

Conjugacy between bosonic and classical Ornstein–Uhlenbeck semigroups

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Two and a half decades of the Macroscopic Fluctuation Theory
ICTS, Bengaluru, 2026

MFT originated from the study of (classical) *lattice systems* driven out of equilibrium by *boundary reservoirs* (e.g. the SSEP in contact with reservoirs at different chemical potentials).

In today's talk: we consider a **quantum** system, consisting of m bosonic modes (harmonic oscillators), which one can think of at the sites of a *lattice*.

The contact with the *reservoirs* is modelled by the **Lindblad terms** of the generator.

To study the *hydrodynamic limit* first, and *large deviations* next, one would have to take $m \rightarrow \infty$: but we are NOT interested in this limit today!

The bosonic Ornstein–Uhlenbeck generator

We consider m quantum harmonic oscillators; they are described in terms of the annihilation and creation operators a_j, a_j^* , $j = 1, \dots, m$, satisfying the CCR

$$[a_j, a_k^*] = \delta_{jk}, \quad [a_j, a_k] = [a_j^*, a_k^*] = 0.$$

We consider them as operators action on the Hilbert space $\mathfrak{h} = \ell^2(\mathbb{Z}_+^m)$. (Note: they are *unbounded*.)

By the GKLS theorem (Gorini–Kossakowski–Lindblad–Sudarshan), the time evolution of an open quantum system is a quantum Markov semigroup, with generator of Lindblad form:

Bosonic OU generator (Heisenberg picture)

$$\mathcal{L}f = i[h, f] + \sum_{\alpha} \left([\ell_{\alpha}^*, f] \ell_{\alpha} + \ell_{\alpha}^* [f, \ell_{\alpha}] \right)$$

- h : a quadratic Hamiltonian in a, a^* ;
- ℓ_{α} : Lindblad operators, linear in a, a^* : $\ell_{\alpha} = \langle\langle \mathcal{X}_{\alpha}, \begin{pmatrix} a \\ a^* \end{pmatrix} \rangle\rangle$, $\mathcal{X}_{\alpha} \in \mathbb{C}^{2m}$.

Guiding question

For **isolated** systems (i.e. no Lindblad terms) with a quadratic Hamiltonian, Gaussian states stay Gaussian at all times t , and the "*coherent states*" $|v\rangle = \mathcal{W}(v)|0\rangle$ stay coherent, following exactly the classical trajectory $v(t)$.

Is there a **dissipative** analogue (i.e. with Lindblad terms) of this correspondence, working for the bosonic OU semigroup?

Answer

Yes: through the Glauber–Sudarshan *normal-order map*, which intertwines the quantum and the classical OU dynamics.

The classical analogue: the Ornstein–Uhlenbeck process

Phase space $\mathbb{R}^{2m} \ni \xi = (x, y)$. The classical OU generator is

$$Lf(\xi) = \frac{1}{2} \sum_{i,j=1}^{2m} A_{ij} \partial_{\xi_i} \partial_{\xi_j} f(\xi) + \sum_{i,j=1}^{2m} M_{ij} \xi_j \partial_{\xi_i} f(\xi),$$

It has:

- constant *diffusion* (noise) *matrix* A , such that $A = A^\top \geq 0$;
- linear *drift term*, with matrix M .

It corresponds to Brownian motion of a particle subject to linear friction and noise.

Question: Why is the quantum generator $\mathcal{L}$ a "quantization" of the classical generator L ?

To make the relation more apparent, we need to write both of them in an equivalent form.

We start by introducing some **NOTATION**:

$\mathcal{R} := \mathbb{C}^m$, considered as a *real* space of $\dim=2m$,
with *Euclidean product* $\langle u, v \rangle = u^+ v + v^+ u$
and *symplectic form* $\omega(u, v) = \langle iu, v \rangle$

$\mathfrak{R} :=$ real-linear maps $\varphi: \mathcal{R} \rightarrow \mathcal{R}$;

Note: they have the form: $\varphi z = U_\varphi z + V_\varphi \bar{z}$
(U_φ, V_φ : complex $m \times m$ matrices.)

