Out of equilibrium dynamics of complex systems

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Plan of Lectures

1. Introduction
2. Coarsening
3. **Disorder**
4. Active Matter
5. Integrability

Third lecture

Plan of lecture

— Definition & examples

— Properties

— List of methods

— Thouless-Anderson-Palmer equations

  – Local order parameters & landscapes (beyond Ginzburg-Landau)

  – Statistical averages

  – Real replicas

— Replica theory

— Relaxation dynamics (experiments, numerics)

— Relaxation dynamics (theory)

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Randomness

Impurities

No material is perfect and totally free of impurities

(vacancies, substitutions, amorphous structures, etc.)

First distinction

— Weak randomness : phase diagram respected, criticality may change

— Strong randomness : phases modified

Second distinction

— Annealed : fluctuating (easier)

— Quenched : frozen, static (harder)

$$\tau_0 \ll t_{\rm obs} \ll \tau_{\rm eq}^{disor}$$

Quenched disorder

Variables frozen in time-scales over which other variables fluctuate

Time scales

$$\tau_0 \ll t_{\rm obs} \ll \tau_{\rm eq}^{disor}$$

$\tau_{\rm eq}^{disor}$ could be the diffusion time-scale for magnetic impurities the magnetic moments of which will be the variables of a **magnetic system**,

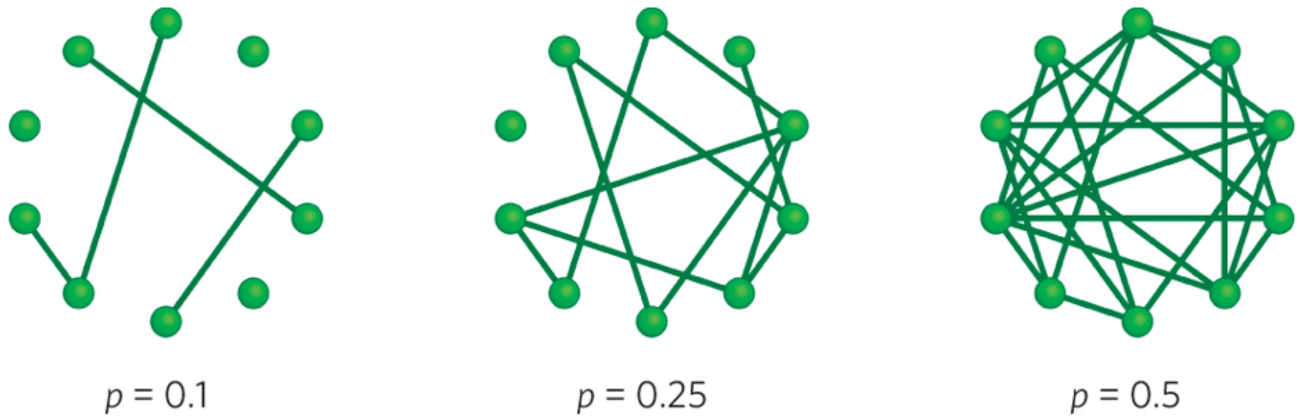
or the flipping time of impurities that create random fields acting on other magnetic variables.

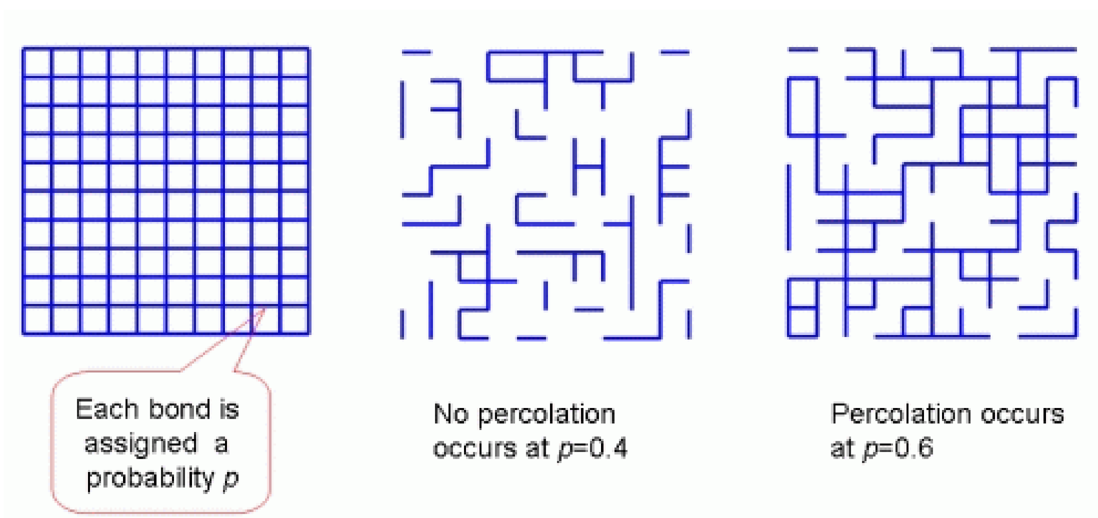
Weak disorder (modifies the critical properties but not the phases) *vs.* **strong** disorder (that modifies both).

e.g. **random ferromagnets** *vs.* **spin-glasses**.

Geometrical problems

Random graphs & Percolation





Spin-glasses

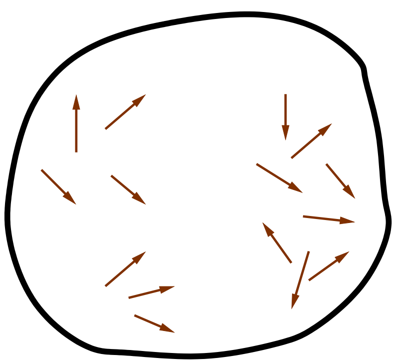
Magnetic impurities (spins) randomly placed in an inert host

$\vec{r}_i$ are random and time-independent since

the impurities do not move during experimental time-scales $\Rightarrow$

quenched randomness

Magnetic impurities in a metal host



spins can flip but not move

RKKY potential

$$V(r_{ij}) \propto \frac{\cos 2k_F r_{ij}}{r_{ij}^3} s_i s_j$$

very rapid oscillations about 0

positive & negative

slow power law decay.

Spin-glasses

Models on a lattice with random couplings

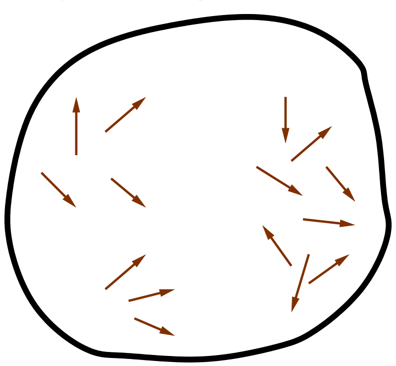
Ising (or Heisenberg) spins $s_i = \pm 1$ sitting on a lattice

J_{ij} are random and time-independent since

the impurities do not move during experimental time-scales $\Rightarrow$

quenched randomness

Magnetic impurities in a metal host



spins can flip but not move

Edwards-Anderson model

$$H_J[\{s_i\}] = -\sum_{\langle ij \rangle} J_{ij} s_i s_j$$

J_{ij} drawn from a pdf with

zero mean & finite variance

Spin-glasses

Magnetic impurities (spins) randomly placed in an inert host

Spin Glasses

Their traits arise from disorderly, discordant magnetic interactions among atoms. Mathematical models of spin glasses are prototypes for complex problems in computer science, neurology and evolution

by Daniel L. Stein

Neural networks

Models on graphs with random couplings

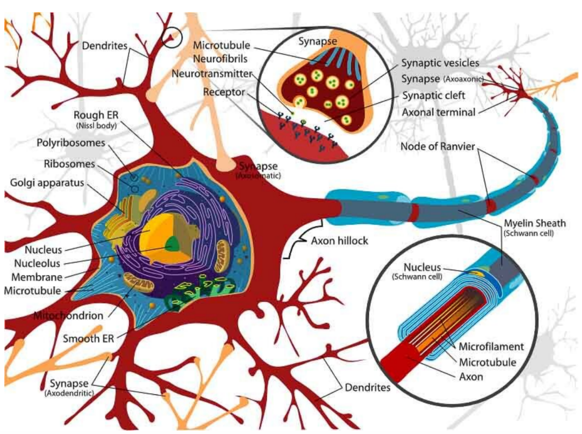
The neurons are Ising spins $s_i = \pm 1$ on a graph

J_{ij} are random and time-independent since

the synapsis do not change during experimental time-scales $\Rightarrow$

quenched randomness

The neural net



spins can flip but not move

Hopfield model

$$H_J[\{s_i\}] = -\sum_{\langle ij \rangle} J_{ij} s_i s_j$$

memory stored in the synapsis

$$J_{ij} = 1/N_p \sum_{\mu=1}^{N_p} \xi_i^\mu \xi_j^\mu$$

the patterns ξ_i^μ

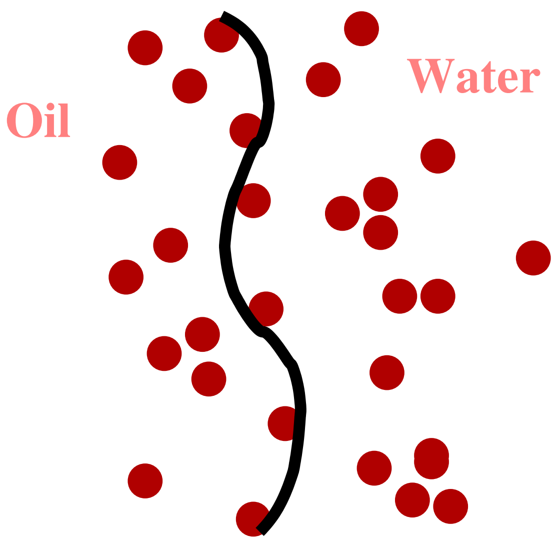
are drawn from a pdf with

zero mean & finite variance

Pinning by impurities

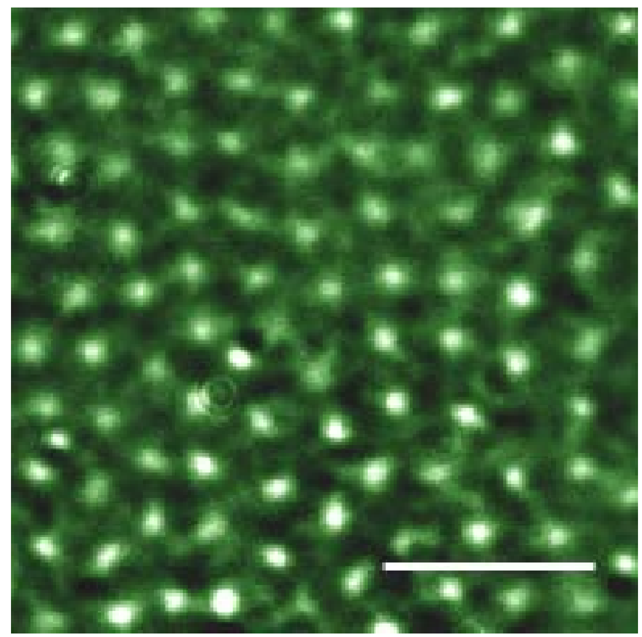
Competition between elasticity and quenched randomness

d-dimensional elastic manifold in a transverse N-dimensional **quenched random potential**.



Interface between two phases ;
vortex line in type-II supercond ;
stretched polymer.

Distorted Abrikosov lattice



Goa et al. 01

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Randomness

Properties

— Spatial inhomogeneity

— Frustration

(spectrum pushed up, degeneracy of ground state)

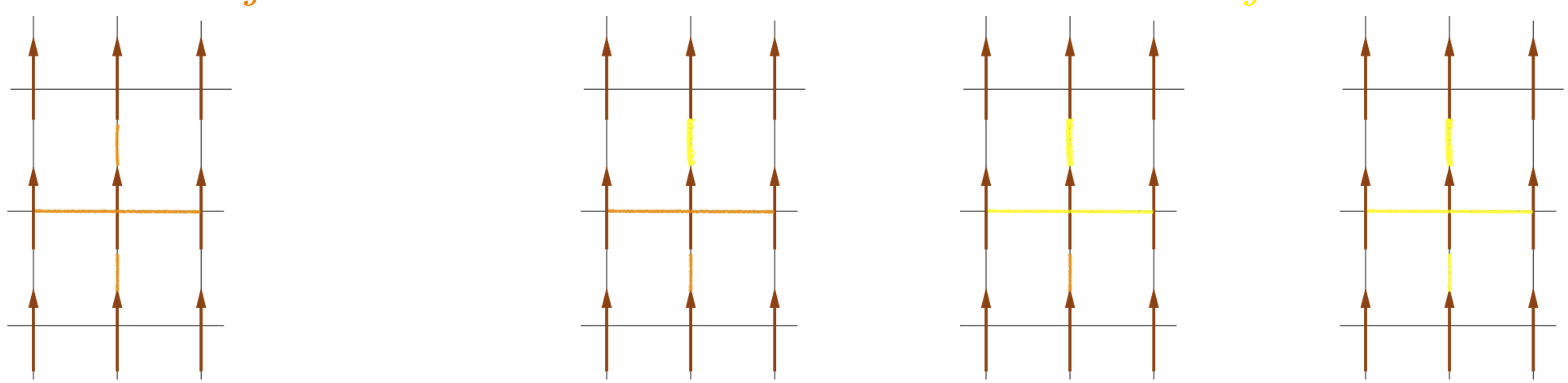
— Probability distribution of couplings, fields, etc.

