ICTS-RRI Maths Circle, Saturday 13 January, 2024

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2:30 pm

Divisibility

The division algorithm is a fundamental mathematical concept that provides a systematic way to divide one integer by another, with the result expressed as a quotient and a remainder. The division algorithm ensures that when you divide a by b, you can express a as a multiple of b plus a remainder, and the remainder is always less than the absolute value of the divisor b. We say that b divides a when the remainder is zero.

Tests of divisibility are rules or criteria that help determine whether a given number is divisible by another without performing the actual division. You may have already come across these in your school.

Let's consider a problem: What is the remainder when 2^{100} is divided by 101?

We can solve such problems without actually performing the division. Here are some warm-up problems to help you get started.

1. Check if the following numbers are divisible by 4:
   (a) 632,
   (b) 7896,
   (c) 245,
   (d) 1584
2. Check if the following numbers are divisible by 6:
   (a) 846,
   (b) 729,
   (c) 312,
   (d) 990
3. Check if the following numbers are divisible by 8:
   (a) 1248,
   (b) 572,
   (c) 896,
   (d) 633
4. Investigate the divisibility by 5 of the sum of the squares of the first 10 positive integers: $1^2 + 2^2 + 3^2 + \ldots + 10^2$.

5. Find the smallest positive integer that is divisible by 8, consists of only the digits 1 and 0, and each digit appears at least once.
6. Investigate the divisibility by 9 of the sum of the cubes of the first 5 positive integers: $1^3 + 2^3 + 3^3 + 4^3 + 5^3$.
7. Determine the largest five-digit number that is divisible by 10, has all distinct digits, and the sum of its digits is 25.

3:45 pm: Break for refreshments
4:00 pm onwards

Modular Arithmetic and Test for Divisibility by 7

Let's take a step back and try to understand how these tests for divisibility can be constructed. One way to go about this is by examining the remainders when powers of 10, 10^0, 10^1, 10^2, $\ldots$, are divided by a specific number.

Compute the remainders when the powers of 10 (10^0, 10^1, 10^2, $\ldots$) are divided by 3. Do you observe any patterns or repetitions in the remainders? Repeat this exercise for the number 7. Based on your observation, propose a divisibility test for determining whether a number is divisible by 7?

What can be said about the remainders when powers of a different number, say 8 (8^0, 8^1, 8^2, $\ldots$) are divided by 3 or 7? Do you notice any patterns?

Explore Further

1. Is there a number that gives a remainder of 1 when divided by 3, remainder 2 when divided by 4, remainder of 3 when divided by 5, and a remainder of 4 when divided by 6?
2. Show that every fourth Fibonacci number is divisible by 3.
3. Given n is an integer not divisible by 3, prove that $n^2 - 1$ is always divisible by 3.
4. Check whether $1^{11} + 2^{11} + 3^{11} + \ldots + 10^{11}$ is divisible by 11. What can you say about the divisibility of $1^n + 2^n + 3^n + \ldots + (n-1)^n$ by n where n is any natural number?
5. Arrange the ten digits; 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 to create a number that is divisible by all the numbers from 2 to 18. How many such arrangments can you make?