To $\varphi \in \mathfrak{R}$ we associate the $2m \times 2m$ matrix

$$\mathcal{F}_\varphi = \begin{pmatrix} U_\varphi & V_\varphi \\ \bar{V}_\varphi & \bar{U}_\varphi \end{pmatrix}, \quad \text{so that} \quad \mathcal{F}_\varphi \begin{pmatrix} v \\ \bar{v} \end{pmatrix} = \begin{pmatrix} \varphi v \\ \overline{\varphi v} \end{pmatrix}.$$

Let also $I = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

A) For the *classical* generator L :

- we have the Poisson bracket $\{f, g\}$; denote $\text{ad}_f = \{f, \cdot\}$ ("adjoint" action)
- introduce the complex variable $z = x + iy$, $\bar{z} = x - iy \in \mathbb{R} := \mathbb{C}^m$
- the Poisson bracket, in terms of z and $\bar{z}$, is given by $(f, g : \mathbb{R} \rightarrow \mathbb{C})$

$$\{f, g\} = \begin{pmatrix} \partial_z f \\ \partial_{\bar{z}} f \end{pmatrix}^\top \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_z g \\ \partial_{\bar{z}} g \end{pmatrix},$$

equivalently, in terms of the variables z and $\bar{z}$,

$$\left\{ \begin{pmatrix} z \\ \bar{z} \end{pmatrix}, \begin{pmatrix} z \\ \bar{z} \end{pmatrix}^\top \right\} = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then, the **classical OU generator** L , in these complex variables, becomes

$$L = \begin{pmatrix} \text{ad}_z \\ \text{ad}_{\bar{z}} \end{pmatrix}^+ \mathcal{F}_Q \begin{pmatrix} \text{ad}_z \\ \text{ad}_{\bar{z}} \end{pmatrix} + i \begin{pmatrix} z \\ \bar{z} \end{pmatrix}^+ I\mathcal{F}_B \begin{pmatrix} \text{ad}_z \\ \text{ad}_{\bar{z}} \end{pmatrix}, \quad Q = Q^* \geq 0.$$

(where Q and B are associated to A and M)

B) For the quantum generator:

- for an operator f on $\mathfrak{h}$, denote $\text{ad}_f = [f, \cdot]$ ("adjoint" action)
- denote also $\text{l}_f g = fg$, $\text{r}_f g = gf$ ("left" and "right" multiplication)
- CCR, in matrix form, become

$$\left[\begin{pmatrix} a \\ a^* \end{pmatrix}, \begin{pmatrix} a \\ a^* \end{pmatrix}^+ \right] = -I,$$

Then, the **quantum OU generator** $\mathcal{L}$ becomes

$$\mathcal{L} = - \begin{pmatrix} \text{ad}_a \\ \text{ad}_{a^*} \end{pmatrix}^+ \mathcal{F}_{\mathcal{Q}} \begin{pmatrix} \text{ad}_a \\ \text{ad}_{a^*} \end{pmatrix} - \frac{1}{2} \left(\begin{pmatrix} \text{l}_a \\ \text{l}_{a^*} \end{pmatrix} + \begin{pmatrix} \text{r}_a \\ \text{r}_{a^*} \end{pmatrix} \right)^+ I \mathcal{F}_{\mathcal{B}} \begin{pmatrix} \text{ad}_a \\ \text{ad}_{a^*} \end{pmatrix}.$$

where $\mathcal{Q}, \mathcal{B} \in \mathfrak{K}$ are determined by h and $\{\ell_\alpha\}$ as follows:

decomposing

$$\sum_{\alpha} (\mathcal{X}_{\alpha}^{+} \cdot) \mathcal{X}_{\alpha} = \mathcal{F}_{X_1} + I\mathcal{F}_{X_2},$$

and letting

$$h = \frac{1}{2} \begin{pmatrix} a \\ a^* \end{pmatrix}^+ \mathcal{F}_H \begin{pmatrix} a \\ a^* \end{pmatrix}$$

we set

$$\mathcal{Q} := X_1, \quad \mathcal{B} := 2X_2 + iH,$$

Note: Classically, $\mathcal{Q} \geq 0$ and $\mathcal{B}$ are **arbitrary**.

