has an enhanced symmetry given by the permutation group of four objects

Dijkgraaf, Verlinde and Verlinde, Comm. Math. Phys. 115, 649 (1988)

Why does this symmetry appear at the critical point of our three-spin model?

Why does the four-state Potts model appear?

Qualitative argument: Consider the Hamiltonian

$$H_3 = - \sum_n Z_n Z_{n+1} Z_{n+2} - h \sum_n X_n$$

In the limit $h \ll 1$, this has four ground states given by the spin configurations

$$(Z_{3j-2}, Z_{3j-1}, Z_{3j}) = (\uparrow, \uparrow, \uparrow), (\uparrow, \downarrow, \downarrow), (\downarrow, \uparrow, \downarrow), (\downarrow, \downarrow, \uparrow)$$

for all j. Hence the three sublattice magnetizations can take four possible values

$$(\langle m_A \rangle, \langle m_B \rangle, \langle m_C \rangle) = (1, 1, 1)\ (1, -1, -1), (-1, 1, -1), (-1, -1, 1)$$

These define the corners of a tetrahedron, and the symmetry group of a tetrahedron is the permutation group of four objects

Energy level spacing distribution

To probe the integrability or nonintegrability of the model, we studied the distribution of the energy level spacing in a particular symmetry sector of the model

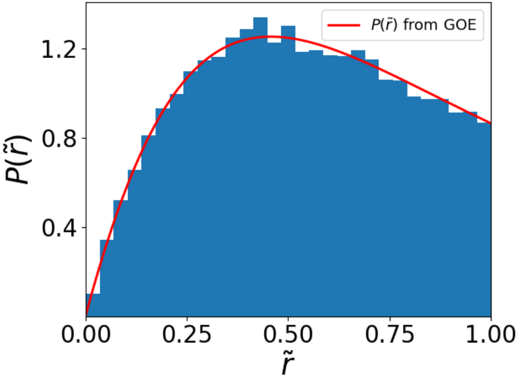
$$
\begin{aligned}
s_n &= E_{n+1} - E_n \\
\tilde{r} &= \frac{\min(s_n, s_{n-1})}{\max(s_n, s_{n-1})}
\end{aligned}
$$

We find that the average value of $\tilde{r}$ is 0.533 which describes the Gaussian orthogonal ensemble (GOE). Hence the model seems to be nonintegrable

We also find that the probability distribution $P(\tilde{r})$ is in good agreement with the Wigner-Dyson distribution

$$
P(r) = \frac{27}{4} \frac{r + r^2}{(1 + r + r^2)^{5/2}} \Theta(1 - r)
$$

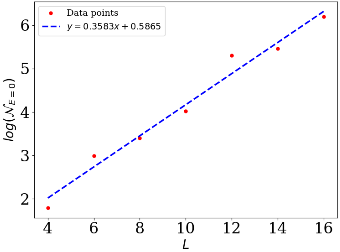
Energy level spacing distribution



Probability distribution of $\tilde{r}$ in the symmetry sector with $(D_1, D_2, D_3) = (1, -1, -1)$ for an $18-$site system with open boundary conditions, for $h = 1$ (the plot does not vary much with h)

Zero energy states

For $H_3 = -\sum_n \, [Z_n \, Z_{n+1} \, Z_{n+2} \; + \; h \, X_n],$ the operator $C = \prod_{n=1}^{L} \; Y_n$ anticommutes with H_3. This implies that the energies must appear in $\pm \, E$ pairs. If L is even, there are also zero energy states for any value of h. The number of such states grows as 1.43^L



This is consistent with an index theorem which says that the number of zero energy states must grow at least as fast as $(\sqrt{2})^L$

Schecter and Iadecola, Phys. Rev. B 98, 035139 (2018)

Zero energy states

It turns out that the zero energy states are of two types

Writing $H_3 = Z + h\,X,$ where

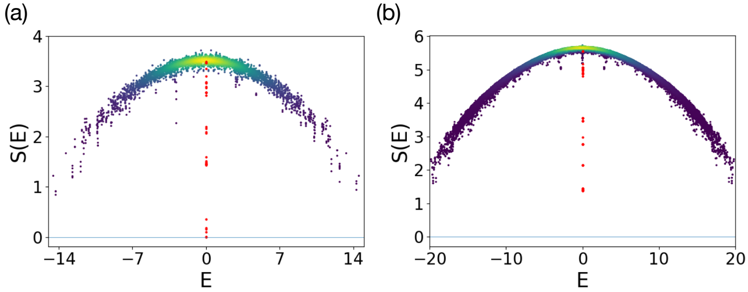
$$Z = -\sum_n Z_n\, Z_{n+1}\, Z_{n+2} \quad \text{and} \quad X = -\sum_n X_n$$

we find that there are type-I states $|\psi\rangle$ which satisfy $Z\,|\psi\rangle = 0$ and $X\,|\psi\rangle = 0$ separately, while type-II states $|\psi\rangle$ satisfy $H_3\,|\psi\rangle = 0,$ but not $Z|\psi\rangle = 0$ and $X\,|\psi\rangle = 0$ separately