— Self-averageness

Heterogeneity

Each variable, spin or other, feels a different local field, $h_i = \sum_{j=1}^{z} J_{ij} s_j$, contrary to what happens in a ferromagnetic sample, for instance.

$J_{ij} > 0$ $\qquad\qquad$ $J_{ij} < 0$



Homogeneous $\qquad\qquad$ **Heterogeneous**

$h_i = 4J \;\; \forall\, i$ $\qquad$ $h_j = 2J$ $\qquad$ $h_k = -2J$ $\qquad$ $h_l = -4J$.

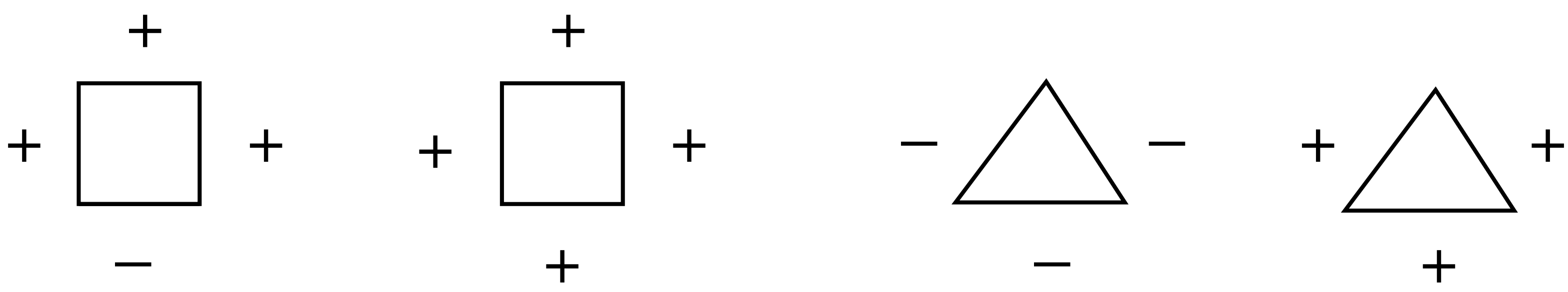
Each sample is *a priori* different but,

do they all have a different thermodynamic and dynamic behavior ?

Frustration

Properties

$$H_J[\{s\}] = -\sum_{\langle ij \rangle} J_{ij} s_i s_j$$ **Ising model**



Disordered **Geometric**

$E_{\rm GS}^{\rm frust} > E_{\rm GS}^{\rm FM}$ and $S_{\rm GS}^{\rm frust} > S_{\rm GS}^{\rm FM}$

Frustration enhances the **ground-state** energy and entropy

One can expect to have **metastable states** too

One cannot satisfy all couplings simultaneously if $\prod_{\rm loop} J_{ij} < 0$

Self-averageness

The disorder-induced free-energy density distribution approaches a Gaussian with vanishing dispersion in the thermodynamic limit :

$$\lim_{N\to\infty} f_N(\beta J) = f_\infty(\beta J) \qquad \text{independently of disorder}$$

— Experiments : all *typical* samples behave in the same way.

— Theory : one can perform a (hard) average of disorder, $[\dots]$,

$$-\beta N f_\infty(\beta J) = \lim_{N\to\infty} [\ln \mathcal{Z}_N(\beta J)]$$

From here, we see that, e.g., the energy density is self-averaging

Replica theory

$$-\beta f_\infty(\beta J) = \lim_{N\to\infty} \lim_{n\to 0} \frac{[\mathcal{Z}_N^n(\beta J)] - 1}{Nn}$$

Self-averageness

The question

Given two samples with different quenched randomness

(e.g. different interaction strengths J_{ij}s or random fields h_i)

but drawn from the same (kind of) distribution

is their behaviour going to be totally different ?

Which quantities are expected to be the same and which not ?

Self-averageness

Observables & distributions

Given a quantity A_J, which depends on the quenched randomness J, it is distributed according to

$$P(A) = \int dJ\; p(J)\; \delta(A - A_J)$$

This pdf is expected to be narrower and narrower (more peaked) as $N \to \infty$

Therefore, one will observe $A = A_{\rm typ}$ such that $\max_A P(A)$

However, it is difficult to calculate $A_{\rm typ}$, what about calculating $[A] = \int dA\; P(A)\; A$?

Self-averageness

Example : the disordered Ising chain

$$H_J[\{s_i\}] = -\sum_i J_i s_i s_{i+1} \qquad J_i \text{ *i.i.d.* with any pdf } p(J_i)$$

Compute the partition function Z by introducing $\sigma_i = s_i s_{i+1}$

$$Z[\{\beta J_i\}] = \sum_{s_i=\pm 1} e^{\beta \sum_i J_i s_i s_{i+1}} = \sum_{\sigma_i=\pm 1} e^{\beta \sum_i J_i \sigma_i} = \prod_{i=1}^{N} 2\coth(\beta J_i)$$

(boundary condition effects negligible for $N \to \infty$)

It is a **product** of N *i.i.d.* random numbers

The free-energy is $-\beta F[\{\beta J_i\}] = \sum_{i=1}^{N} \ln\coth(\beta J_i) + N \ln 2$

It is a **sum** of N *i.i.d.* random numbers

Self-averageness

Example : the disordered Ising chain

$$H_J[\{s_i\}] = -\sum_i J_i s_i s_{i+1} \qquad J_i \text{ \textit{i.i.d.} with any pdf } p(J_i)$$

The partition function & the free energy density are different objects

$$Z[\{\beta J_i\}] = \prod_{i=1}^{N} 2\coth(\beta J_i) \qquad -\beta f[\{\beta J_i\}] = \frac{1}{N}\sum_{i=1}^{N} \ln\coth(\beta J_i) + \ln 2$$

Take J_i to be *i.i.d* with zero mean $[J_i] = 0$ & finite variance $[J_i^2] = \sigma^2$ and use the **Central Limit Theorem** :

$X = \frac{1}{N}\sum_i x_i$ is Gaussian distributed with average $\langle X\rangle = \langle x_i\rangle$ and variance $\langle (X - \langle X\rangle)^2\rangle = \sigma^2/N$

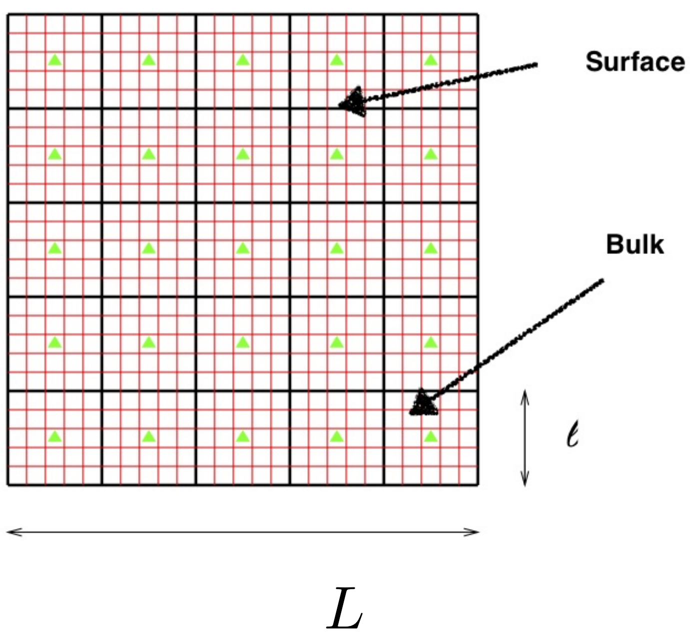
Therefore f_J is Gaussian distributed and its variance vanishes for $N \to \infty$

Moreover, $f_J^{\text{typ}} = [f_J]$

Self-averageness

Systems with short-range interactions

Divide a, say, cubic system of volume $V = L^d$ in n sub-cubes, of volume $v = \ell^d$ with $V = nv$



$$-\beta F_J \approx \sum_{k=1}^{L/\ell} \ln \sum_{\text{bulk}_k} e^{-\beta H_J(\text{bulk}_k)}$$

For $L \gg \ell$ the CLT

$\Rightarrow$ f_J is Gaussian distributed and

$$f_J^{\text{typ}} = [\, f_J \,]$$

Self-averageness

Qquenched vs. annealed

Go back to the one dimensional disordered Ising chain and show that

the partition function and the spatial correlations

are not self-averaging.

The annealed free-energy is defined as $-\beta F^{\rm annealed} = \ln[Z_J]$

The quenched free-energy is defined as $-\beta F^{\rm quenched} = [\ln Z_J]$

Jenssen's inequality applied to the convex function $-\ln y$ implies

$$-\ln[Z_J] \leq -[\ln Z_J]$$

and for the free-energies one deduces

$$F^{\rm annealed} = -\beta^{-1}\ln[Z_J] \leq -\beta^{-1}[\ln Z_J] = F^{\rm quenched}$$

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Methods

disordered systems

Statics

Method	System
TAP Thouless-Anderson-Palmer Replica theory	fully-connected (complete graph) Gaussian approx. to field-theories
Cavity or Peierls approx.	dilute (random graph)
Bubbles & droplet arguments functional RG[1]	finite dimensions

Dynamics

Generating functional for classical field theories (MSRJD).

Schwinger-Keldysh closed-time path-integral for quantum dissipative models (the previous is recovered in the $\hbar \to 0$ limit).

Perturbation theory, renormalization group techniques, self-consistent approximations

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Randomness

Properties

— Spatial inhomogeneity

Not all sites behave in the same way

— Frustration

Impossibility to satisfy all conditions imposed by the Hamiltonian

(spectrum pushed up, degeneracy of ground state)

— Annealed vs quenched

Couplings, fields, etc. fluctuate or are frozen

$f^{\text{annealed}} \leq f^{\text{quenched}}$

— Quenched disorder : static pdfs of couplings, fields, etc.

— Self-averageness

$\lim_{N\to\infty}[f^{\text{quenched}}] = \lim_{N\to\infty} f^{\text{typ}}$

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Low temperature phases

Phenomenology : homogeneity vs inhomogeneity

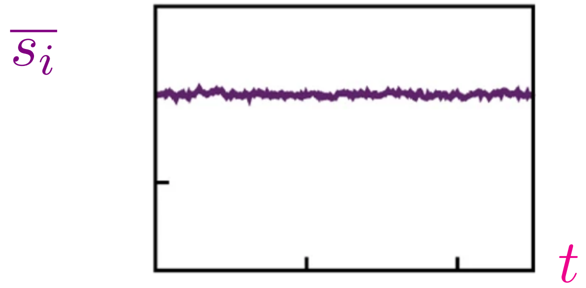
In a **ferromagnet in equilibrium** at temperature $T < T_c$, $\langle s_i \rangle = m(T) \ \forall i$ or $\langle s_i \rangle = -m(T) \ \forall i$ in the two homogeneous, symmetric and degenerate equilibrium states

Low temperature phases

Phenomenology : homogeneity vs inhomogeneity

In a **ferromagnet in equilibrium** at temperature $T < T_c$, $\langle s_i \rangle = m(T)\ \forall i$ or $\langle s_i \rangle = -m(T)\ \forall i$ in the two homogeneous, symmetric and degenerate equilibrium states

If one were to follow the time evolution of each spin in one of the two equilibrium states at $T < T_c$, one would see $\overline{s_i}(t) = m(T) + \delta_i(t)$ with $\delta_i(t)$ small time-dependent fluctuation and the overline states for a running time average $\overline{s_i}(t) = \tau^{-1} \int_t^{t+\tau} dt'\, s_i(t')$



Low temperature phases

Phenomenology : homogeneity vs inhomogeneity

In a **spin-glass in equilibrium** at temperature $T < T_c$, one expects $\langle s_i \rangle = m_i(T)$, with a different value for each i, in each inhomogeneous and degenerate equilibrium state.

There may be many different ensembles $\{m_i(T)\}$ that are equilibrium states (degeneracy, similar to what we saw in the frustrated magnets for the ground states but here in the full low T phase)

There is also the up-down symmetry $\{m_i(T)\} \mapsto \{-m_i(T)\}$

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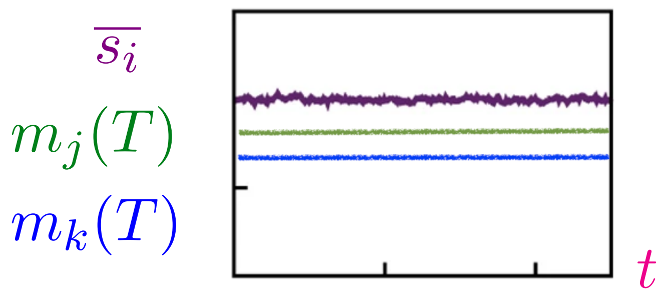
If one were to follow the time evolution of each spin in one of the possibly many equilibrium states at $T < T_c$, one would see $\overline{s_i}(t) = m_i(T) + \delta_i(t)$ with $\delta_i(t)$ small time-dependent fluctuation and the overline states for a running time average $\overline{s_i}(t) = \tau^{-1} \int_t^{t+\tau} dt' \, s_i(t')$



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Mean-field theory

Fully connected Ising models

General model

$$H_J[\{s_i\}] = -\frac{1}{2}\sum_{i\neq j} J_{ij} s_i s_j \qquad \text{with Ising variables} \qquad s_i = \pm 1$$

$\mathcal{O}(1)$ scaling of the local fields $\Rightarrow$ scaling of J_{ij}

What is a local field ?