Quantum mechanically, $\mathcal{Q}$ and $\mathcal{B}$ are **not** arbitrary: since $\sum_{\alpha} (\mathcal{X}_{\alpha}^{+} \cdot) \mathcal{X}_{\alpha} \geq 0$, one automatically has

$$2\mathcal{F}_{\mathcal{Q}} \pm \frac{1}{2}(I\mathcal{F}_{\mathcal{B}} + \mathcal{F}_{\mathcal{B}}^*I) \geq 0,$$

which in particular implies $\mathcal{Q} \geq 0$.

(This is the trace, on the generator, $\mathcal{L}$ of the *complete positivity* of the semigroup $\mathcal{P}_t$.)

Hence: *classical* OU generator $L \implies$ *quantum* OU generator $\mathcal{L}$,
under the "*Rule*":

Poisson brackets $\rightsquigarrow$ commutators;

multiplication by $z \rightsquigarrow \frac{1}{2}(l_a + r_a)$.

What we know classically

Assume $Q = Q^* \geq 0$ and $\text{Spec}(B) \subset \{\text{Re } \lambda < 0\}$.

1. There is a strongly continuous (*Feller*) Markov semigroup (P_t) on $C_0(\mathcal{R}; \mathbb{C})$ with generator L .
2. It has an *invariant probability* (unique if $Q > 0$): the **Gaussian measure** γ with mean 0 and covariance

$$C = 2 \int_0^\infty dt e^{tB^*} Q e^{tB}.$$

(Equivalently, C is a solution of the **Liapounov equation** $B^*C + CB = -2Q$.)

3. P_t is given by the **Mehler formula**

$$(P_t f)(z) = \int \gamma_{t,z}(dw) f(w),$$

$\gamma_{t,z}$ = Gaussian with mean $e^{tB^\#} z$ and covariance $C_t = 2 \int_0^t ds e^{sB^*} Q e^{sB}$.

4. P_t is determined by its action on the “**Weyl functions**” $W(v)(z) = \exp\{v^\top \bar{z} - v^+ z\}$:

$$P_t W(v) = e^{-\frac{1}{2} \langle v, C_t v \rangle} W(e^{tB} v).$$

Questions

Can we say the same for the quantum OU generator $\mathcal{L}$?

- Can we build a quantum Markov semigroup $\mathcal{P}_t$ with generator $\mathcal{L}$?
- Can we describe its invariant state?
- Is there a quantum analogue of the Mehler formula?
- And, most importantly: what is the relation with the classical semigroup P_t ?

Bosonic Gaussian states

Weyl operators: for $v \in \mathcal{R}$,

$$\mathcal{W}(v) = \exp \{ v^\top a^* - v^\dagger a \} \quad (\text{unitary}).$$

Definition

The *bosonic Gaussian state* with mean μ and covariance $\mathcal{C}$ is the state ϱ (self-adjoint, positive, trace class, $\text{Tr } \varrho = 1$) such that

$$\text{Tr} (\varrho \mathcal{W}(v)) = \exp \left\{ -i \omega(\mu, v) - \frac{1}{2} \langle v, \mathcal{C} v \rangle \right\}, \quad v \in \mathcal{R}.$$

Note

ϱ exists only if $\mathcal{F}_{\mathcal{C}} - \frac{1}{2}I \geq 0$. This is a genuinely quantum feature: it expresses *Heisenberg's uncertainty principle*.

