Flatness approach for the control of PDEs

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Outline

1. Flatness approach

Controllability

A system

$$\frac{dx}{dt} = f(x, u)$$

with $x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$, is exactly controllable in time T if for any x_0, x_T, one may find a function $u = u(t)$, called a control input, such that

$$x(0) = x_0 \quad \text{and} \quad x(T) = x_T.$$

What is Flatness?

- Powerful criterions enable us to decide whether a nonlinear control system is controllable or not, but most of them do not provide any indication on how to design an explicit control input steering the system from a point to another one.
- There exists, however, a large class of systems, the so-called flat systems, for which explicit control inputs can be found.
- Roughly, the flatness approach consists in a parameterisation of the trajectory by some (flat) output. It was introduced in 1995 by **M. Fliess, J. Lévine, Ph. Martin, P. Rouchon** for (linear or nonlinear) ODE, and it is still very popular thanks to its numerous applications in Engineering.

Let us consider a smooth control system

$$\dot{x} = f(x,u), \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \tag{1.1}$$

given together with an output $y \in \mathbb{R}^m$ depending on x, u and a finite number of derivatives of u

$$y = h(x, u, \dot{u}, ..., u^{(r)}).$$

We say that y is a flat output if x and u can be expressed as functions of y and of a finite number of its derivatives

$$x = g(y, \dot{y}, ..., y^{(p)}), \tag{1.2}$$

$$u = h(y, \dot{y}, ..., y^{(q)}). \tag{1.3}$$

In (1.2)-(1.3), p and q denote some nonnegative integers, and g and h denote some smooth functions. Conversely, it is assumed that a pair (x, u) as in (1.2)-(1.3) solves (1.1).

Control constraints

We aim to solve the control problem

$$\dot{x} = f(x, u), \quad (1.4)$$

$$x(0) = x_0, \quad x(T) = x_T \quad (1.5)$$

• The ODE (1.4) is satisfied whenever x and u are parameterized by y.
• To satisfy the constraints $x(0) = x_0, \quad x(T) = x_T$, it remains to design a smooth output y such that

$$x(0) = g(y, \dot{y}, ..., y^{(p)})(0) = x_0, \quad (1.6)$$

$$x(T) = h(y, \dot{y}, ..., y^{(q)})(T) = x_T \quad (1.7)$$

The last conditions are easy to satisfy.

Example 1

- Consider the double integrator (linearized pendulum)

$$\dot{x}_1 = x_2 \quad (1.8)$$

$$\dot{x}_2 = u \quad (1.9)$$

where the state is $x = (x_1, x_2) \in \mathbb{R}^2$ and the control is $u \in \mathbb{R}$.

- Pick $y = x_1$. It is a flat output, as $x_1 = y$, $x_2 = \dot{y}$, and $u = \ddot{y}$.
- To steer the system from $x_0 = (0, 0)$ to $x_T = (1, 0)$ in time T, we have to pick a function $y \in C^\infty([0, T], \mathbb{R})$ such that

$$y(0) = 0, \quad \dot{y}(0) = 0, \quad y(T) = 1, \quad \dot{y}(T) = 0.$$

- Clearly, $y(t) = t^2(2T - t)^2/T^4$ is convenient.

Example 2

- Consider the nonlinear system

$$\dot{x}_1 = u_1 \tag{1.10}$$

$$\dot{x}_2 = u_2 \tag{1.11}$$

$$\dot{x}_3 = x_2 u_1 \tag{1.12}$$

where the state is $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and the control is $(u_1, u_2) \in \mathbb{R}^2$.

- How to find a flat output? Eliminating the input u_1 in (1.10)-(1.12) yields $x_2 = \dot{x}_3/\dot{x}_1$, so that x_2 can be expressed as a function of x_1, x_3 and their derivatives. The same is true for u_2 thanks to (1.11).
- Pick as output $y = (y_1, y_2) = (x_1, x_3)$.
- Then y is a flat output. Indeed, we have

$$(x_1, x_2, x_3) = (y_1, \frac{\dot{y}_2}{\dot{y}_1}, y_2) \tag{1.13}$$

$$(u_1, u_2) = (\dot{y}_1, \frac{\ddot{y}_2\dot{y}_1 - \dot{y}_2\ddot{y}_1}{(\dot{y}_1)^2}). \tag{1.14}$$

- Note that $\dot{y}_1$ vanishes somewhere if $x_1(0) = x_1(T)$. Thus y has to be designed in such a way that $\dot{y}_2/\dot{y}_1$ is well defined and smooth. If we choose for y_1 and y_2 some analytic functions, it is sufficient to impose that $\dot{y}_2$ vanishes at (at least) the same order as $\dot{y}_1$ where $\dot{y}_1$ vanishes.

Example 2 (2)

- Pick $x_0 = (0,0,0)$ and $x_T = (0,0,1)$. Any candidate $y = (y_1, y_2)$ is such that $\dot{y}_1(\bar{t}) = 0$ for some $\bar{t} \in [0, T]$
- Recall that $(x_1, x_2, x_3) = (y_1, \frac{\dot{y}_2}{\dot{y}_1}, y_2)$
- We have to find a pair (y_1, y_2) of analytic functions such that $\dot{y}_2/\dot{y}_1$ is (well defined) analytic on $[0, T]$ with

$$
\begin{aligned}
& y_1(0) = y_2(0) = 0, \quad \dot{y}_2(0) = 0, \quad \text{and } \dot{y}_1(0) \neq 0. \\
& y_1(T) = 0, \;\; y_2(T) = 1, \;\; \dot{y}_2(T) = 0, \quad \text{and } \dot{y}_1(T) \neq 0, \\
& \dot{y}_1(\frac{T}{2}) = \dot{y}_2(\frac{T}{2}) = 0, \quad \ddot{y}_1(\frac{T}{2}) \neq 0, \quad \text{and } \dot{y}_1(t) \neq 0 \text{ for } t \neq \frac{T}{2}.
\end{aligned}
$$

- The function

$$
(y_1(t), y_2(t)) = \left(t(T-t), -\frac{120}{T^5}(\frac{t^5}{5} - \frac{3Tt^4}{8} + \frac{T^2 t^3}{6}) \right)
$$

is convenient.