It is the field felt by a selected site

$$h_i = \frac{1}{2}\sum_{j(\neq i)} J_{ij} s_j$$

and we require it to be $\mathcal{O}(1)$

Mean-field theory

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$\mathcal{O}(1)$ scaling of the local fields $\Rightarrow$ scaling of J_{ij}

In the Curie-Weiss ferromagnetic case

$$J_{ij} = \frac{J}{N} \;\text{ such that }\; h_i = \frac{J}{2N} \sum_{j(\neq i)} s_j = \mathcal{O}(1)$$

in the two ferromagnetic $s_i = 1 \; \forall i$ or $s_i = -1 \; \forall i$ phases

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In the Sherrington-Kirkpatrick disordered case

$$J_{ij} = \mathcal{O}(\frac{J}{\sqrt{N}}) \;\; \text{such that} \;\; h_i \sim \frac{J}{2\sqrt{N}} \sum_{j(\neq i)} s_j = \mathcal{O}(1)$$

in the PM or spin-glass phases $s_i = \pm 1 \; \forall i$

Mean-field theory

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In the Sherrington-Kirkpatrick disordered case

$$J_{ij} = \mathcal{O}(\frac{J}{\sqrt{N}}) \text{ such that } h_i \sim \frac{J}{2\sqrt{N}} \sum_{j(\neq i)} s_j = \mathcal{O}(1)$$

in the PM or spin-glass phases, say, $s_i = \pm 1$ with equal probability

One can use a Gaussian pdf

$$P(J_{ij}) = (2\pi\sigma^2)^{-1/2} \exp[-J_{ij}^2/(2\sigma^2)] \text{ with } \sigma^2 = J^2/N$$

Mean-field theory

Fully connected Ising models

Even more general models (recall the K-sat problem)

$$H_J[\{s_i\}] = -\frac{1}{3!} \sum_{i \neq j \neq k} J_{ijk} s_i s_j s_k \qquad \text{with Ising variables} \qquad s_i = \pm 1$$

$\mathcal{O}(1)$ scaling of the local fields $\Rightarrow$ scaling of J_{ijk}

In the $p = 3$ Curie-Weiss ferromagnetic case

$$J_{ijk} = \frac{J}{N^{p-1}} \quad \text{such that } h_i \sim \frac{J}{2N^{p-1}} \sum_{jk(\neq i)} s_j s_k = \mathcal{O}(1)$$

in the two ferromagnetic $s_i = 1 \; \forall i$ or $s_i = -1 \; \forall i$ phases

In the $p = 3$ disordered case

$$J_{ijk} = \mathcal{O}\Big(\frac{J}{\sqrt{N^{p-1}}}\Big) \quad \text{such that } h_i \sim \frac{J}{2\sqrt{N^{p-1}}} \sum_{j \neq k(\neq i)} s_j s_k = \mathcal{O}(1)$$

in the PM or spin-glass phases $s_i = \pm 1$ with equal probability

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— Frustration

Impossibility to satisfy all conditions imposed by the Hamiltonian

(spectrum pushed up, degeneracy of ground state)

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Gaussian pdf of J_{ij} with $\sigma^2 = J^2/N$

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— Complex free-energy landscapes : beyond Ginzburg-Landau

Mean-field theory

Fully connected Curie-Weiss Ising model for PM-FM

Normalize J by the size of the system N to have $\mathcal{O}(1)$ local fields

$$H = -\frac{J}{2N} \sum_{i \neq j} s_i s_j - h \sum_i s_i$$

The partition function reads $\mathcal{Z} = \int_{-1}^{1} du \, e^{-\beta N \mathbf{f}(u)}$ with $Nu = \sum_i s_i$

$$\mathbf{f}(u) = -\frac{J}{2} u^2 - hu + T \left[\frac{1+u}{2} \ln \frac{1+u}{2} + \frac{1-u}{2} \ln \frac{1-u}{2} \right]$$

Energy terms and entropic contribution stemming from $\mathcal{N}(\{s_i\})$ yielding the same u value.

Use the **saddle-point**, $\lim_{N \to \infty} f_N(\beta J, \beta h) = \mathbf{f}(u_{sp})$, with

$$u_{sp} = \tanh\left(\beta J u_{sp} + \beta h\right) = \langle u \rangle = m$$

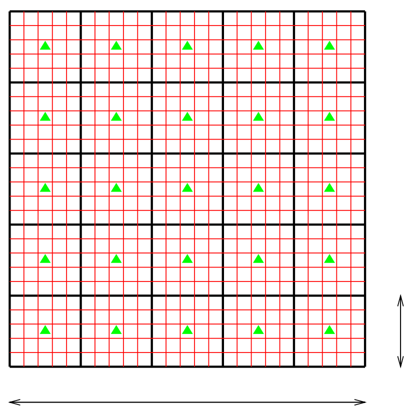
Ginzburg-Landau for PM-FM

Continuous scalar statistical field theory with local aspects

Coarse-grain the spin

$\phi(\vec{r}) = V_{\vec{r}}^{-1} \sum_{i \in V_{\vec{r}}} s_i$

Set $h = 0$



The partition function is $\mathcal{Z} = \int \mathcal{D}\phi \, e^{-\beta V \mathbf{f}(\phi)}$ with V the volume and

$$\mathbf{f}(\phi) = \int d^d r \, \left\{ \frac{1}{2} [\nabla \phi(\vec{r})]^2 + \frac{T-J}{2} \phi^2(\vec{r}) + \frac{\lambda}{4} \phi^4(\vec{r}) \right\}$$

Elastic + potential energy with the latter inspired by the results for the fully-connected model (entropy around $\phi \sim 0$ and symmetry arguments.

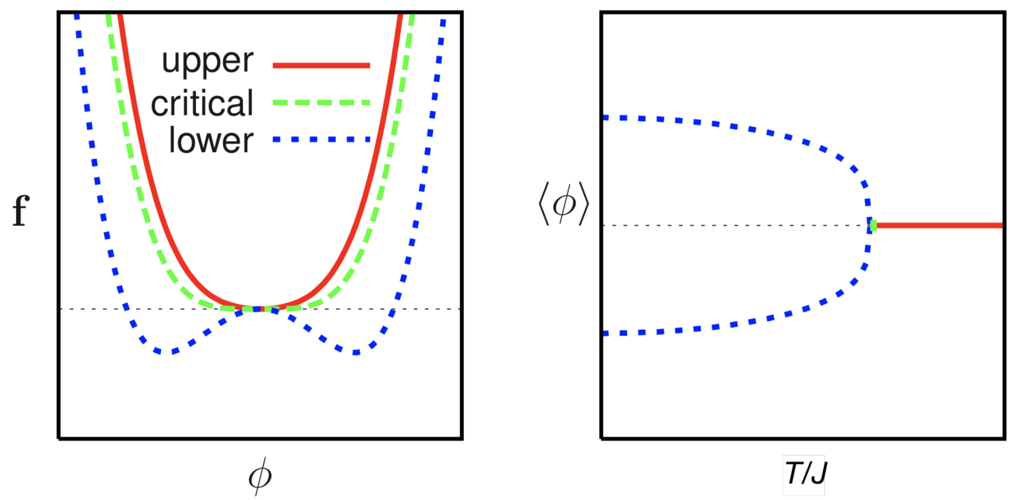
Uniform saddle point in the $V \to \infty$ limit : $\phi_{sp}(\vec{r}) = \langle \phi(\vec{r}) \rangle = m$

The free-energy density is $\lim_{V \to \infty} f_V(\beta, J) = \mathbf{f}(\phi_{sp})$

2nd order phase-transition

Continuous scalar statistical field theory

bi-valued equilibrium states related by symmetry



Ginzburg-Landau free-energy

Scalar order parameter

MFT for disordered spin models

Fully connected SG : Sherrington-Kirkpatrick model

$$H = -\frac{1}{2}\sum_{i\neq j} J_{ij} s_i s_j - \sum_i h_i s_i$$

with J_{ij} i.i.d. Gaussian variables, $[J_{ij}] = 0$ and $[J_{ij}^2] = J^2/N = \mathcal{O}(1/N)$.

One finds the naive free-energy landscape

$$N\mathbf{f}(\{m_i\}) = -\frac{1}{2}\sum_{i\neq j} J_{ij} m_i m_j + T\sum_{i=1}^{N} \frac{1+m_i}{2}\ln\frac{1+m_i}{2} + \frac{1-m_i}{2}\ln\frac{1-m_i}{2}$$

and the (naive) TAP equations

$$m_{i\,sp} = \tanh(\beta \sum_{j(\neq i)} J_{ij} m_{j\,sp} + \beta h_i)$$

that determine the restricted averages $m_i = \langle s_i \rangle = m_{i\,sp}$.

MFT for disordered spin models

Fully connected SG : A simple proof

The more traditional one assumes independence of the spins,

$$P(\{s_i\}) = \prod_i p_i(s_i)$$

with $p_i(s_i) = \frac{1+m_i}{2}\delta_{s_i,1} + \frac{1-m_i}{2}\delta_{s_i,-1}$

and uses this form to express $\langle H \rangle - T\langle S \rangle$ with $S = \ln \mathcal{N}(\{s_i\})$

The energetic contribution is straightforward to evaluate

The entropic contribution is the one we already computed for the Curie-Weiss model, taking care of keeping the indices i

A more powerful proof expresses $\mathbf{f}$ as the Legendre transform of $-\beta F(h_i)$ with $m_i = N^{-1}\partial[-\beta F(h_i)]/\partial h_i$ and takes care of a "problem" to be solved in the next slides

Georges & Yedidia 91

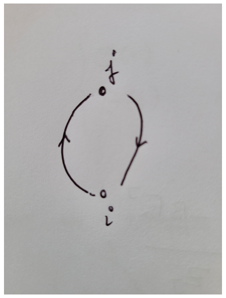
MFT for disordered spin models

Missing : the Onsager reaction term

These equations are not completely correct.

The *Onsager reaction term* is missing.

This term represents the reaction of the spin i to itself



The magnetisation in i produces a field $h'_{j(i)} = J_{ji} m_i = J_{ij} m_i$ on spin j

This field induces a magnetisation $m'_{j(i)} = \chi_{jj} h'_{j(i)} = \chi_{jj} J_{ij} m_i$ on the spin j.

This magnetisation produces a field $h'_{i(j)} = J_{ij} m'_{j(i)} = J_{ij} \chi_{jj} J_{ij} m_i$ on site i.

The equilibrium fluctuation-dissipation relation between susceptibilities and connected correlations implies $\chi_{jj} = \beta \langle (s_j - \langle s_j \rangle)^2 \rangle = \beta(1 - m_j^2)$ and one then has $h'_{i(j)} = \beta(1 - m_j^2) J_{ij}^2 m_i$

MFT for disordered spin models

The Onsager reaction term

The idea of Onsager – or *cavity method* – is that one has to study the ordering of the spin i in the absence of its own effect on the rest of the system.

The total field produced by the sum of $h'_{i(j)} = \beta(1 - m_j^2) J_{ij}^2 m_i$ over all the spins j with which it can connect, has to be subtracted from the mean-field created by the other spins in the sample, *i.e. the total local field* should be

$$h_i^{\rm loc} = \sum_{j(\neq i)} J_{ij} m_j - \beta m_i \sum_{j(\neq i)} J_{ij}^2 (1 - m_j^2)$$

recall that $J_{ij} = \mathcal{O}(1/\sqrt{N})$. Finally, the TAP equations read

$$m_i = \tanh \left\{ \sum_{j(\neq i)} \left[\beta J_{ij} m_j - \beta^2 m_i J_{ij}^2 (1 - m_j^2) \right] \right\}$$

MFT for disordered spin models

Orders of magnitude

The Thouless-Anderson-Palmer (TAP) equations read

$$m_i = \tanh\left\{ \sum_{j(\neq i)} [\beta J_{ij} m_j - \beta^2 m_i J_{ij}^2 (1 - m_j^2)] \right\}$$

The first term in the rhs $\sum_{j(\neq i)} J_{ij} m_j \simeq \frac{1}{\sqrt{N}} \sqrt{N} = \mathcal{O}(1)$ because of the central limit theorem.