The quantum Markov semigroup

As in the classical case, $\mathcal{P}_t$ is determined by its action on the Weyl operators.

Theorem

There is a unique *quantum Markov semigroup* $\mathcal{P}_t$ on the W^* -algebra $\mathcal{B}(\mathfrak{h})$ (weakly* continuous, completely positive, consisting of weakly* continuous maps, $\mathcal{P}_t \mathbf{1} = \mathbf{1}$) such that

$$\mathcal{P}_t \mathcal{W}(v) = e^{-\frac{1}{2} \langle v, \mathcal{C}_t v \rangle} \mathcal{W}(e^{t\mathcal{B}} v), \quad \mathcal{C}_t = 2 \int_0^t ds e^{s\mathcal{B}^*} \mathcal{Q} e^{s\mathcal{B}}.$$

If $\text{Spec}(\mathcal{B}) \subset \{\text{Re } \lambda < 0\}$, the *bosonic Gaussian state* ϱ with mean 0 and covariance

$$\mathcal{C} = 2 \int_0^\infty dt e^{t\mathcal{B}^*} \mathcal{Q} e^{t\mathcal{B}} \quad (\text{same formula as classically})$$

is *invariant* for $\mathcal{P}_t$.

Some remarks:

- The constraint $\mathcal{F}_C \pm \frac{1}{2}I \geq 0$ holds *automatically*.
- With enough Lindblad operators, $\text{Span}\{\mathcal{X}_\alpha\}_\alpha = \mathbb{C}^{2m}$, the invariant state is unique, and moreover $\mathcal{F}_C \pm \frac{1}{2}I > 0$.
- Saying that $\mathcal{L}$ is the **generator** is more delicate: $\mathcal{L}$ does not really act on $\mathcal{B}(\mathfrak{h})$; it holds only in a *weak sense*.

Extension to L^2 spaces

Classically, a standard argument extends P_t to a strongly continuous contraction semigroup $P_t^{(2)}$ on the Hilbert space $L^2(\mathcal{R}, \gamma)$.

One can do the same for the *quantum model*: by complete positivity, the Kadison–Schwarz inequality and the invariance of ϱ , we get:

Theorem

$\mathcal{P}_t$ extends to a strongly continuous contraction semigroup $\mathcal{P}_t^{(2,s)}$ on $L_s^2(\mathfrak{h}, \varrho)$, $s \in \{0, 1\}$, the completion of $\mathcal{B}(\mathfrak{h})$ with respect to

$$\langle f, g \rangle_{\varrho,1} = \text{Tr}(\varrho f^* g) \quad (\text{GNS}), \quad \langle f, g \rangle_{\varrho,0} = \text{Tr}(\varrho g f^*).$$

(The contraction in one line (for $s = 1$): Kadison–Schwarz gives $(\mathcal{P}_t f)^*(\mathcal{P}_t f) \leq \mathcal{P}_t(f^* f)$; taking the trace against the invariant ϱ :

$$\|\mathcal{P}_t f\|_{\varrho,1}^2 \leq \text{Tr}(\varrho \mathcal{P}_t(f^* f)) = \text{Tr}(\varrho f^* f) = \|f\|_{\varrho,1}^2$$

For $s = 0$ use $f f^*$ instead.)

The generator

Why useful: the non-commutative polynomials $\mathcal{A}_0$ in a_j, a_j^* (unbounded operators!) form a **dense** subset of $L_s^2(\mathfrak{h}, \varrho)$, and $\mathcal{L}$ makes perfect sense on $\mathcal{A}_0$.

(Likewise, $\mathcal{A}_0 = \mathbb{C}[z, \bar{z}]$ is dense in $L^2(\mathcal{R}, \gamma)$.)