2. Flatness approach for the control of PDE

Introduction

- Let $\Omega \subset \mathbb{R}^N$ be a smooth, bounded open set, and let $\gamma \subset \partial\Omega$ be a (nonempty) open set. Consider a boundary-initial value problem

$$P(D)z = 0, \qquad t \in (0,T),\ x \in \Omega \tag{2.1}$$
$$B_1(D)z = 1_\gamma u, \qquad t \in (0,T),\ x \in \partial\Omega \tag{2.2}$$
$$B_2(D)z = 0, \qquad t \in (0,T),\ x \in \partial\Omega, \tag{2.3}$$
$$z(x,0) = z_0(x), \qquad x \in \Omega. \tag{2.4}$$

  Here $D = (-i\partial_t, -i\partial_{x_1}, ..., -i\partial_{x_N})$, and P, B_1, B_2 are polynomial functions.

- We say that system (2.1)-(2.4) is exactly controllable in some space H in time $T > 0$, if for any $z_0, z_1 \in H$, one may pick a control input u s.t. the solution of (2.1)-(2.4) satisfies

$$z(x,T) = z_1(x), \quad x \in \Omega$$

- If the above property holds for any z_0, but (solely) for $z_1 = 0$, we say that system (2.1)-(2.4) is null controllable.

Introduction (2)

$$\begin{aligned} \theta_t - \Delta\theta \qquad [+f(\theta, \nabla\theta)] &= 0, && x \in \Omega,\ t \in (0,T), \\ \theta &= 1_\gamma u, && x \in \partial\Omega,\ t \in (0,T), \\ \theta(x,0) &= \theta_0(x), && x \in \Omega. \end{aligned}$$

- It was proved by Lebeau-Robbiano (1995) and Fursikov-Imanuvilov (1996) that the above system is null controllable for any $T > 0$ and any control region γ.
- To derive such a result, a parabolic Carleman estimate due to Fursikov-Imanuvilov can be used

$$\begin{aligned} &\int_0^T\int_\Omega [(s\theta)^{-1}(|\Delta v|^2 + |v_t|^2) + \lambda^2(s\theta)|\nabla v|^2 + \lambda^4(s\theta)^3|v|^2]e^{-2s\varphi}dxdt \\ &\le C_1\left(\int_0^T\int_\Omega |v_t + \Delta v|^2 e^{-2s\varphi}dxdt + \int_0^T\int_\gamma \lambda(s\theta)|\partial_n\psi||\partial_n v|^2 e^{-2s\varphi}d\sigma dt\right) \end{aligned}$$

for $s \ge s_0$ and $\lambda \ge \lambda_0$, with the weights

$$\varphi(x,t) := \frac{e^{\frac{3}{2}\lambda\|\psi\|_{L^\infty}} - e^{\lambda\psi(x)}}{t(T-t)}, \quad \theta(x,t) := \frac{e^{\lambda\psi(x)}}{t(T-t)}$$

Boundary control of the heat equation

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth open set, $\Gamma_0 \subset \partial\Omega$ be a (nonempty) open set, and $T > 0$.

We are concerned with the **null controllability problem**:
given θ_0, find a function $u = u(t, x)$ s.t. the solution of

$$\begin{aligned} \theta_t - \Delta\theta &= 0 && (t, x) \in (0, T) \times \Omega, \\ \frac{\partial\theta}{\partial\nu} &= 1_{\Gamma_0} u(t, x) && (t, x) \in (0, T) \times \partial\Omega, \\ \theta(0, x) &= \theta_0(x), && x \in \Omega \end{aligned}$$

satisfies

$$\theta(T, x) = 0 \qquad x \in \Omega$$

Null controllability of the heat equation

- Duality methods (observability estimate for the adjoint eq.)
  - **Fattorini-Russell '71**, **Luxembourg-Korevarr '71**, **Dolecki '73** (1D, using biorthogonal families and complex analysis)
  - **Lebeau-Robbiano '95**, **Imanuvilov-Fursikov '96** (ND, $\forall(\Omega, \Gamma_0, T)$, using Carleman estimates)
- Direct methods
  - **Jones '77**, **Littman '78** (construction of a fundamental solution with compact support in time, $\Gamma_0 = \partial\Omega$)
  - **Guo-Littman '95** (solution of ill-posed problems)
  - **Laroche-Martin-Rouchon 2000** (approximate controllability using a flatness approach)

Here, we shall revisit the flatness approach, obtain the **null controllability**, and show its relevance to numerics.

2.1 Null controllability of the heat equation in 1D

P. Martin, L. Rosier, P. Rouchon, *Null controllability of the heat equation using flatness*, Automatica 50 (2014), 3067–3076

Flatness for PDE

- The flatness method was applied by **Laroche-Martin-Rouchon** in 2000 to derive the approximate controllability of
  1. the 1D heat eq;
  2. the beam equation;
  3. the linearized KdV equation.
- They proved that prepared initial data can be driven to 0 by using control inputs that are **Gevrey**.

Classe $C\{M_n\}$ (see W. Rudin, Real and complex analysis)

- Let $M_0, M_1, M_2, ..$ be positive real numbers such that $M_0 = 1$ and $M_n^2 \leq M_{n-1} M_{n+1}$ for all n
- Let $C\{M_n\}$ be the class of functions $f \in C^\infty(\mathbb{R})$ such that there exist some positive constants $C = C(f)$, $R = R(f)$ s.t.

$$\|f^{(n)}\|_{L^\infty(\mathbb{R})} \leq C \frac{M_n}{R^n}, \qquad \forall n \geq 0,$$

where $f^{(n)} = d^n f / dx^n$

Theorem

Each class $C\{M_n\}$ is an algebra with respect to pointwise multiplication.

- A class $C\{M_n\}$ is said to be quasi-analytic if the conditions $f \in C\{M_n\}$ and $f^{(n)}(0) = 0$ for all $n \geq 0$ imply $f(x) = 0$ for all $x \in \mathbb{R}$.

Theorem

$C\{M_n\}$ is quasi-analytic iff the only function in $C\{M_n\}$ with compact support is the trivial one.