The second term $\sum_{j(\neq i)} J_{ij}^2 (1 - m_j^2) \simeq \frac{1}{N} N = \mathcal{O}(1)$ because all terms in the sum are positive definite ($m_j \leq 1 \ \forall j$)

Recall that $m_i = \langle s_i \rangle$

MFT for disordered spin models

Orders of magnitude

The Thouless-Anderson-Palmer (TAP) equations read

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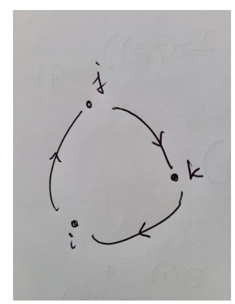
Exercise

Check that higher order loops are negligible,

since sub-leading in powers of N



MFT for disordered spin models

Orders of magnitude

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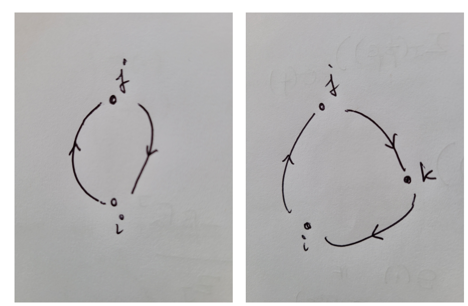
Exercise

Check that in the Curie Weiss model $J_{ij} = J/N$ there is no need of Onsager terms

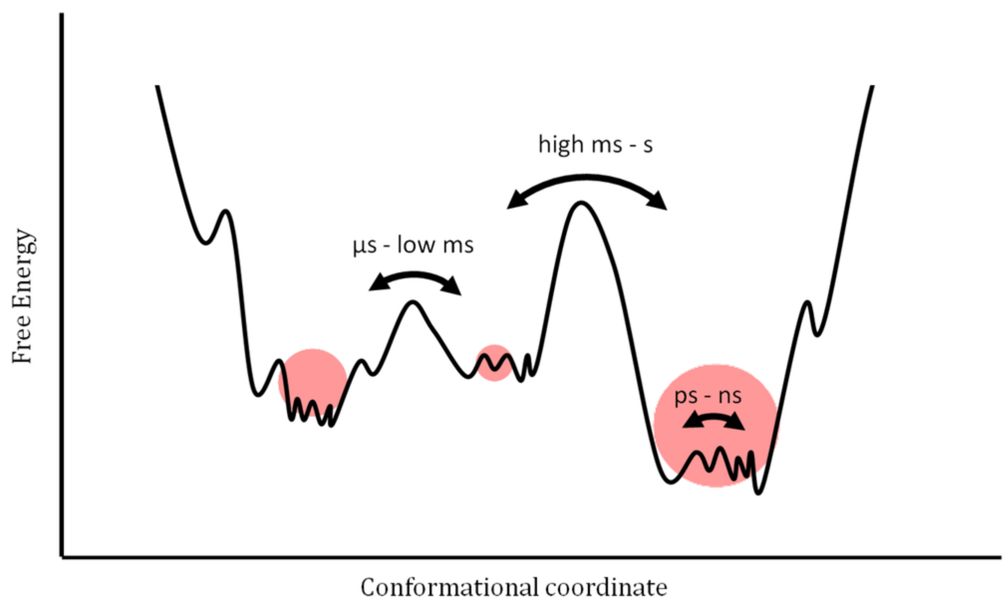


Landscape

Free-energy density at fixed randomness

The TAP equations are the extremization conditions on the *TAP free-energy*

$$F_J^{\rm tap}(\{m_i\}) = -\frac{1}{2}\sum_{i\neq j} J_{ij} m_i m_j - \frac{\beta}{4}\sum_{i\neq j} J_{ij}^2 (1-m_i^2)(1-m_j^2) + T\sum_{i=1}^{N}\left[\frac{1+m_i}{2}\ln\frac{1+m_i}{2} + \frac{1-m_i}{2}\ln\frac{1-m_i}{2}\right]$$



At low temperatures $\{m_i\}$

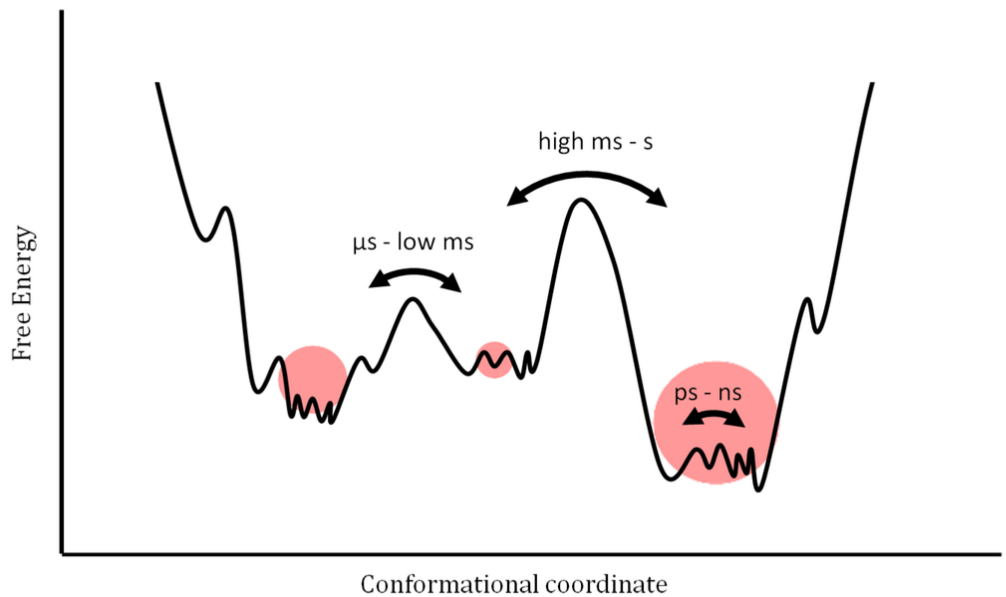
Landscape

Free-energy density at fixed randomness

The TAP equations are the extremization conditions on the *TAP free-energy*

$$\frac{\delta F_J^{\rm tap}(\{m_i\})}{\delta m_j} = 0$$

The stability of the solutions is determined by the Hessian $\dfrac{\delta^2 F_J^{\rm tap}(\{m_i\})}{\delta m_j \delta m_k}$



At low temperatures $\{m_i\}$

Landscape

Free-energy density at fixed randomness

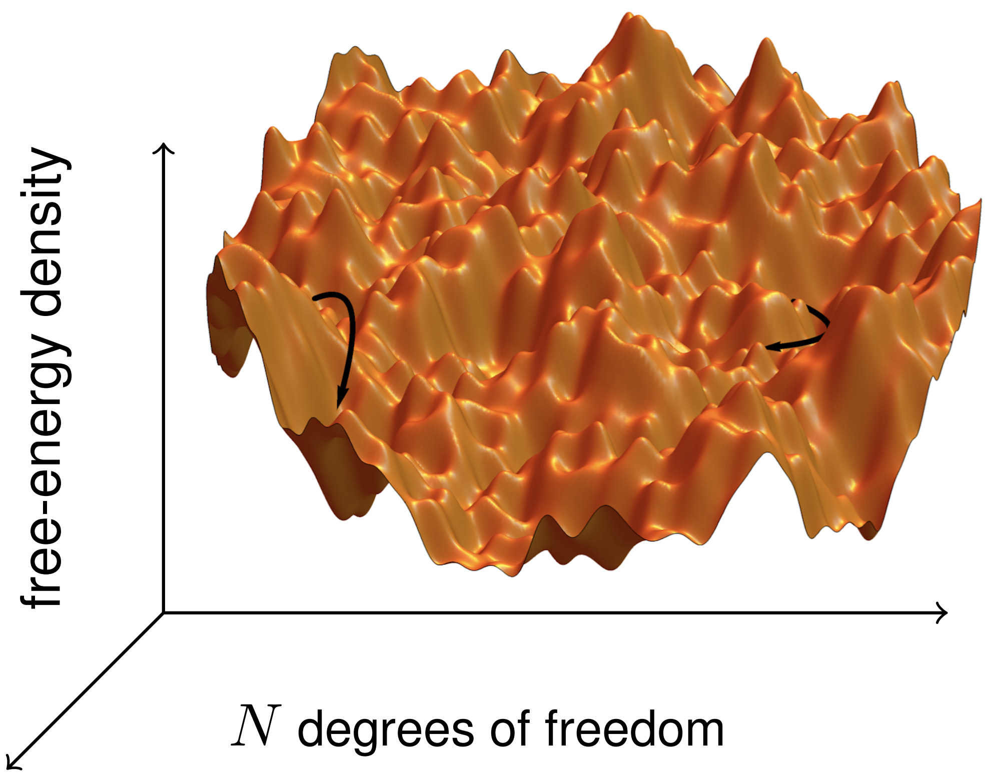


Figure adapted from a picture by **C. Cammarota**

Topography of the landscape on the N-dimensional substrate made by the N order parameters ?

Features

At fixed randomness

- There are N **local order parameters**, $m_i, \ i = 1, \ldots, N$
- The saddle-points are **heterogeneous**: m_i differ from site to site
- At high temperatures only one trivial solution $\{m_i = 0\}$
- At low temperatures the TAP equations have **many solutions** $\{{m_i}^\alpha\}$, which are extrema of the TAP free-energy landscape, *i.e.* saddles of all types, $\alpha = 1, \ldots, \mathcal{N}_J$
- For each solution $\{{m_i}^\alpha\}$, there is also $\{-{m_i}^\alpha\}$ but apart from this trivial doubling, the remaining solutions are not related by symmetry
- The TAP free-energy can take different values at different $\{{m_i}^\alpha\} \Rightarrow f^\alpha_{\rm tap}$

Features

All this is reshuffled for another realization of disorder

- Still N **local order parameters**, $m_i, \; i = 1, \ldots, N$
- The TAP equations have other solutions $\{m_i{}^\alpha\}$, extrema of the TAP free-energy landscape, F_J^{tap}, labelled by $\alpha = 1, \ldots, \mathcal{N}_J$
- A **global order parameter** ? The simplest guess $\frac{1}{N}\sum_{i=1}^{N} m_i^\alpha$ cannot be since it is $= 0$ One expects as many positive as negative m_is and similarity in all respects. Another try

$$q_{\text{EA}}^\alpha = \frac{1}{N}\sum_{i=1}^{N} (m_i^\alpha)^2$$

- "Typicality expected" (though see below for equilibrium states)

Features

Numbers of metastable states

- N **local order parameters**, $m_i, \; i = 1, \ldots, N$
- The TAP equations have many solutions $\{m_i{}^\alpha\}$, extrema of the TAP free-energy landscape, $\alpha = 1, \ldots, \mathcal{N}_J$
- One can count how many saddles of each kind exist and their **complexity**

$$\mathcal{N}_J = \prod_{i=1}^{N} \int_{-1}^{1} dm_i \; \delta(m_i - m_i^\alpha) \qquad\qquad \Sigma = \ln \mathcal{N}$$

- how many of these at each level of free-energy density, by inserting a delta-function $\delta(f_J^{\rm tap}(\{m_i^\alpha\}) - f) \Rightarrow \mathcal{N}_J(f)$
- How many with a given stability $\mathcal{N}_J(f, K)$ with K the number of positive eigenvalues of the Hessian, with adequate delta-functions

Summary

Local & global order parameters

$$m_i^\alpha \equiv \langle s_i \rangle_\alpha$$

$= 0$ at $T \geq T_c$

$\neq 0$ at $T < T_c$

α labels the TAP state

Magnetization

$m^\alpha = \frac{1}{N} \sum_i m_i^\alpha = 0$ at all temperatures and for all α

Edwards-Anderson order parameter

$$q_{\mathrm{EA}}^\alpha \equiv \frac{1}{N} \sum_i (m_i^\alpha)^2 = \frac{1}{N} \sum_i \langle s_i \rangle_\alpha^2$$

$= 0$ at $T \geq T_c$

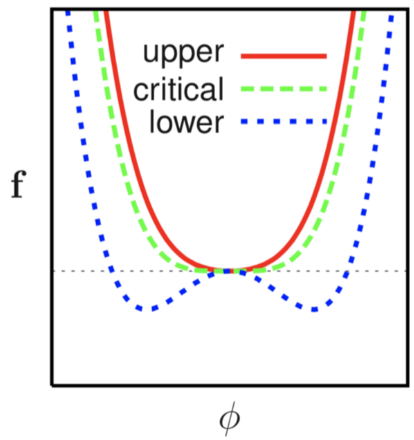
$\neq 0$ at $T < T_c$

Statistical averages

At fixed interactions

The average of a generic observable is $\langle O \rangle = \sum_\alpha w_\alpha \langle O \rangle_\alpha$

In the **FM case**, each state ($\langle \phi \rangle = \pm \phi_0$) has weigth $w_\pm = 1/2$ and the sum is $\langle O \rangle = \frac{1}{2} \langle O \rangle_+ + \frac{1}{2} \langle O \rangle_-$ with $\langle O \rangle_\pm$ the average in each of the states. For instance, the averaged magnetization vanishes if one sums over the $\pm$ states or it is different from zero if one restricts the sum to only one of them.



FM case

The dashed blue line with

two minima $\pm|\phi_0|$

If we have many more ?

Statistical averages

At fixed interactions

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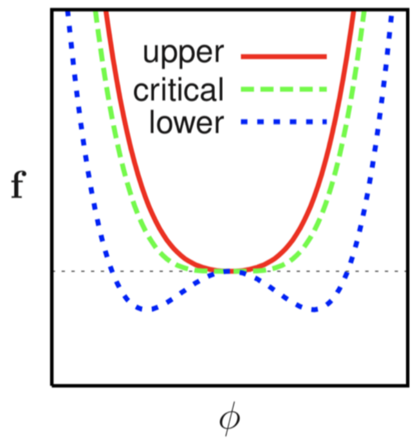


FM case $f_+ = f_-$

$$w_\pm = \frac{e^{-\beta N f_\pm}}{e^{-\beta N f_+} + e^{-\beta N f_-} + e^{-\beta N f_0}} \simeq \frac{1}{2}$$

$$w_0 = \frac{e^{-\beta N f_0}}{e^{-\beta N f_+} + e^{-\beta N f_-} + e^{-\beta N f_0}} \ll w_\pm$$

Statistical averages

At fixed randomness

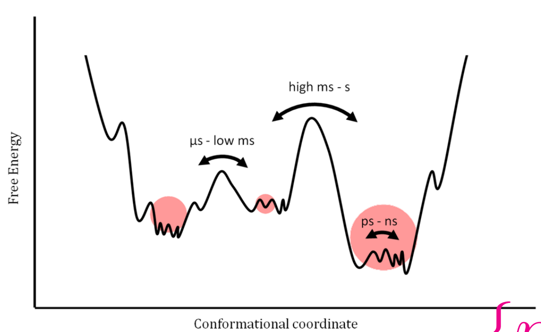
The average of a generic observable is $\langle O \rangle = \sum_\alpha w_\alpha \langle O \rangle_\alpha$

For systems with **quenched randomness** $w_\alpha^J = \frac{e^{-\beta N \mathbf{f}_\alpha^J}}{\sum_\gamma e^{-\beta N \mathbf{f}_\gamma^J}}$

where we added a super-script to the weight w

^{J} indicates that the weights depend on the the disorder realization

and α is a label that identifies the TAP solution



$\{m_i\}$

One can sum over all saddles irrespectively of their stability. Higher lying ones will be exponentially suppressed or will dominate depending on $\Sigma_J(f, K)$

Statistical averages

At fixed randomness

The average of a generic observable is $\langle O \rangle = \sum_\alpha w_\alpha \langle O \rangle_\alpha$

For systems with **quenched randomness** $w_\alpha^J = \frac{e^{-\beta N \mathbf{f}_\alpha^J}}{\sum_\gamma e^{-\beta N \mathbf{f}_\gamma^J}}$

The sum over α, in the case in which there are an exponential in N number of TAP solutions, can be replaced by an integral over $\mathbf{f}$

$$\langle O \rangle = \mathcal{Z}^{-1}(\beta, J) \int d\mathbf{f} \, e^{-\beta[N\mathbf{f} - T \ln \mathcal{N}_J(\mathbf{f},\beta)]} \, O(\mathbf{f}, \beta)$$

$\mathcal{N}_J$ is the number of solutions to the TAP eqs. with free-energy density $\mathbf{f}$.