On $\mathcal{A}_0$ the bosonic OU generator $\mathcal{L}$ makes perfect sense! We have:

Theorem: $\mathcal{A}_0$ is a core for the generator $\mathcal{L}^{(2,s)}$ of $\mathcal{P}_t^{(2,s)}$, and on $\mathcal{A}_0$ the generator acts as the Lindblad operator

$$\mathcal{L}f = i[h, f] + \sum_{\alpha} \left([\ell_{\alpha}^*, f] \ell_{\alpha} + \ell_{\alpha}^* [f, \ell_{\alpha}] \right).$$

Towards the comparison *quantum* / *classical*: symplectic diagonalization

To relate the quantum and the classical OU semigroups we need the *normal* and *antinormal* order maps.

To define them, we first need a linear algebra lemma (known as **Williamson's Theorem**):

Lemma (symplectic diagonalization)

Given $\mathcal{C} > 0$, there exist σ symplectic (i.e. $\sigma^* i \sigma = i$) and D real, diagonal and > 0 , such that

$$\mathcal{C} = \sigma^* D \sigma.$$

Bogoliubov transformation

Let $\sigma \in \text{Sp}(\mathcal{R})$. Write $\sigma z = Uz + V\bar{z}$. The corresponding **Bogoliubov transformation** is

$$\begin{pmatrix} b \\ b^* \end{pmatrix} = \begin{pmatrix} U & V \\ \bar{V} & \bar{U} \end{pmatrix} \begin{pmatrix} a \\ a^* \end{pmatrix}$$

(It will allow us reduce everything to the case of a **diagonal** covariance.)

Main Observation: since σ is symplectic, b, b^* still **satisfy the CCR**.

On the phase space, the same transformation defines new variables

$$\begin{pmatrix} w \\ \bar{w} \end{pmatrix} = \begin{pmatrix} U & V \\ \bar{V} & \bar{U} \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix}.$$

Normal/Antinormal maps

Note: the covariance matrix is $\mathcal{C} > 0$ (as a consequence of $\mathcal{C} \geq \frac{1}{2}I$). Hence, we can take its symplectic diagonalization: $\mathcal{C} = \sigma^* D \sigma$.

Then, consider the Bogoliubov's transformation $a, a^* \longrightarrow b, b^*$, and the corresponding change of variables $z, \bar{z} \longrightarrow w, \bar{w}$.

Definition: The **normal** / **antinormal** maps Φ / Ψ are defined as follows. In multi-index notation, they are the bijections

$$\Phi: \mathcal{A}_0 \rightarrow \mathcal{A}_0 \quad , \quad \Psi: \mathcal{A}_0 \rightarrow \mathcal{A}_0$$

uniquely determined by their action on monomials

$$\Phi((b^*)^k b^h) = w^h \bar{w}^k \quad , \quad \Psi(w^h \bar{w}^k) = b^h (b^*)^k.$$

- Φ : *normal* order (creation operators on the left).
- Ψ : *antinormal* order (creation operators on the right).

At the level of polynomials, these maps are "trivial"; the whole point is whether they extend to the Hilbert spaces $L_s^2(\mathfrak{h}, \rho) \leftrightarrow L^2(\mathcal{R}, \gamma)$.

This depends on the choice of $s \in \{0, 1\}$ and of the "correct" the Gaussian probability γ on the classical side.

Theorem

(A) Set $C_+ = C - \frac{1}{2}\sigma^*\sigma$ and let γ_+ be the Gaussian probability with mean 0 and covariance C_+ . Then Φ extends to a contraction

$$\Phi: L_1^2(\mathfrak{h}, \varrho) \longrightarrow L^2(\mathcal{R}, \gamma_+).$$

(B) Set $C_- = C + \frac{1}{2}\sigma^*\sigma$ and let γ_- be the Gaussian measure with mean 0 and covariance C_- . Then Ψ extends to a contraction

$$\Psi: L^2(\mathcal{R}, \gamma_-) \longrightarrow L_0^2(\mathfrak{h}, \varrho).$$

Note: $C_{\pm} \geq 0$ precisely because $C \geq \frac{1}{2}I$ (which implies $D \geq \frac{1}{2}$) (uncertainty principle).