The Denjoy-Carleman Theorem

Theorem (Denjoy-Carleman)

Suppose $M_0 = 1$, $M_n^2 \le M_{n-1}M_{n+1}$ for $n = 1, 2, 3, \ldots$. Then the class $C\{M_n\}$ is NOT quasi-analytic iff $\sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n} < \infty$

Assume that

$$|f^{(n)}(x)| \le C\frac{(n\,!)^s}{R^n} \quad \forall x \in \mathbb{R},\ \forall n \in \mathbb{N}$$

so that $f \in C\{M_n\}$ with $M_n := (n!)^s$.

1. if $s \le 1$, $C\{M_n\}$ is quasi-analytic : f cannot be a nontrivial function with compact support
2. if $s > 1$, $C\{M_n\}$ is NOT quasi-analytic : f can be a nontrivial function with a compact support

Gevrey functions (1)

- A function $y \in C^\infty([0,T])$ is **Gevrey of order** $s \geq 0$ if there exist $R, C > 0$ such that
$$|y^{(p)}(t)| \leq C\frac{p!^s}{R^p}, \quad \forall p \in \mathbb{N},\ \forall t \in [0,T]$$
- The larger s, the less regular y is ($s = 1 \iff y \in C^\omega$)
- For $s < 1$, y is entire (complex analytic on $\mathbb{C}$)
- $\theta \in C^\infty([t_1,t_2]\times[0,1])$ is **Gevrey of order** s_1 **in** x **and** s_2 **in** t if

$$|\partial_x^{p_1}\partial_t^{p_2}\theta(t,x)| \leq C\frac{(p_1!)^{s_1}(p_2!)^{s_2}}{R_1^{p_1}R_2^{p_2}} \qquad \forall p_1, p_2 \in \mathbb{N},\ \forall (t,x) \in [t_1,t_2]\times[0,1]$$

Gevrey functions (2)

- The set of Gevrey functions of order $s \geq 0$ on $[0, T]$ is an algebra for the multiplication of functions $(fg)(t) = f(t)g(t)$
- There are Gevrey functions of order $s > 1$ with compact support. Easy to construct from the "Gevrey step function"

$$\phi_s(t) = \begin{cases} 1 & \text{if } t \leq 0, \\ \dfrac{e^{-\frac{1}{(1-t)^r}}}{e^{-\frac{1}{(1-t)^r}} + e^{-\frac{1}{t^r}}} & \text{if } 0 < t < 1, \\ 0 & \text{if } t \geq 1 \end{cases}$$

which is Gevrey of order $s = 1 + r^{-1}$ for $r > 0$.

Gevrey functions of order less than 1

We denote by $G^\sigma([0,T])$ the space of functions of Gevrey order $\sigma \geq 0$ on $[0,T]$. If $\sigma \leq 1$, those functions are analytic, and entire if $\sigma < 1$.

Proposition

Let $T > 0$ and $f \in G^\sigma([0,T])$, $\sigma \geq 0$, and set

$$\begin{aligned} g &:= \inf\{s \geq 0;\ f \in G^s([0,T])\}, \\ \rho &:= \inf\{k > 0,\ \exists r_0 > 0,\ \forall r > r_0,\ \max_{|z|=r} |f(z)| < \exp(r^k)\}. \end{aligned}$$

Assume $g < 1$. Then f is an entire function of order $\rho \leq (1-g)^{-1}$. If, in addition, $\rho \geq 1$, then

$$\rho = (1-g)^{-1}.$$

• Note that the assumption $\rho \geq 1$ is needed, as for polynomial functions $g = \rho = 0$.
• For instance, a function which is Gevrey of order not less than $1/2$ ($g = 1/2$), and for which $\rho \geq 1$, is an entire function of order $\rho = 2$ (Ex. $x \mapsto exp(Cx^2)$) .

More about Laroche-Martin-Rouchon result

- The heat control problem reads:

$$\begin{aligned} \theta_t - \theta_{xx} &= 0, \qquad x \in (0,1) \\ \theta_x(t,0) &= 0, \quad \theta_x(t,1) = u(t), \\ \theta(0,x) &= \theta_0. \end{aligned}$$

- Laroche, Martin, and Rouchon proved in 2000 that for initial data decomposed as

$$\theta_0(x) = \sum_{i \geq 0} y_i \frac{x^{2i}}{(2i)!}$$

with

$$|y_i| \leq C \frac{i!^s}{R^i}, \qquad i \geq 0$$

with $s \in (1,2)$, C, $R > 0$, the system can be driven to 0 with a control $u(t)$ which is Gevrey of order s.

Flat output, trajectory, control,...

Take $y = \theta(t, 0)$ as output. It is **flat**, in the sense that the map $\theta \to y$ is a **bijection** between appropriate spaces of functions.

Seek a candidate solution (analytic in x) in the form

$$\theta(t, x) = \sum_{i \geq 0} a_i(t) \frac{x^i}{i!}$$

Plugging this sum in the heat eq. gives $\sum_{i \geq 0} [a_{i+2} - a_i{}'\,] \frac{x^i}{i!} = 0$ ($' = d/dt$), and hence

$$a_{i+2} = a_i', \quad i \geq 0.$$

Since $a_0(t) = \theta(t, 0) = y(t)$ and $a_1(t) = 0$, we arrive at

$$a_{2i+1} = 0, \quad a_{2i} = y^{(i)}, i \geq 0,$$

$$\theta(t, x) = \sum_{i \geq 0} y^{(i)}(t) \frac{x^{2i}}{(2i)!}, \quad u(t) = \theta_x(t, 1) = \sum_{i \geq 1} \frac{y^{(i)}(t)}{(2i-1)!}$$

Gevrey functions

Since $\theta(t,x) = \sum_{i\geq 0} y^{(i)}(t)\frac{x^{2i}}{(2i)!}$, it remains to find $y \in C^\infty([0,T])$ s.t. the series converges and

$$y^{(i)}(0) = y_i, \quad y^{(i)}(T) = 0, \quad i \geq 0.$$

Impossible to do with an analytic function, but possible (as we shall see later) with a function Gevrey of order $s > 1$.

Our first aim is to show that the above computations are fully justified when y is Gevrey of order $s \in [0,2)$.