Hence type-I states do not change at all as h is varied. Further, they have very low entanglement entropy between two halves of the system

Hence they qualify as many-body scars which violate ETH

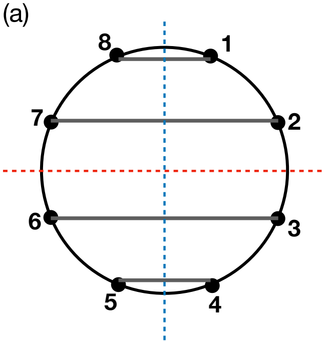
Type-I zero energy states



Half-chain entanglement spectrum for all the energy states of systems with 12 and 18 sites. The red points denote type-I scar states

Type-I scar states

We have analytically found some scar states built out of products of singlets (these form a subset of RVB states)



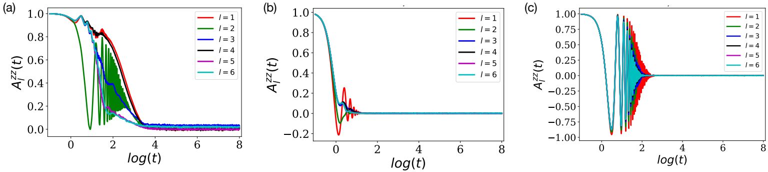
Some scar states for an $8-$ site system. The number of these states grows linearly with L. There are also scar states involving products of singlets and triplets

Autocorrelation functions near one end of the system

In an attempt to see if there are strong zero modes, we have looked at the infinite-temperature autocorrelation functions

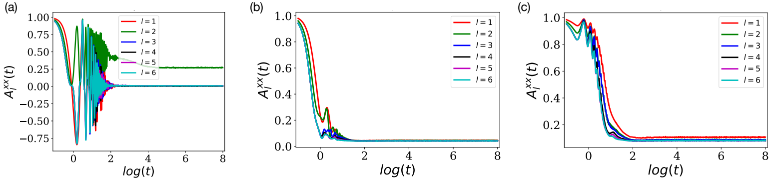
$$A_l^{zz}(t) = \frac{1}{2^L}\,\mathrm{Tr}[\sigma_l^z(t)\sigma_l^z]$$
$$A_l^{xx}(t) = \frac{1}{2^L}\,\mathrm{Tr}[\sigma_l^x(t)\sigma_l^x]$$

for sites l close to one end of the system



A^{zz} autocorrelation at sites $l = 1, 2, \cdots, 6$ for (a) $h = 0.2$, (b) $h = 1$, (c) $h = 5$. We see long-lived oscillations (note the log scale for time)

Autocorrelation functions near one end



A^{xx} autocorrelation at sites $l = 1, 2, \cdots, 6$ for (a) $h = 0.2$, (b) $h = 1$, (c) $h = 5$. We again see long-lived oscillations

The oscillations can be understood using perturbative arguments for $h \ll 1$ and $h \gg 1$. But the decays at very long times are not understood analytically

The anomalous behavior at very long times for $l = 2$ and $h = 0.2$ is also not understood

Summary

- We have studied a one-dimensional model with three-spin Ising interactions and a transverse field. The model has duality, and there is a quantum phase transition at the self-dual point
- The model has a $Z_2 \times Z_2$ symmetry, giving rise to four different symmetry sectors. Three of these sectors are degenerate in energy, while the fourth sector is non-degenerate
- The quantum critical point seems to lie in the universality class of the four-state Potts model
- The energy level spacing distribution indicates that the model is nonintegrable
- The model has an exponentially large number of zero energy states. A subset of these are scar states.
  How does the number of scar states grow with system size?
- The autocorrelation functions near one end of the system show anomalous relaxation with time. Why does this happen?

References and acknowledgments

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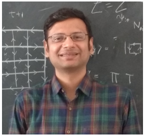

Udupa et al, Phys. Rev. B 108, 214430 (2023)

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