Idea of the proof:

There is an orthogonal basis of $L_1^2(\mathfrak{h}, \varrho)$ of non-commutative Hermite polynomials $(h, k \in \mathbb{Z}_+^m)$

$$\mathfrak{e}_{h,k} := \sum_{\ell \in \mathbb{Z}_+^m} (-1)^{|\ell|} \ell! \binom{h}{\ell} \binom{k}{\ell} (d - \frac{1}{2})^\ell (b^*)^{k-\ell} b^{h-\ell} \quad \text{with} \quad \|\mathfrak{e}_{h,k}\|_{\varrho,1}^2 = h! k! (d - \frac{1}{2})^h (d + \frac{1}{2})^k$$

The map Φ sends $\mathfrak{e}_{h,k}$ to the classical Hermite polynomial for γ_+ :

$$e_{h,k} := \sum_{\ell \in \mathbb{Z}_+^m} (-1)^{|\ell|} \ell! \binom{h}{\ell} \binom{k}{\ell} d - \frac{1}{2}^\ell w^{h-\ell} \bar{w}^{k-\ell}, \quad \text{with} \quad \|e_{h,k}\|_{\gamma_+}^2 = h! k! (d - \frac{1}{2})^{h+k}$$

So Φ maps an orthogonal basis of $L_1^2(\mathfrak{h}, \varrho)$ to an orthogonal family in $L^2(\mathfrak{h}, \gamma_+)$, shrinking every norm: hence, it is a contraction.

Same argument for Ψ .

We can interpret this result as the analogue of *Glauber–Sudarshan* of quantum optics:

$$\rho = \frac{1}{\pi} \int dv p(v) |v\rangle \langle v| \mapsto p(v) \quad p(v) \text{ is the } P\text{-function of } \rho$$

- γ_+ is the *P-function* of ϱ (and γ_- is its "Husimi function") both with respect to the coherent states of the *b-modes* (NOT with respect to the original modes a, a^*).
- Φ is the *Glauber–Sudarshan map* on observables: quantum expectations of normally ordered observables become classical averages:

$$\text{Tr}(\rho f) = \int d\gamma_+ \Phi(f)$$

Main result: intertwining

Theorem

Assume $\text{Spec}(\mathcal{B}) \subset \{\text{Re } \lambda < 0\}$. Set $B = \mathcal{B}$ and let

$$Q_{\pm} = \mathcal{Q} \pm \frac{1}{4}(\sigma^* \sigma B + B^* \sigma^* \sigma) \geq 0.$$

Let $L_{\pm}$ be the classical OU generators with data $Q_{\pm}$ and B , with semigroups $P_t^{(2,\pm)}$ on $L^2(\mathcal{R}, \gamma_{\pm})$. Then, for all $t \geq 0$:

$$\Phi \circ \mathcal{P}_t^{(2,1)} = P_t^{(2,+)} \circ \Phi$$

$$\mathcal{P}_t^{(2,0)} \circ \Psi = \Psi \circ P_t^{(2,-)}$$

$$\begin{array}{ccc} L_1^2(\mathfrak{h}, \varrho) & \xrightarrow{\mathcal{P}_t^{(2,1)}} & L_1^2(\mathfrak{h}, \varrho) \\ \Phi \downarrow & & \downarrow \Phi \\ L^2(\mathcal{R}, \gamma_+) & \xrightarrow{P_t^{(2,+)}} & L^2(\mathcal{R}, \gamma_+) \end{array}$$

$$\begin{array}{ccc} L^2(\mathcal{R}, \gamma_-) & \xrightarrow{P_t^{(2,-)}} & L^2(\mathcal{R}, \gamma_-) \\ \Psi \downarrow & & \downarrow \Psi \\ L_0^2(\mathfrak{h}, \varrho) & \xrightarrow{\mathcal{P}_t^{(2,0)}} & L_0^2(\mathfrak{h}, \varrho) \end{array}$$

Note: $C_{\pm}$ solves the Lyapunov equation $B^* C_{\pm} + C_{\pm} B = -2Q_{\pm}$, so $\gamma_{\pm}$ is invariant for $P_t^{(2,\pm)}$.