Flatness property

Proposition

Let $s \in [0,2)$ and $y \in C^\infty([t_1,t_2])$ $(-\infty < t_1 < t_2 < \infty)$ be Gevrey of order s on $[t_1,t_2]$. Let

$$\theta(t,x) := \sum_{i\geq 0} \frac{x^{2i}}{(2i)!} y^{(i)}(t).$$

Then θ is Gevrey of order s in t and $s/2$ in x on $[t_1,t_2]\times[0,1]$ and it solves the ill-posed problem

$$\begin{aligned} \theta_t - \theta_{xx} &= 0, \qquad (t,x) \in [t_1,t_2]\times[0,1], \\ \theta(t,0) = y(t),\ \theta_x(t,0) &= 0. \end{aligned}$$

Thus $u(t) = \theta_x(t,1) = \sum_{i\geq 1} \frac{y^{(i)}(t)}{(2i-1)!}$ is Gevrey of order s on $[t_1,t_2]$.

Proof of the flatness property

We want to prove that the formal series

$$\partial_t^m \partial_x^n \theta(t,x) = \sum_{2i \geq n} \frac{x^{2i-n}}{(2i-n)!} y^{(i+m)}(t) \tag{2.5}$$

is uniformly convergent on $[t_1, t_2] \times [0,1]$ with an estimate of its sum of the form

$$|\partial_t^m \partial_x^n \theta(t,x)| \leq C \frac{m!^s}{R_1^m} \frac{n!^{\frac{s}{2}}}{R_2^n}. \tag{2.6}$$

Since y is Gevrey of order s, we have for all $(t,x) \in [t_1, t_2] \times [0,1]$

$$\begin{aligned}
\left| \frac{x^{2i-n}}{(2i-n)!} y^{(i+m)}(t) \right| &\leq \frac{M}{R^{i+m}} \frac{(i+m)!^s}{(2i-n)!} \\
&\leq \frac{M}{R^{i+m}} \frac{(2^{i+m} i!\, m!)^s}{(2i-n)!} \\
&\leq \frac{M}{R_1^{i+m}} \frac{(2^{-2i}\sqrt{\pi i}\,(2i)!)^{\frac{s}{2}}}{(2i-n)!} m!^s \\
&\leq M \frac{(\pi i)^{\frac{s}{4}}}{R_1^i (2i-n)!^{1-\frac{s}{2}}} n!^{\frac{s}{2}} \frac{m!^s}{R_1^m},
\end{aligned}$$

where we have set $R_1 = 2^{-s}R$, used $(i+j)! \leq 2^{i+j} i! j!$ and $(2i)! \sim \frac{2^{2i}}{\sqrt{\pi i}} i!^2$.

Thus, the formal series in

$$\partial_t^m \partial_x^n \theta(t,x) = \sum_{2i \geq n} \frac{x^{2i-n}}{(2i-n)!} y^{(i+m)}(t)$$

is uniformly convergent on $[t_1, t_2] \times [0,1]$ for all $m, n \in \mathbb{N}$, and hence $\theta \in C^\infty([t_1, t_2] \times [0,1])$.

On the other hand, picking any $R_2 \in (0, \sqrt{R_1})$, since

$$M \sum_{2i \geq n} \frac{(\pi i)^{\frac{s}{4}}}{R_1^i (2i-n)!^{1-\frac{s}{2}}} \leq M(\frac{\pi}{2})^{\frac{s}{4}} R_1^{-\frac{n}{2}} \sum_{j \geq 0} \frac{j^{\frac{s}{4}} + n^{\frac{s}{4}}}{R_1^{\frac{j}{2}} j!^{1-\frac{s}{2}}} \leq C R_2^{-n},$$

for some constant $C > 0$ independent of n, we conclude that

$$|\partial_t^m \partial_x^n \theta(t,x)| \leq C \frac{n!^{\frac{s}{2}}}{R_2^n} \frac{m!^s}{R_1^m},$$

which proves that θ is Gevrey of order s in t and $s/2$ in x, as desired.

Main result

Theorem

Let $\theta_0 \in L^2(0,1)$ *and* $T > 0$. *Pick* $\tau \in (0,T)$ *and* $s \in (1,2)$. *There exists* $y \in C^\infty([\tau,T])$ *Gevrey of order* s *on* $[\tau,T]$ *such that, setting*

$$u(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \tau \\ \sum_{i\geq 0} \frac{y^{(i)}(t)}{(2i-1)!} & \text{if } \tau < t \leq T, \end{cases}$$

the solution θ *of*

$$\begin{aligned} \theta_t - \theta_{xx} &= 0, \qquad x \in (0,1) \\ \theta_x(t,0) = 0,\ \theta_x(t,1) &= u(t), \\ \theta(0,x) &= \theta_0(x) \end{aligned}$$

satisfies $\theta(T,.) = 0$.

Short proof of the null controllability of the heat equation

1. Decompose the initial state θ_0 as a Fourier series of cosines, namely

$$\theta_0(x) = \sum_{n\geq 0} c_n \sqrt{2}\cos(n\pi x) \quad \text{in } L^2(0,1)$$

where $2|c_0|^2 + \sum_{n\geq 1} |c_n|^2 = \int_0^1 |\theta_0(x)|^2 dx < \infty$.
2. Denote the free evolution ($u = 0$) by $\bar{\theta}$. It reads

$$\bar{\theta}(t,x) = \sum_{n\geq 0} c_n e^{-n^2\pi^2 t}\sqrt{2}\cos(n\pi x),$$

and it can be proved that $\bar{\theta}$ is Gevrey of order 1 in t and $1/2$ in x in $[\tau, T] \times [0,1]$ for all $0 < \tau < T$.
3. In particular, the trace

$$\bar{\theta}(t,0) = \sqrt{2}\sum_{n\geq 0} c_n e^{-n^2\pi^2 t}$$

is analytic on $(0,+\infty)$, hence Gevrey of order 1 on the interval $[\tau, T]$ for any $\tau \in (0,T)$.