For $N \to \infty$ the integral is dominated by the saddle point

$$\frac{1}{T} = \frac{1}{N} \frac{\partial \ln \mathcal{N}_J(\mathbf{f}, \beta)}{\partial \mathbf{f}}\bigg|_{\mathbf{f}_{sp}} = \frac{1}{N} \frac{\partial \Sigma_J(\mathbf{f}, \beta)}{\partial \mathbf{f}}\bigg|_{\mathbf{f}_{sp}}$$

complexity

Statistical averages

Consequences

The **equilibrium free-energy** f is given by the saddle-point evaluation of the partition sum:

$$f = \mathbf{f}_{sp} - \frac{T}{N} \ln \mathcal{N}_J(\mathbf{f}_{sp}, \beta)$$

The rhs is the **Landau free-energy** of the problem, with $\mathbf{f}_{sp}$ playing the role of the energy and $N^{-1} \ln \mathcal{N}_J(\mathbf{f}_{sp}, \beta)$ of the entropy

The contribution of the complexity or configurational entropy contribution is negative and in some cases higher lying extrema (metastable states) can dominate the partition sum with respect to lower lying ones if $\ln \mathcal{N}_J(\mathbf{f}_{sp}, \beta) \propto N$

This feature is proposed to describe **super-cooled liquids**.

A global observable

Effect of multi-states

What is the expression of the global order parameter once one takes into account the multi-states ?

$$q \equiv \frac{1}{N}\sum_i \langle s_i \rangle^2 = \frac{1}{N}\sum_i (\sum_\alpha w_\alpha^J m_i^\alpha)^2 = \frac{1}{N}\sum_i \sum_\alpha w_\alpha^J m_i^\alpha \sum_\beta w_\beta^J m_i^\beta$$

note that this is different from $q_{\rm EA}^\alpha = \frac{1}{N}\sum_i (m_i^\alpha)^2$

Defining now $\boxed{q_{\alpha\beta} \equiv \frac{1}{N}\sum_i m_i^\alpha m_i^\beta}$ an overlap between different states

and $\boxed{P_J(q') \equiv \sum_{\alpha\beta} w_\alpha^J w_\beta^J \, \delta(q' - q_{\alpha\beta})}$

we obtain $\boxed{q \equiv \frac{1}{N}\sum_i \langle s_i \rangle^2 = \int dq' \; P_J(q') \; q'}$

Real replicas

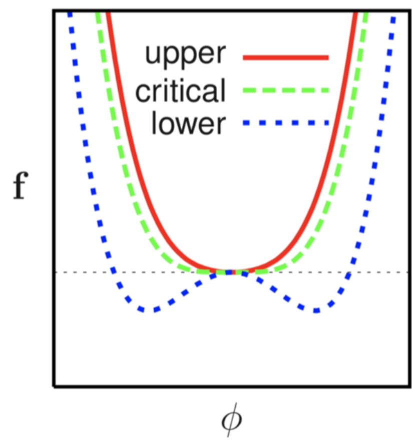
How to see the TAP states ? overlaps between replicas

Take one sample and run it, with e.g. Monte Carlo, until it reaches **equilibrium**, measure the spin configuration $\{s_i\}$.

Re-initialize the same sample (same J_{ij}), run it again until it reaches **equilibrium**, & measure the spin configuration $\{\sigma_i\}$.

Construct the overlap $q_{s\sigma} \equiv N^{-1} \sum_{i=1}^{N} s_i \sigma_i$.

In a **PM system** the overlap will typically vanish as, say, $N^{-1/2}$



Many repetitions

for a system with $N \gg 1$

$$P(q_{s\sigma}) = \delta(q_{s\sigma})$$

Real replicas

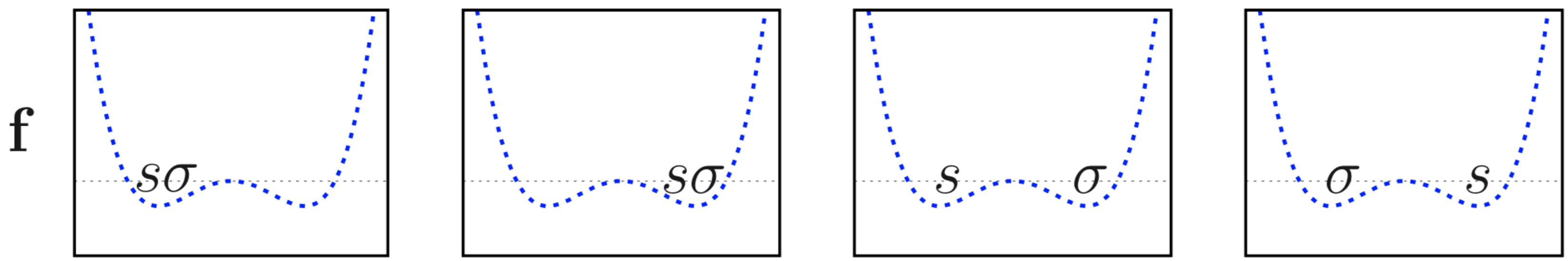
How to see the TAP states ? overlaps between replicas

Take one sample and run it, with e.g. Monte Carlo, until it reaches **equilibrium**, measure the spin configuration $\{s_i\}$.

Re-initialize the same sample (same J_{ij}), run it again until it reaches **equilibrium**, & measure the spin configuration $\{\sigma_i\}$.

Construct the overlap $q_{s\sigma} \equiv N^{-1} \sum_{i=1}^{N} s_i \sigma_i$.

In a **FM system** there are four possibilities



$q_{s\sigma} = m^2$ m^2 $-m^2$ $-m^2$

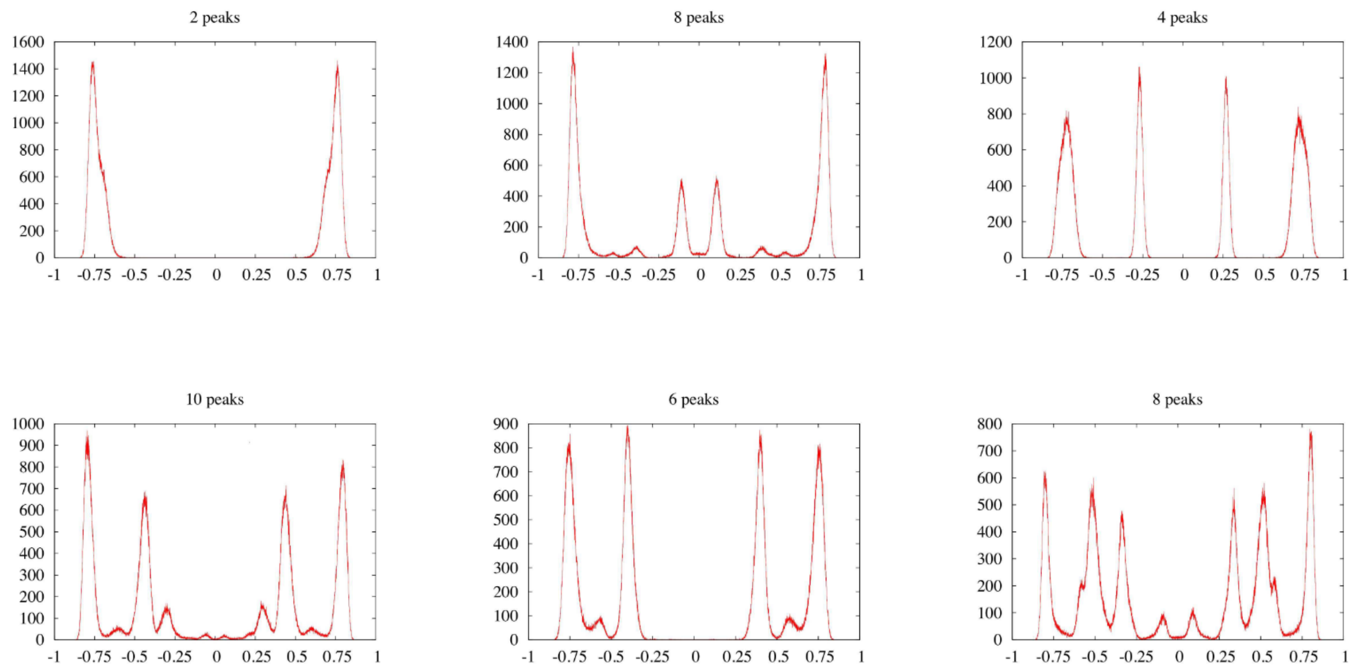
Many repetitions $P(q_{s\sigma}) = \frac{1}{2}\delta(q_{s\sigma} - m^2) + \frac{1}{2}\delta(q_{s\sigma} + m^2)$

Real replicas

Pdf of overlaps between replicas at fixed randomness

Sherrington-Kirkpatrick model with $N = 4096$ at $T = 0.4\,T_c$

$$H_J[\{s_i\}] = -\frac{1}{2}\sum_{i\neq j} J_{ij} s_i s_j \qquad q_{s\sigma} = \frac{1}{N}\sum_i s_i \sigma_i \qquad P_J(q_{s\sigma})$$



Finite size corrections in the Sherrington-Kirkpatrick model

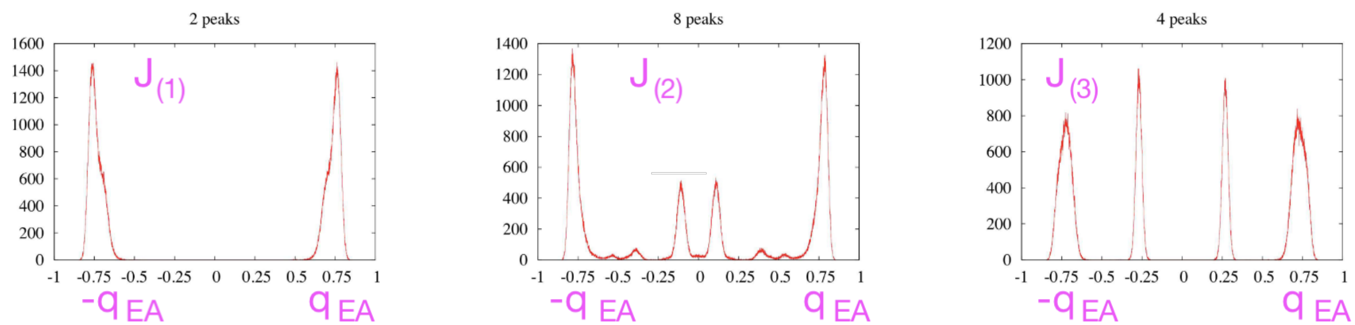
Aspelmeier, Billoire, Marinari & Moore (2007)

Real replicas

Ooverlaps between replicas at fixed randomness

Sherrington-Kirkpatrick model with $N = 4096$ at $T = 0.4\,T_c$

$$H_J[\{s_i\}] = -\frac{1}{2}\sum_{i\neq j} J_{ij} s_i s_j \qquad q_{s\sigma} = \frac{1}{N}\sum_i s_i\sigma_i \qquad P_J(q_{s\sigma})$$



$J_{ij}^{(1)}$ $\qquad$ $J_{ij}^{(2)}$ $\qquad$ $J_{ij}^{(3)}$

Data in each panel for a different realization of the random couplings

Each sample has peaks at $q_{s\sigma} = \pm q_{\mathrm{EA}} \simeq \pm 0.75$:

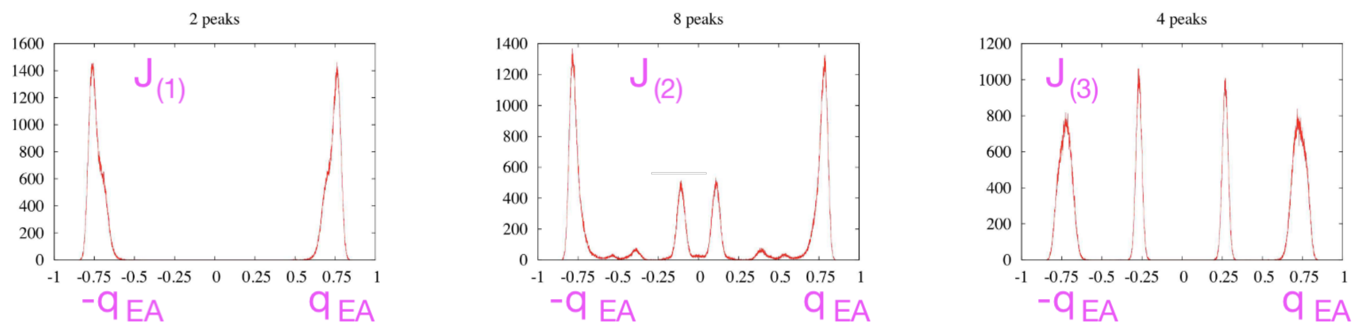
two configurations in the same (or the spin-reversed) state

Real replicas

Pdf of overlaps between replicas at fixed randomness

Sherrington-Kirkpatrick model with $N = 4096$ at $T = 0.4\,T_c$

$$H_J[\{s_i\}] = -\frac{1}{2}\sum_{i \neq j} J_{ij} s_i s_j \qquad q_{s\sigma} = \frac{1}{N}\sum_i s_i \sigma_i \qquad P_J(q_{s\sigma})$$



$J_{ij}^{(1)}$ $\qquad$ $J_{ij}^{(2)}$ $\qquad$ $J_{ij}^{(3)}$

Data in each panel for a different realization of the random couplings

$0.75 \simeq q_{\rm EA} < 1$ and the width of the peaks at $q_{s\sigma} = \pm q_{\rm EA}$:

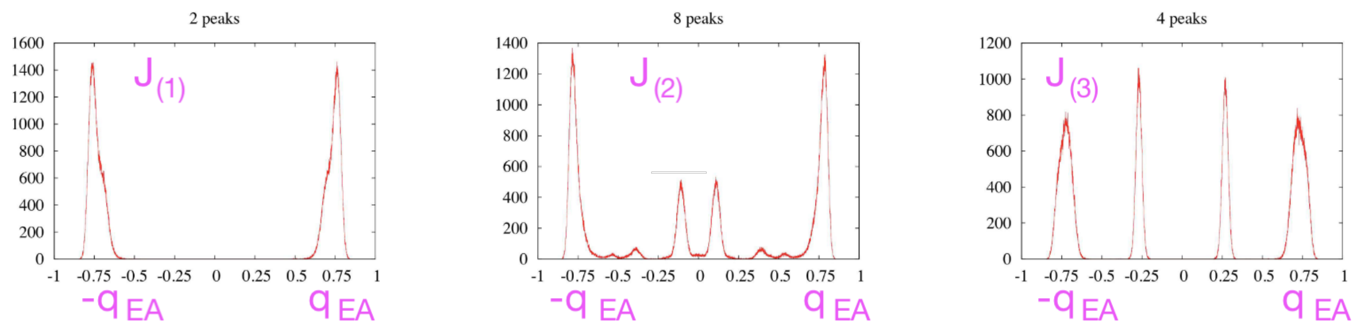
due to $0 < T < T_c$ and finite N, respectively

Real replicas

Pdf of overlaps between replicas at fixed randomness

Sherrington-Kirkpatrick model with $N = 4096$ at $T = 0.4\,T_c$

$$H_J[\{s_i\}] = -\frac{1}{2}\sum_{i\neq j} J_{ij} s_i s_j \qquad q_{s\sigma} = \frac{1}{N}\sum_i s_i \sigma_i \qquad P_J(q_{s\sigma})$$



$J_{ij}^{(1)}$ $\qquad J_{ij}^{(2)}$ $\qquad J_{ij}^{(3)}$

Data in each panel for a different realization of the random couplings

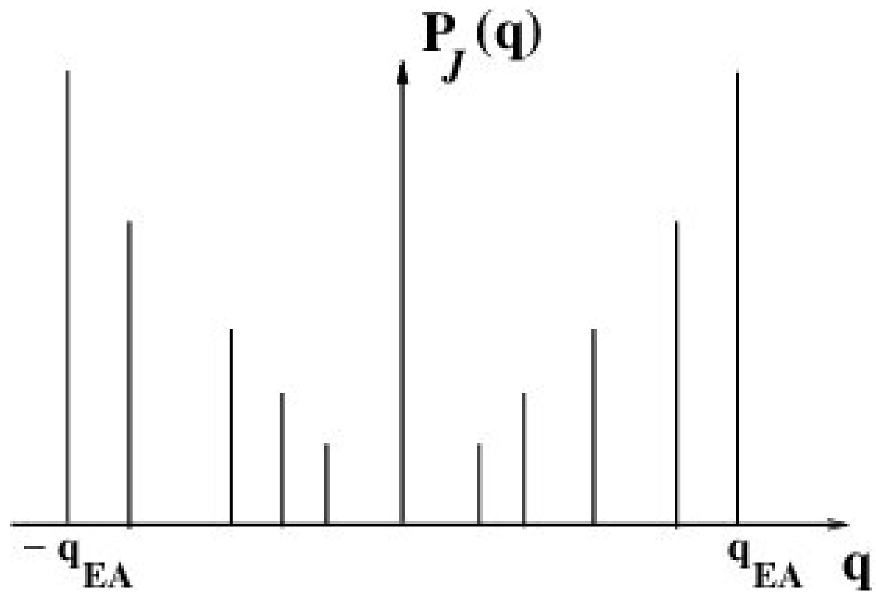
Most samples also have peaks at $|q_{s\sigma}| < q_{\rm EA}$:

replicas $\{s_i\}$ and $\{\sigma_i\}$ falling in different states

Real replicas

Pdf of overlaps between replicas at fixed randomness

SK model with $N \to \infty$ at $T < T_c$

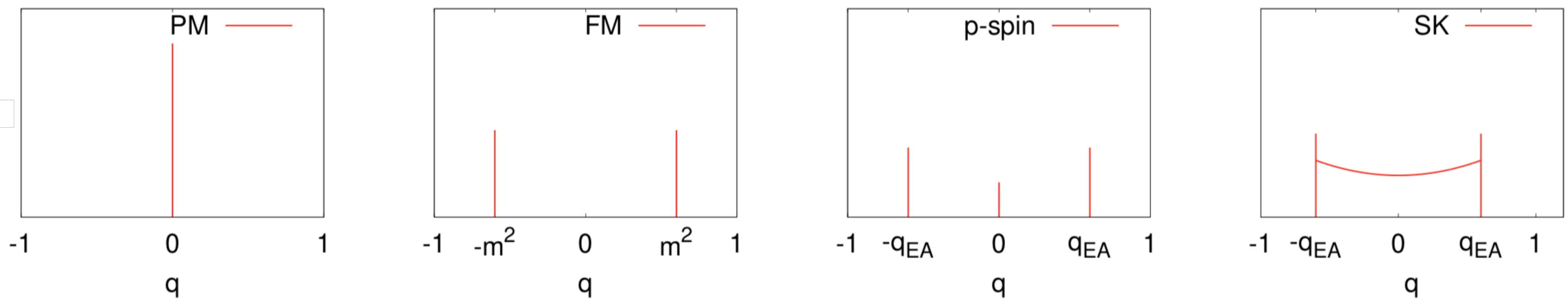


What happens if one averages $P_J(q)$ over disorder

Real replicas

Disordered averaged pdf of overlaps $P(q) = [P_J(q)]$

Parisi 79-82 prescription for the replica symmetry breaking Ansatz yields



High temperature FM Structural glasses Spin-glasses

Thermodynamic quantities, in particular the equilibrium free-energy density are expressed as functions of $P(q)$.

The equilibrium free-energy density predicted by the replica theory was confirmed by **Guerra & Talagrand 00-04** indepedent mathematical-physics mthods.

Typical vs. averaged

TAP vs. Replicas

Precursors

Look at an integer parameter n

and its $n \to 0$ limit

In 1972 Fortuin and Kasteleyn studied the **Potts model** with n components :

- $n = 2$ Ising
- $n = 1$ percolation
- $n = 0$ random resistors

Use the identify $x^n = \exp(n \ln x)$ and expand around $n = 0$:

$$\lim_{n \to 0} x^n = 1 + n \ln x + \mathcal{O}(n^2)$$

Replica method

A sketch

$$-\beta[f_J] = \lim_{N\to\infty} \frac{[\ln Z_N(\beta, J)]}{N} = \lim_{N\to\infty} \lim_{n\to 0} \frac{[Z_N^n(\beta, J)] - 1}{Nn}$$

Z_N^n partition function of n independent copies of the system : **replicas**.

Gaussian average over disorder : coupling between replicas

$$\sum_a \sum_{i\neq j} J_{ij} s_i^a s_j^a \ \Rightarrow \ \sum_{i\neq j} \left(\sum_a s_i^a s_j^a\right)^2$$

Quadratic decoupling with the Hubbard-Stratonovich trick

$$Q_{ab} \sum_i s_i^a s_i^b + \frac{1}{2} Q_{ab}^2$$

Q_{ab} is a 0×0 matrix but it admits an interpretation in terms of **overlaps**

The elements of Q_{ab} can evaluated by saddle-point if one exchanges the limits $N \to \infty$ $n \to 0$ with $n \to 0$ $N \to \infty$.

Replica method

In more detail

Z_N^n partition function of n independent copies of the system : **replicas**.

$$Z_N^n(\beta, J) = \underbrace{\sum_{\{s_i^{(1)}=\pm 1\}} \cdots \sum_{\{s_i^{(n)}=\pm 1\}}}_{\text{notation } \mathrm{Tr}_{\{s_i^a\}}} e^{-\beta \sum_{a=1}^n \sum_{i\neq j} J_{ij} s_i^a s_j^a}$$

One can exchange the order of the trace and the average over disorder

$$[Z_N^n(\beta, J)] = \mathrm{Tr}_{\{s_i^a\}} \int \prod_{i\neq j} dJ_{ij} P(J_{ij}) \; e^{-\beta \sum_{a=1}^n \sum_{i\neq j} J_{ij} s_i^a s_j^a}$$

$$\boxed{[Z_N^n(\beta, J)] = \mathrm{Tr}_{\{s_i^a\}} \; e^{-\beta H_{\text{eff}}[\{s_i^a\}]}}$$

$H_{\text{eff}}[\{s_i^a\}]$ does not have any randomness but couples the replicas

$$\boxed{\sum_a \sum_{i\neq j} J_{ij} s_i^a s_j^a \;\Rightarrow\; \sum_{i\neq j} \left(\sum_a s_i^a s_j^a\right)^2}$$

Replica method

In more detail

$$[Z_N^n(\beta,J)] = \mathrm{Tr}_{\{s_i^a\}}\, e^{-\beta H_{\rm eff}[\{s_i^a\}]}$$

$H_{\rm eff}[\{s_i^a\}]$ does not have any randomness but couples the replicas

$$\sum_{i\neq j}\left(\sum_a s_i^a s_j^a\right)^2 = \sum_{i\neq j}\sum_a\sum_b s_i^a s_j^a s_i^b s_j^b \sim \sum_{ab}\sum_i s_i^a s_i^b \sum_j s_j^a s_j^b$$

One sees Q_{ab} here, introduce their definition via a delta or apply Hubbard-Stratonovich

Once this done, one can exchange the trace (the sum over spin configurations) and the integral over Q_{ab} and end up with

$$[Z_N^n(\beta,J)] \propto \int \prod_{ab} dQ_{ab} e^{-F(Q_{ab})}$$

Replica method

For the SK model

$Q_{ab} = q_{ab}$ and $p = 2$

$$\begin{aligned}
\beta F(q_{ab}) &= -\frac{N\beta^2 J^2}{2}\left[-\sum_{a\neq b} q_{ab}^p + n\right] - N\ln\zeta(q_{ab}) \ , \\
\zeta(q_{ab}) &= \sum_{s_a} e^{-\beta H(q_{ab}, s_a)} \ , \\
H(q_{ab}, s_a) &= -J\sum_{ab} q_{ab} s_a s_b - h\sum_a s_a \ ,
\end{aligned}$$

Replica method

In more detail

$$[Z_N^n(\beta, J)] = \text{Tr}_{\{s_i^a\}}\, e^{-\beta H_{\text{eff}}[\{s_i^a\}]} \propto \int \prod_{ab} dQ_{ab} e^{-F(Q_{ab})}$$

$H_{\text{eff}}[\{s_i^a\}]$ and Q_{ab} do not have any randomness but couple the replicas

The elements of Q_{ab} can be evaluated by saddle-point if one exchanges the limits $N \to \infty$ $n \to 0$ with $n \to 0$ $N \to \infty$.

At the saddle-point level one identifies $Q_{ab}^{sp} = N^{-1}\langle \sum_i s_i^a s_i^b \rangle$

The spin glass transition is from the paramagnetic state with $Q_{a\neq b} = 0$ to a spin glass state with $Q_{a\neq b} \neq 0$ as the temperature is decreased.