Where do $Q_{\pm}$ come from?

They can be obtained by a *formal computation* with Weyl operators.

One checks

$$\Phi \mathcal{W}(v) = e^{-\frac{1}{4}\langle v, \sigma^* \sigma v \rangle} W(v)$$

Then

$$\Phi \mathcal{P}_t \mathcal{W}(v) = e^{-\frac{1}{2}\langle v, \mathcal{C}_t v \rangle - \frac{1}{4}\langle e^{tB} v, \sigma^* \sigma e^{tB} v \rangle} W(e^{tB} v)$$

while

$$P_t^+ \Phi \mathcal{W}(v) = e^{-\frac{1}{4}\langle v, \sigma^* \sigma v \rangle - \frac{1}{2}\langle v, C_t^+ v \rangle} W(e^{tB} v)$$

Equality forces

$$C_t^+ = \mathcal{C}_t + \frac{1}{2}(e^{tB^*} \sigma^* \sigma e^{tB} - \sigma^* \sigma)$$

Differentiating at $t = 0$ gives Q_+ , while letting $t \rightarrow \infty$ gives C_+ . Similarly for Q_- and C_- .

(The rigorous proof goes through the core $\mathcal{A}_0$, the identity $\Phi[r, f] = \frac{1}{i}\{\Phi r, \Phi f\}$ for r linear, and the resulting identity between generators on polynomials.)

Application 1: sectoriality

Under suitable assumptions (e.g. $\text{Span}\{\mathcal{X}_\alpha\}_\alpha = \mathbb{C}^{2m}$):

Theorem

$-\mathcal{L}^{(2,s)}$ is sectorial, with the same angle of sectoriality θ_s as the $2m \times 2m$ matrix $-\mathcal{G}_s$ acting on $\mathbb{C}^{2m}$, where

$$\mathcal{G}_s = \left(\mathcal{F}_\mathcal{C} - \frac{1}{2}(-1)^s I\right) \mathcal{F}_\mathcal{B}.$$

Hence $\mathcal{P}_t^{(2,s)}$ extends to a holomorphic semigroup in the sector of angle $\frac{\pi}{2} - \theta_s$.

An infinite-dimensional property, computed from a finite matrix.

Sectorial of angle θ means that the numerical range of $\mathcal{L}$ lies in the sector $|\arg z| \leq \theta$.

The classical analogue, for the OU operator on $L^2(\gamma)$, is due to [Chill, Fasangova, Metafune and Pallara]; we transfer it to the quantum case via the intertwining.

Application 2: the spectrum

The spectrum of the classical OU generator is known explicitly [Metafune, Pallara, Priola]; via the intertwining it transfers to the quantum case:

Theorem

Let $\lambda_1, \dots, \lambda_r$ be the distinct eigenvalues of $\mathcal{B}^*$. Then, for $s \in \{0, 1\}$,

$$\text{Spec}(\mathcal{L}^{(2,s)}) = \left\{ \sum_{j=1}^r n_j \lambda_j : n \in \mathbb{Z}_+^r \right\}.$$

(Note: The relaxation depends only on the drift $\mathcal{B}$, not on the noise $\mathcal{Q}$.)

Mechanism: $\mathcal{L}$ preserves the finite-dimensional spaces of polynomials of degree $\leq n$, where the eigenvalues can be computed; one then shows that the resolvent is compact, so there is nothing else in the spectrum.

Thank you!