Short proof of the null controllability of the heat equation

4. To solve the null control problem, it is sufficient to apply the flatness property by solving the (ill-posed) problem

$$\begin{aligned} \theta_t - \theta_{xx} &= 0, && (t,x) \in (0,T) \times (0,1), \\ \theta(t,0) = y(t) := \phi_s\left(\frac{t-\tau}{R}\right)\bar{\theta}(t,0), && && t \in (0,T) \\ \theta_x(t,0) &= 0, && t \in (0,T). \end{aligned}$$

where $0 < R < T - \tau$ and ϕ_s is the “Gevrey step function” of order s (defined before):

$$\phi_s(t) = \begin{cases} 1 & \text{if } t \leq 0 \\ 0 & \text{if } t \geq 1 \end{cases}$$

This is closely related to the approach developed in Guo-Littman ’95 in the semilinear case. Here, however, the solution can be given explicitly:

$$\theta(t,x) = \bar{\theta}(t,x) = \sum_{n \geq 0} c_n e^{-n^2\pi^2 t} \sqrt{2} \cos(n\pi x), \qquad \text{if } \ t < \tau$$

$$\theta(t,x) = \sum_{i \geq 0} \frac{x^{2i}}{(2i)!} y^{(i)}(t), \qquad \text{if } \ t > \tau \text{ (or } 0 < t \leq T)$$

Numerical simulations (N=1) Trajectory

Initial state: $\theta_0 := 1_{(1/2,1)}(x) - 1_{(0,1/2)}(x)$
Parameters: $\tau = 0.3$, $R = 0.2$, $T = \tau + R = 0.5$, $s = 1.6$

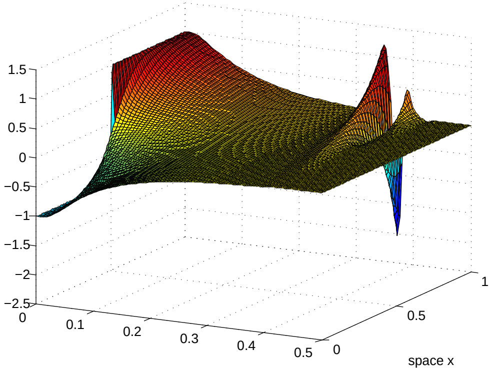


Fig.1. $\theta(t, x)$

Computations by Philippe Martin

Numerical simulations (N=1) Control

Initial state: $\theta_0 := 1_{(1/2,1)}(x) - 1_{(0,1/2)}(x)$
Parameters: $\tau = 0.3$, $R = 0.2$, $T = \tau + R = 0.5$, $s = 1.6$

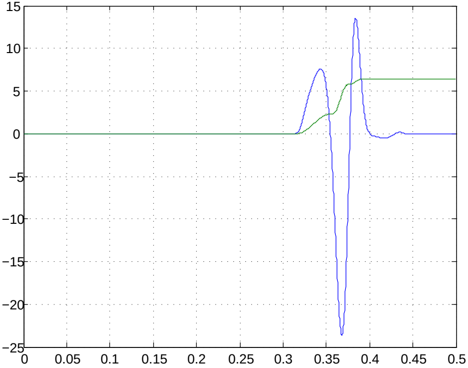


Fig. 2. $\bar{u}(t)$ (blue) and $||\bar{u}(t)||_{L^2(0,t)}$ (green)

Computations by Philippe Martin

Numerical computation of a large number of derivatives

- With this approach, we have to compute $N \geq 20$ derivatives of some functions, e.g.
$$\varphi(t) = \exp(-t^{-k}(1-t)^{-k})$$
where $k = (s-1)^{-1}$.
- Purely numerical methods (using e.g. finite differences) are not appropriate!
- Symbolic computations limited to $N \leq 20$.
- We compute the derivatives **by induction** as follows: Derivating φ yields
$$p^{k+1}\dot{\varphi} = k\,\dot{p}\,\varphi$$
where $p(t) = t(1-t)$.
Derivating i times that identity and using Leibniz' rule results in
$$\color{red}{p^{k+1}\varphi^{(i+1)}} + \sum_{j=1}^{i} \begin{pmatrix} i \\ j \end{pmatrix} (p^{k+1})^{(j)}\varphi^{(i+1-j)} = k(\dot{p}\varphi^{(i)} + i\ddot{p}\varphi^{(i-1)})$$
This equation gives $\varphi^{(i+1)}$ in terms of $\varphi^{(0)}$, ..., $\varphi^{(i)}$, and of $(p^{k+1})^{(j)}$, $j \leq i$.
- In practice, $N = 140$ derivatives can be computed on line.

2.2 Null controllability of the heat equation on cylinders

P. Martin, L. Rosier, P. Rouchon, *Null controllability of the heat equation using flatness*, Automatica 50 (2014), 3067–3076.

The dimension N

- So far, the flatness approach was applied to 1D PDE (and for radial solution of 2D problems). The expansion of the solution as a power series in all the spatial coordinates seems not to work well, even in 2D.
- Here, we shall see that we can deal with the null controllability of the heat equation on a cylinder

  $$\Omega = \omega \times (0,1) \subset \mathbb{R}^N$$

  where $\omega \subset \mathbb{R}^{N-1}$ is a smooth, bounded open set, and $N \geq 2$. We thus consider the control problem ($x = (x', x_N)$)

$$\begin{array}{rcll} \theta_t - \Delta\theta & = & 0, & (t,x) \in (0,T) \times \Omega \\ \dfrac{\partial\theta}{\partial\nu}(t,x',1) & = & u(t,x'), & (t,x') \in (0,T) \times \omega \\ \dfrac{\partial\theta}{\partial\nu}(t,x) & = & 0 & (t,x) \in (0,T) \times (\partial\Omega \setminus \omega \times \{1\}) \\ \theta(0,x) & = & \theta(x), & x \in \Omega \end{array}$$

- For $N = 3$, this is nothing but the control of the temperature of a metallic rod by the heat flux on one lateral section.