Replica method

SK model : replica symmetric Ansatz

Permutation symmetry between replicas $\Rightarrow$

Insert $Q_{a\neq b} = q$ and $Q_{aa} = 1$ in the effective Hamiltonian

Saddle-point with respect to q and $n \to 0$

$$\boxed{q = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}}\, e^{-z^2/2}\, \tanh\left(\beta J \sqrt{q} z\right)}$$

Note the similarity with the equation for m in the Curie-Weiss model

$q = 0$ for $T \geq T_c = J$

$q \neq 0$ for $T < T_c = J$

Problem I Is this solution stable ? No

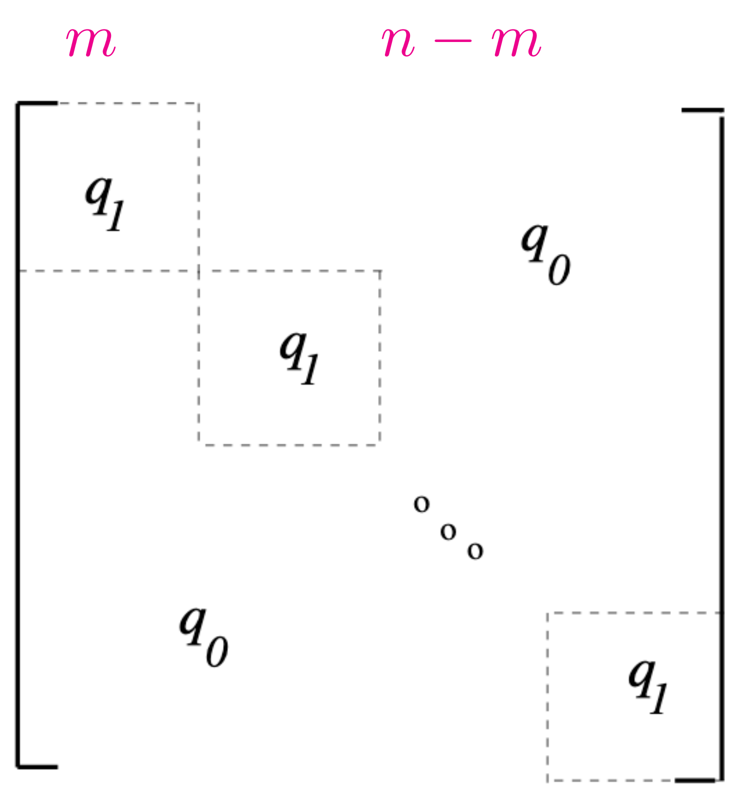
Problem II Does it have a zero-temperature vanishing entropy ? No

Problem III Ground state energy density $e = -0.77 \pm 0.01$ while the replica symmetric value $e = -0.798$, is three standard deviations smaller (in units of J)

Replica method

SK model: one step replica symmetry breaking

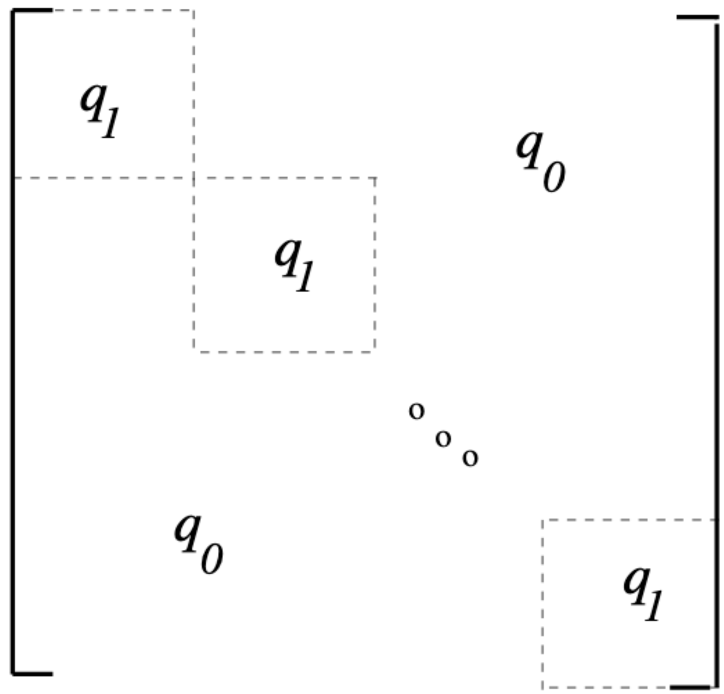
Permutation symmetry broken



$n \times n$ matrix divided in diagonal blocks of size $m \times m$ and the rest

Replica method

SK model : one step replica symmetry breaking



Problem I Stability : improved but not solved

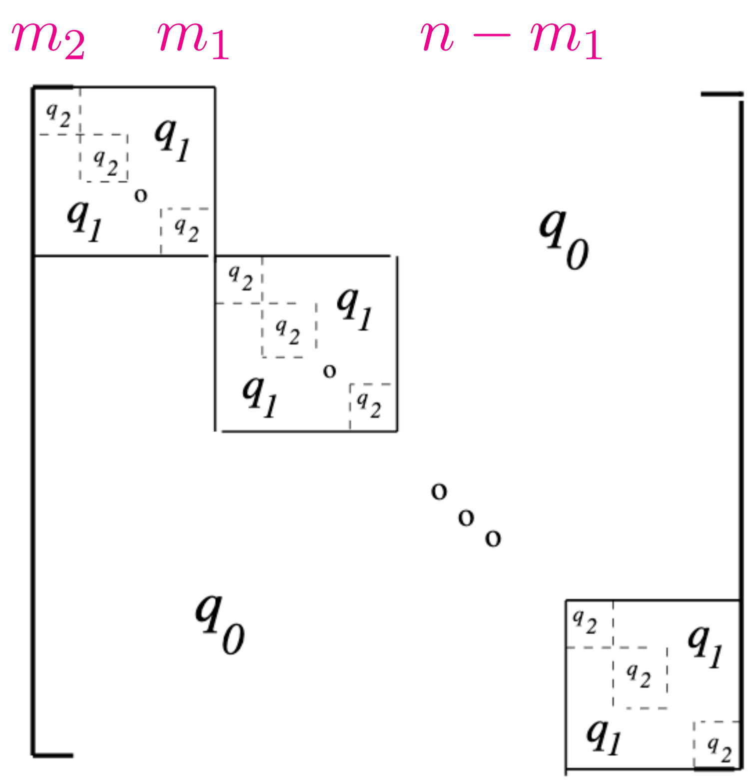
Problem II Zero-temperature entropy : improved but not solved

Problem III e closer to numerical value

Replica method

SK model: two step replica symmetry breaking

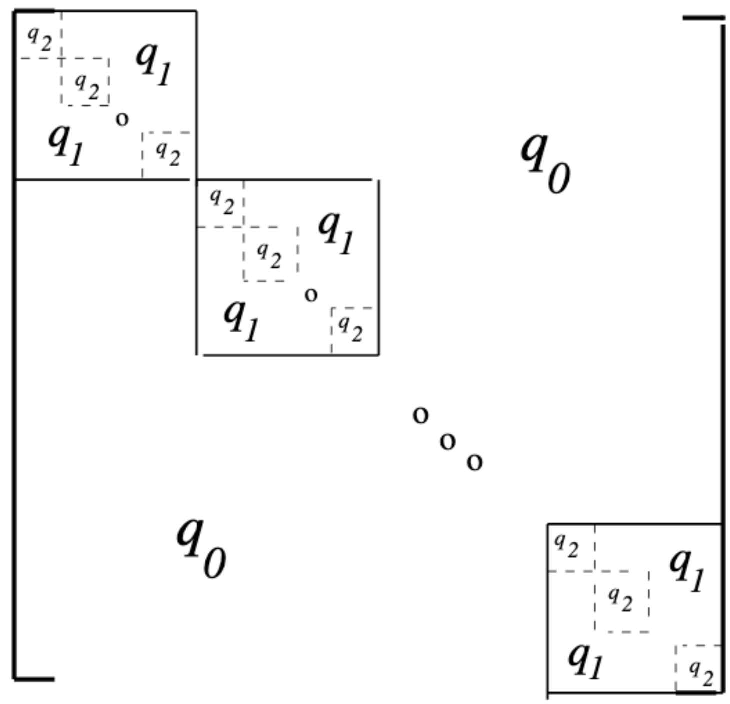
Permutation symmetry broken



$n \times n$ matrix divided in diagonal blocks of size $m_2 \times m_2$, and the rest in blocks of size $m_1 \times m_1$ and the rest

Replica method

SK model: two step replica symmetry breaking



Problem I Stability : improved but not solved

Problem II Zero-temperature entropy : improved but not solved

Problem III e closer to numerical value

Replica method

SK model: full replica symmetry breaking

Blocks of size m_i with parameter q_i

e.g. for replica symmetric case one block a single q.

∞ number of breaking steps, that is, of blocks

$m_i \mapsto x$ and the parameter $q_i \mapsto q(x)$

$$[\langle s_i \rangle^2] = \int_0^1 dx\, q(x) = \int \frac{dx}{dq}\, dq\, q(x) = \int dq\, P(q)\, q$$

with $$\boxed{P(q) = \frac{dx}{dq}}$$

Problem I Stability : solved

Problem II Zero-temperature entropy : solved $S = 0$

Problem III e in agreement with numerical value within numerical accuracy $e = -0.7633$

MFT for disordered spin models

Phase transition

For large N one expects $J_{ij}^2 \simeq [J_{ij}^2] = J^2/N$ with $J = \mathcal{O}(1)$

Simplification $m_i = \tanh\left\{\beta \sum_{j(\neq i)} J_{ij} m_j - \beta^2 m_i \frac{J^2}{N} \sum_{j(\neq i)} (1 - m_j^2)\right\}$

A 2nd order phase transition $\Rightarrow m_i \simeq 0$ at $T \stackrel{<}{\sim} T_c$ then using $\tanh y \sim y$

The TAP equations become $m_i \sim \beta \sum_{j(\neq i)} J_{ij} m_j - \beta^2 J^2 m_i$

Diagonalize this eq. going to the basis of eigenvectors of the J_{ij} matrix

The eqs read $m_\lambda \sim \beta \Big(J_\lambda - \beta J^2 \Big) m_\lambda$

The notation we use is such that

J_λ is an eigenvalue of the J_{ij} matrix associated to the eigenvector $\vec{v}_\lambda$

m_λ represents the projection of $\vec{m}$ on the eigenvector $\vec{v}_\lambda$, $m_\lambda = \vec{v}_\lambda \cdot \vec{m}$

with $\vec{m}$ the N-vector with components m_i, $\vec{m} = (m_1, \ldots, m_N)$

MFT for disordered spin models

Phase transition

If we add a weak external field the eqs read $m_\lambda \sim \beta(J_\lambda - \beta J^2) m_\lambda + \beta h_\lambda^{\rm ext}$

The variation with respect to the field at linear order is

$$\left. \frac{\partial m_\lambda}{\partial h_\lambda^{\rm ext}} \right|_{\vec{h}^{\rm ext}=\vec{0}} = \beta (J_\lambda - \beta J^2) \left. \frac{\partial m_\lambda}{\partial h_\lambda^{\rm ext}} \right|_{\vec{h}^{\rm ext}=\vec{0}} + \beta$$

and the *staggered susceptibility* (of the projection on $\vec{v}_\lambda$)

$$\chi_\lambda \equiv \left. \frac{\partial m_\lambda}{\partial h_\lambda^{\rm ext}} \right|_{\vec{h}^{\rm ext}=\vec{0}} = \beta \ \left(1 - \beta J_\lambda + (\beta J)^2 \right)^{-1}$$

Random matrix theory tells us that the eigenvalues of the random matrix $J_{ij} = \mathcal{O}(1/\sqrt{N})$ are distributed with the Wigner semi-circle law and the largest eigenvalue is $J_\lambda^{\rm max} = 2J$

> The staggered susceptibility of staggered magnetization in the direction of the largest eigenvalue diverges at $\beta_c J = 1$ the correct value

MFT for disordered spin models

Phase transition

If we add a weak external field the eqs read $m_\lambda \sim \beta(J_\lambda - \beta J^2) m_\lambda + \beta h_\lambda^{\rm ext}$

The variation with respect to the field at linear order is

$$\left.\frac{\partial m_\lambda}{\partial h_\lambda^{\rm ext}}\right|_{\vec h^{\rm ext}=\vec 0} = \beta(J_\lambda - \beta J^2) \left.\frac{\partial m_\lambda}{\partial h_\lambda^{\rm ext}}\right|_{\vec h^{\rm ext}=\vec 0} + \beta$$

and the *staggered susceptibility* (of the projection on $\vec v_\lambda$)

$$\chi_\lambda \equiv \left.\frac{\partial m_\lambda}{\partial h_\lambda^{\rm ext}}\right|_{\vec h^{\rm ext}=\vec 0} = \beta \left(1 - \beta J_\lambda + (\beta J)^2\right)^{-1}$$

Random matrix theory tells us that the eigenvalues of the random matrix J_{ij} are distributed with the Wigner semi-circle law

For $J_{ij} = \mathcal{O}(1/\sqrt{N})$ the largest eigenvalue is $J_\lambda^{\rm max} = 2J$

The staggered susceptibility for the largest eigenvalue diverges at $\beta_c J = 1$

Without the reaction term the divergence is at the inexact value $T^* = 2T_c$

Plan of lecture

— Definition & examples

— Properties

— List of methods

— Thouless-Anderson-Palmer equations

– Local order parameters & landscapes (beyond Ginzburg-Landau)