Expansion of the solution

- The good way to solve the problem is to consider “hybrid” expansions of θ mixing Fourier series in x' (no control on $\partial\omega$) and power series in x_N (control at $x_N = 1$).
- Introduce an orthonormal basis in $L^2(\omega)$, $(e_j)_{j\geq 0}$, constituted of eigenvectors for the Neumann Laplacian in $\omega \subset \mathbb{R}^{N-1}$, i.e.

$$\begin{aligned} -\Delta' e_j &= \lambda_j e_j \qquad \text{in } \omega \\ \frac{\partial e_j}{\partial \nu'} &= 0 \qquad \text{on } \partial\omega \end{aligned}$$

  where $\Delta' = \partial^2_{x_1} + \cdots \partial^2_{x_{N-1}}$, $\nu' =$ outward unit normal to ω, $0 = \lambda_0 < \lambda_1 \leq \lambda_j \leq \lambda_{j+1} \leq \cdots$
- Decompose $\theta(t, x', 0)$ as

$$\theta(t, x', 0) = \sum_{j\geq 0} z_j(t) e_j(x').$$

  We claim that the system is **flat**, with $(z_j(t))_{j\geq 0}$ as “flat output”. Indeed, given a sequence $(z_j(t))_{j\geq 0}$ of smooth functions, we seek a formal solution of the heat equation in the form

$$\theta(t, x', x_N) = \sum_{i\geq 0} \frac{x_N^i}{i!} a_i(t, x')$$

  where the a_i's are still to be defined.

Expansion of the solution (2)

Plugging the formal solution $\theta = \sum_{i\geq 0} \dfrac{x_N^i}{i!} a_i$ in the heat equation gives

$$\sum_{i\geq 0} \frac{x_N^i}{i!}[a_{i+2}(t,x') - (\partial_t - \Delta')a_i(t,x')] = 0$$

so that $a_{i+2} = (\partial_t - \Delta')a_i$ for all $i \geq 0$. Moreover

$$a_0(t,x') = \theta(t,x',0) = \sum_{j\geq 0} z_j(t)e_j(x'), \qquad a_1(t,x') = 0.$$

Therefore, for all $i \geq 0$

$$\begin{aligned} a_{2i+1} &= 0, \\ a_{2i} &= (\partial_t - \Delta')^i a_0 = \sum_{j\geq 0} (\partial_t - \Delta')^i [z_j(t)e_j(x')] = \sum_{j\geq 0} e_j(x')(\partial_t + \lambda_j)^i z_j(t) \\ &= \sum_{j\geq 0} e_j(x')e^{-\lambda_j t} y_j^{(i)}(t) \end{aligned}$$

where we have set $y_j(t) := e^{\lambda_j t} z_j(t)$. We arrive at

$$\theta(t,x',x_N) = \sum_{j\geq 0} e^{-\lambda_j t} e_j(x') \sum_{i\geq 0} y_j^{(i)}(t) \frac{x_N^{(2i)}}{(2i)!}$$

Flatness property

Proposition

Let $s \in (1,2)$, $-\infty < t_1 < t_2 < \infty$, and let $y = (y_j)_{j\geq 0}$ in $C^\infty([t_1, t_2])$ satisfy for some constants $M, R > 0$

$$|y_j^{(i)}(t)| \leq M \frac{i!^s}{R^i}, \qquad \forall i, j \geq 0,\ \forall t \in [t_1, t_2].$$

Then the function

$$\theta(t, x', x_N) = \sum_{j\geq 0} e^{-\lambda_j t} e_j(x') \sum_{i\geq 0} y_j^{(i)}(t) \frac{x_N^{(2i)}}{(2i)!}$$

is well defined in $[t_1, t_2] \times \overline{\Omega}$, and it is Gevrey of order s in t, $1/2$ in $x_1, ..., x_{N-1}$ and $s/2$ in x_N. It solves the ill-posed problem

$$\begin{aligned} \theta_t - \Delta\theta &= 0, \qquad (t, x) \in [t_1, t_2] \times \overline{\Omega}, \\ \theta(t, x', 0) &= \sum_{j\geq 0} e^{-\lambda_j t} y_j(t) e_j(x'), \\ \theta_{x_N}(t, x', 0) &= 0. \end{aligned}$$

Thus $u(t,x') = \theta_{x_N}(t,x',1) = \sum_{j\geq 0} e^{-\lambda_j t} e_j(x') \sum_{i\geq 1} \frac{y_j^{(i)}(t)}{(2i-1)!}$ is Gevrey of order s in t and $1/2$ in $x_1, ..., x_{N-1}$.

The proof is similar to those in dimension 1, but more technical (we need Weyl's formula $\lambda_j \sim j^{\frac{2}{N-1}}$).

Null controllability of the heat equation on cylinders

Consider the control system

$$(S) \qquad \begin{cases} \theta_t - \Delta\theta = 0, & (t,x) \in (0,T) \times \Omega \\ \frac{\partial\theta}{\partial\nu}(t,x',1) = u(t,x'), & (t,x') \in (0,T) \times \omega \\ \frac{\partial\theta}{\partial\nu}(t,x) = 0 & (t,x) \in (0,T) \times (\partial\Omega \setminus \omega \times \{1\}) \\ \theta(0,x) = \theta_0(x), & x \in \Omega \end{cases}$$

Here $\Omega := \omega \times (0,1) \subset \mathbb{R}^N$, $N \geq 2$, is a cylinder.

Main result

Theorem

Let $\Omega = \omega \times (0,1) \subset \mathbb{R}^{N-1} \times \mathbb{R}$, $\theta_0 \in L^2(\Omega)$ and $T > 0$ be given. Pick any $\tau \in (0,T)$ and any $s \in (1,2)$. Then there exists a sequence $(y_j)_{j\geq 0}$ of functions in $C^\infty([\tau, T])$ that are Gevrey of order s on $[\tau, T]$ and such that, setting

$$u(t,x') = \begin{cases} 0 & \text{if } 0 \leq t \leq \tau, \\ \sum_{i,j\geq 0} e^{-\lambda_j t} \frac{y_j^{(i)}(t)}{(2i-1)!} e_j(x') & \text{if } \tau \leq t \leq T, \end{cases}$$

we have

$$\theta(T,.) = 0.$$

Here, (e_j, λ_j) denotes the j^{th} pair of eigenfunction/eigenvalue for the Neumann Laplacian on $\omega \subset \mathbb{R}^{N-1}$.

More about the regularity: the control u is Gevrey of order s in t and $1/2$ in $x_1, ..., x_{N-1}$ on $[0,T] \times \overline{\omega}$.
Furthermore, $\theta \in C([0,T], L^2(\Omega)) \cap C^\infty((0,T] \times \overline{\Omega})$, and θ is Gevrey of order s in t, $1/2$ in $x_1, ..., x_{N-1}$ and $s/2$ in x_N on $[\epsilon, T] \times \overline{\Omega}$ for all $\epsilon \in (0,T)$.

Construction of the trajectory

Assume given $T > 0$, $\tau \in (0, T)$, $s \in (1, 2)$, and $\theta_0 \in L^2(\Omega)$ decomposed as

$$\theta_0(x', x_N) = \sum_{j,n\geq 0} c_{j,n} e_j(x') \sqrt{2} \cos(n\pi x_N).$$

where $-\Delta' e_j(x') = \lambda_j e_j(x')$ in $\omega \subset \mathbb{R}^{N-1}$, $\partial e_j / \partial \nu' = 0$ on $\partial\omega$.

The exact solution θ of the previous control problem such that $\theta(T, .) = 0$ is

$$\begin{aligned}
\theta(t, x', x_N) &= \sum_{j,n\geq 0} c_{j,n} e^{-(\lambda_j + n^2\pi^2)t} e_j(x') \sqrt{2} \cos(n\pi x_N), \qquad 0 \leq t \leq \tau, \\
\text{so that } \theta(t, x', 0) &= \sum_{j\geq 0} e^{-\lambda_j t} e_j(x') \sqrt{2} \sum_{n\geq 0} c_{j,n} e^{-n^2\pi^2 t}, \qquad 0 \leq t \leq \tau, \\
\theta(t, x', x_N) &= \sum_{j\geq 0} e^{-\lambda_j t} e_j(x') \sum_{i\geq 0} y_j^{(i)}(t) \frac{x_N^{2i}}{(2i)!}, \qquad \tau \leq t \leq T,
\end{aligned}$$

where (with $0 < R < T - \tau$)

$$y_j(t) = \phi_s\left(\frac{t-\tau}{R}\right) \sqrt{2} \sum_{n\geq 0} c_{j,n} e^{-n^2\pi^2 t}, \qquad 0 < t \leq T. \tag{2.7}$$

Step 1 (no control)

In practice, only **partial sums** can be computed. They prove to give **accurate** approximations of both the trajectory and the control.
During the free evolution, we take as approximation of θ

$$\overline{\theta}(t,x',x_N) = \sum_{0\le j\le \bar{j}} \sum_{0\le n\le \overline{n}} c_{j,n} e^{-(\lambda_j+n^2\pi^2)t} e_j(x')\sqrt{2}\cos(n\pi x_N), \qquad 0\le t\le \tau.$$

where $\bar{j},\overline{n}\in\mathbb{N}$.
Clearly, for $0\le t\le\tau$,

$$||(\theta-\overline{\theta})(t)||^2_{L^2(\Omega)} = \sum_{(j,n)\notin[0,\bar{j}]\times[0,\overline{n}]} e^{-2(\lambda_j+n^2\pi^2)t}|c_{j,n}|^2,$$

hence $\overline{\theta}\to\theta$ in $C([0,\tau];L^2(\Omega))$ as $\bar{j}\to\infty$ and $\overline{n}\to\infty$.
Better estimates can be derived if the initial data is more regular...

Step 2 (control process)

Introduce the approximations

$$\overline{u}(t,x') = \sum_{0\le j\le \bar{j}} e^{-\lambda_j t} e_j(x') \sum_{0\le i\le \bar{i}} \overline{y_j}^{(i)}(t) \frac{1}{(2i-1)!}, \qquad \tau \le t \le T,$$

$$\overline{\theta}(t,x',x_N) = \sum_{0\le j\le \bar{j}} e^{-\lambda_j t} e_j(x') \sum_{0\le i\le \bar{i}} \overline{y_j}^{(i)}(t) \frac{x_N^{2i}}{(2i)!}, \qquad \tau \le t \le T,$$

with

$$\overline{y_j}(t) = \phi_s\Big(\frac{t-\tau}{T-\tau}\Big) \sum_{0\le n\le \bar{n}} c_{j,n} e^{-n^2\pi^2 t}, \qquad \tau \le t \le T.$$

Then we have

Theorem

$$||\theta(t)-\overline{\theta}(t)||_{L^\infty(\Omega)} \le C_1 f(\bar{i},\bar{j},\bar{n})||\theta_0||_{L^2(\Omega)}, \qquad \forall t\in[\tau,T], \tag{2.8}$$

where $f(\bar{i},\bar{j},\bar{k}) = e^{-C_2 \bar{j}^{\frac{2}{N-1}}} + e^{-C_3 \bar{i}\ln \bar{i}} + e^{-C_4 \bar{n}^2}$. In (2.8) we can pick any $C_2 < A_1\tau$, any $C_3 < 2-s$, and any $C_4 < \pi^2\tau$, while $C_1 = C_1(N,\omega,\tau,s,C_2,C_3,C_4)$.

Corollary

(Stronger norms) *For all $q \in \mathbb{N}^*$, we have*

$$
\begin{aligned}
||\theta(t) - \overline{\theta}(t)||_{W^{q,\infty}((\tau,T)\times\Omega)} &\leq C_1' f(\bar{i},\bar{j},\overline{n})||\theta_0||_{L^2(\Omega)}, \\
||u(t) - \overline{u}(t)||_{W^{q-1,\infty}((\tau,T)\times\Omega)} &\leq C_1'' f(\bar{i},\bar{j},\overline{n})||\theta_0||_{L^2(\Omega)}
\end{aligned}
$$

for some constants C_1', C_1'' depending on $N, \omega, \tau, s, q, C_2, C_3, C_4$.

Real solution associated with the approximated control

Let $\hat{u}$ denote the control defined by 0 for $t \in [0, \tau]$ and by $\bar{u}$ for $t \in [\tau, T]$, and let $\hat{\theta}$ be the corresponding solution issuing from the same initial data θ_0 as θ.