– Statistical averages

– Real replicas

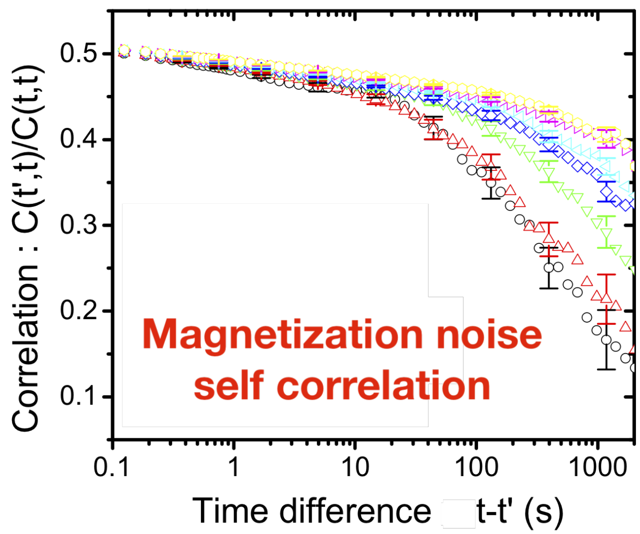
— Replica theory

— **Relaxation dynamics (experiments, numerics)**

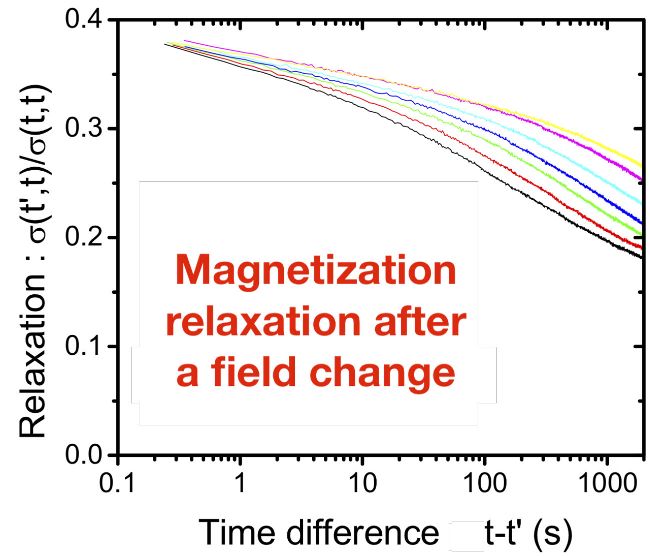
— Relaxation dynamics (theory)

Spin-glasses

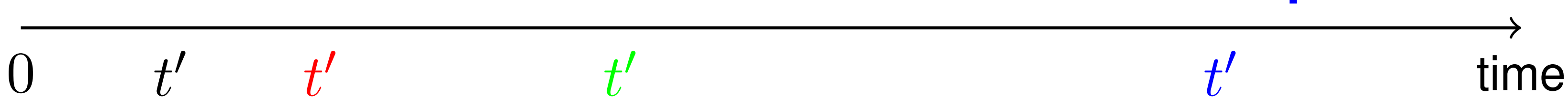
slow relaxation & loss of stationarity (aging)



Correlation



Linear response



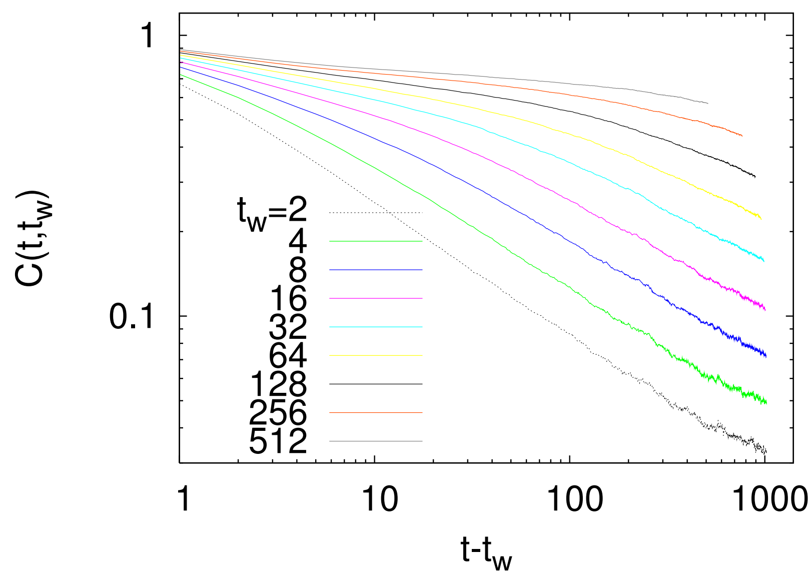
Different curves are measured after log-spaced reference times t' after the quench: **breakdown of stationarity** $\Longrightarrow$ far from equilibrium

No identifiable growing length $\mathcal{R}(t)$: **glassy microscopic mechanisms ?**

Spin-glass experiments **Hérisson & Ocio 02-04**

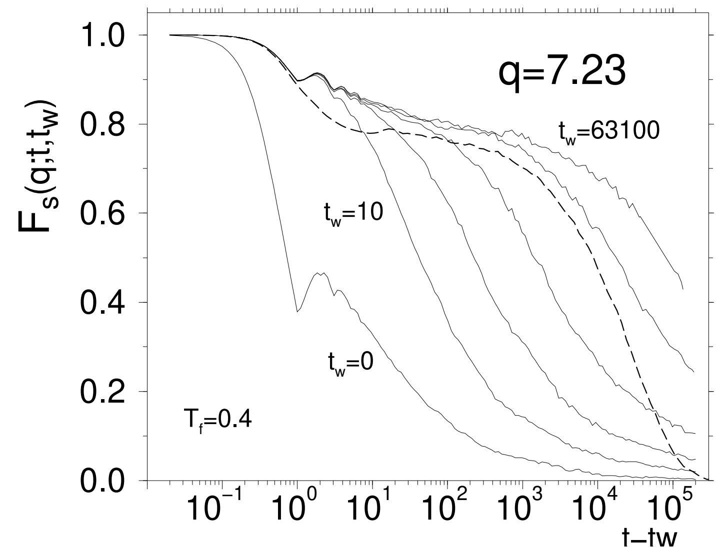
Ferromagnet vs glass

Not so different as long as correlations are concerned



2d Ising model - spin-spin

Sicilia ***et al.*** **07**



Lennard-Jones - density-density

Kob & Barrat 99

One correlation can exhibit stationary and non stationary relaxation

in different two-time regimes

Plan of lecture

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— **Relaxation dynamics (theory)**

Models

Self generated disorder

Exact treatment

- Langevin dynamics of spins with quenched random interactions on a complete graph.
- Particles in interaction moving in an infinitely dimensional space.

Approximate treatment (e.g. no quenched randomness)

- Finite d : self-consistent resummation of infinite subsets of diagrams mode-coupling theory, etc.

The exact treatment of an approximate model is identical to the approximate treatment of the realistic model

Schwinger-Dyson equations

Stochastic dynamics

In the $N \to \infty$ limit exact causal Schwinger-Dyson equations

$$(\partial_t - z_t) C(t,t_w) = \int dt' \left[\Sigma(t,t') C(t',t_w) + D(t,t') R(t_w,t') \right] + 2T R(t',t)$$

$$(\partial_t - z_t) R(t,t_w) = \int dt' \; \Sigma(t,t') R(t',t_w) + \delta(t-t_w)$$

where the self-energy and vertex are functions of C and R and depend on the choice of the model/approximation

Schwinger-Dyson equations

Stochastic dynamics

In the $N \to \infty$ limit exact causal Schwinger-Dyson equations

$$(\partial_t - z_t) C(t,t_w) = \int dt' \left[\Sigma(t,t') C(t',t_w) + D(t,t') R(t_w,t') \right] + 2T R(t',t)$$

$$(\partial_t - z_t) R(t,t_w) = \int dt' \; \Sigma(t,t') R(t',t_w) + \delta(t-t_w)$$

for p spin spherical models they read C and R :

$$D(t,t_w) = \frac{p}{2} C^{p-1}(t,t_w) \; , \quad \Sigma(t,t_w) = \frac{p(p-1)}{2} C^{p-2}(t,t_w) \, R(t,t_w)$$

and the Lagrange multiplier z_t is fixed by $C(t,t)=1$.

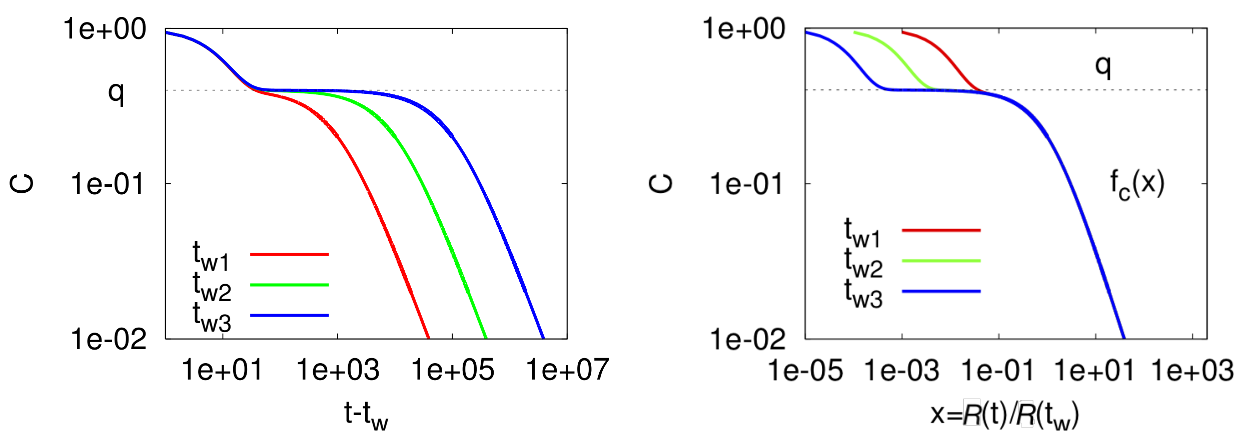
Dynamics of p spin models

Analytic results

- separation of time-scales $$C(t,t_w) = C_{eq}(t-t_w) + C_{ag}(t,t_w)$$ $$\chi(t,t_w) = \chi_{eq}(t-t_w) + \chi_{ag}(t,t_w)$$
- Highly non-trivial relation between χ and C : violations of FDT.
- (Approx) Time-reparametrization invariance $t \to h(t)$ in aging.

LFC & Kurchan 93

(Smart) numerical results

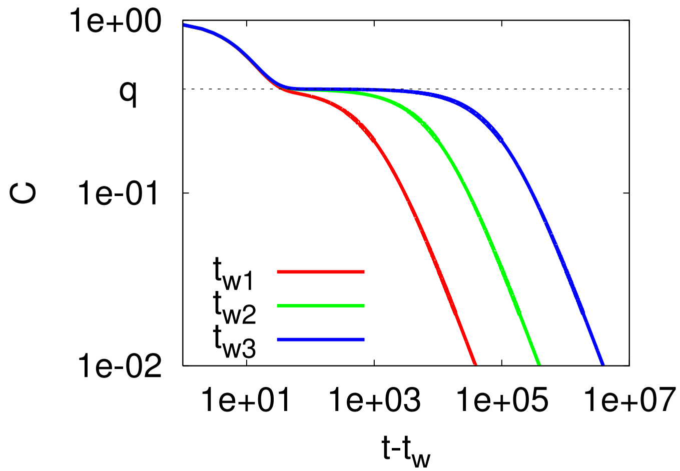


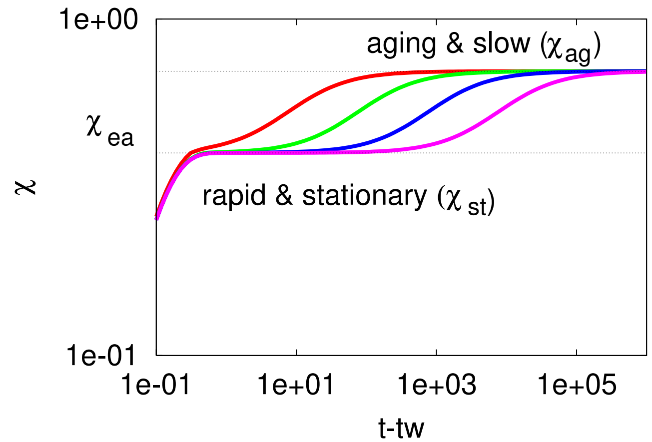
Kim & Latz 00

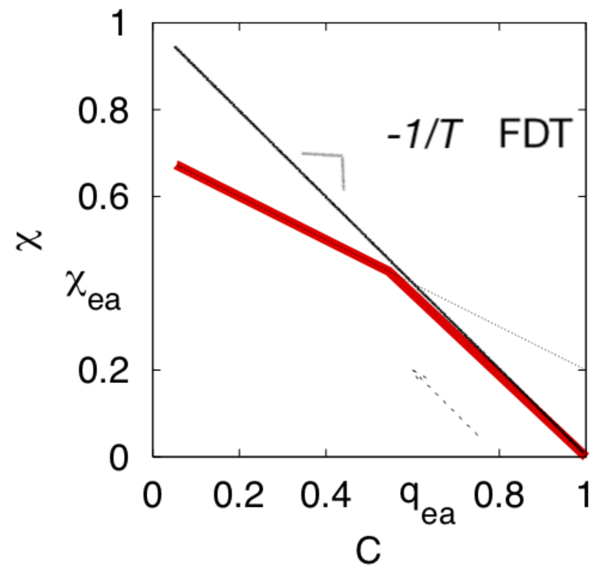
Meaning of $R(t) \sim t^\alpha$?

Glassy dynamics

Fluctuation-dissipation relation : parametric plot







Analytic solution to a mean-field model **LFC & J. Kurchan 93**

Summary

Main issues