Then we have

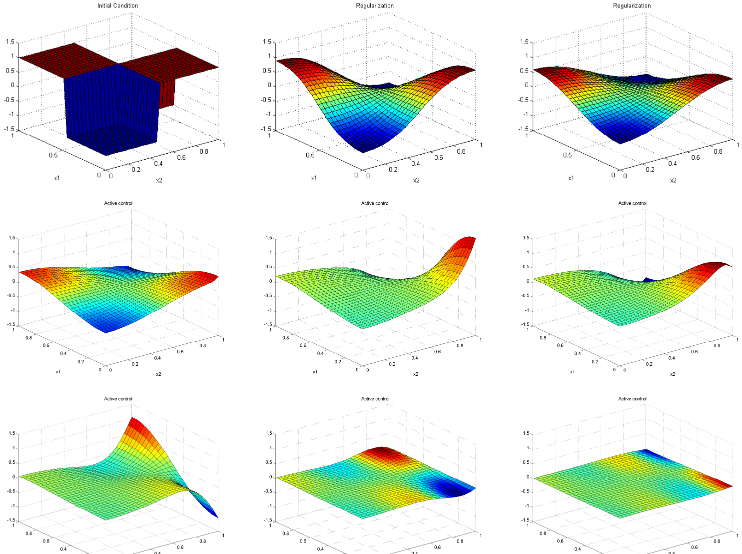
Corollary

$$||\theta(t) - \hat{\theta}(t)||_{L^\infty((0,T)\times\Omega)} \leq C_1''' f(\bar{i}, \bar{j}, \overline{n})||\theta_0||_{L^2(\Omega)} \tag{2.9}$$

for some constants C_1''' depending on $N, \omega, \tau, s, C_2, C_3, C_4$.

Numerical simulations (N=2) Trajectory

Initial state: $\theta_0 := \left(1_{(1/2,1)}(x_1) - 1_{(0,1/2)}(x_1)\right)\left(1_{(0,1/2)}(x_2) - 1_{(1/2,1)}(x_2)\right)$ Parameters: $\tau = 0.05$, $R = 0.25$, $T = \tau + R = 0.3$, $s = 1.65$



Numerical simulations (N=2) Control

Initial state: $\theta_0 := \left(1_{(1/2,1)}(x_1) - 1_{(0,1/2)}(x_1)\right)\left(1_{(0,1/2)}(x_2) - 1_{(1/2,1)}(x_2)\right)$ Parameters: $\tau = 0.05$, $R = 0.25$, $T = 0.3$, $s = 1.65$

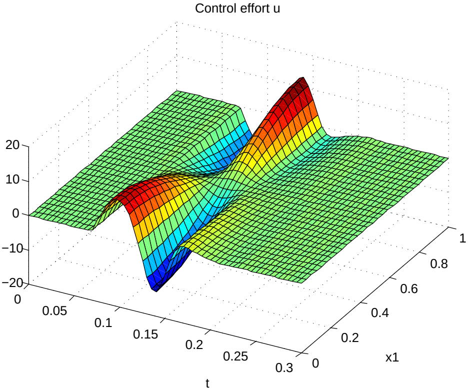


Fig. 4. $\bar{u}(t, x_1)$

Computations by Philippe Martin

2.3. Null controllability of the 1D Schrödinger equation

P. Martin, L. Rosier, P. Rouchon, *Null controllability of the 1D Schrödinger equation using flatness*, Automatica J. IFAC 91 (2018), 208–216.

Controllability of Schrödinger eq. by flatness approach

- For the sake of simplicity, we limit ourselves to the 1D case

$$
\begin{aligned}
i\theta_t + \theta_{xx} &= 0, \quad 0 < x < 1 \\
\theta(t,0) = 0, \quad \theta(t,1) &= u(t) \\
\theta(0,x) &= \theta_0(x).
\end{aligned}
$$

- The null ($\Longleftrightarrow$ exact) controllability can be established by the same flatness approach as for the heat eq. However, the first step (smoothing effect) has to be modified, for the application of a null boundary control does not smooth out the solution as for the heat eq.

- Following an idea in LR-Bing-Yu Zhang (2009), we notice that a **strong** smoothing effect occurs if we consider Schrödinger equation on the whole line with a compactly supported initial data:

$$\begin{aligned} iv_t + v_{xx} &= 0, \quad -\infty < x < \infty \\ v(0,x) &= v_0(x) := \begin{cases} \theta_0(x) & \text{if } x \in (0,1) \\ -\theta_0(-x) & \text{if } x \in (-1,0), \\ 0 & \text{if } x \in (-\infty,-1) \cup (1,+\infty) \end{cases} \end{aligned}$$

Proposition

For any $\theta_0 \in L^2(0,1)$*, the function* $(t,x) \to v(t,x)$ *is* **Gevrey of order** $1/2$ **in** x **and** 1 **in** t *on any rectangle* $[\varepsilon, T] \times [-L, L]$*, for* $0 < \varepsilon < T$*,* $L > 0$*.*

Flatness applied to Schrödinger

Theorem

Let $\theta_0 \in L^2(0,1)$ and $T > 0$. Pick $\tau \in (0,T)$ and $s \in (1,2)$. There exists $y \in C^\infty([\tau, T])$ Gevrey of order s on $[\tau, T]$ such that, setting

$$u(t) = \begin{cases} v(t,1) & \text{if } 0 \leq t \leq \tau \\ \sum_{k\geq 1}(-i)^k \frac{y^{(k)}(t)}{(2k+1)!} & \text{if } \tau < t \leq T, \end{cases}$$

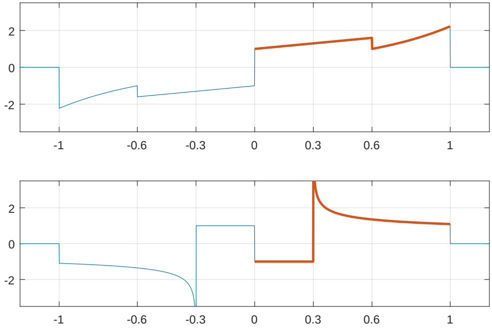
the solution θ of

$$\begin{aligned} i\theta_t + \theta_{xx} &= 0, \qquad x \in (0,1) \\ \theta(t,0) = 0,\ \theta(t,1) &= u(t), \\ \theta(0,x) &= \theta_0(x) \end{aligned}$$

satisfies $\theta(T,.) = 0$. Furthermore, u is in $L^4(0,T)$ and

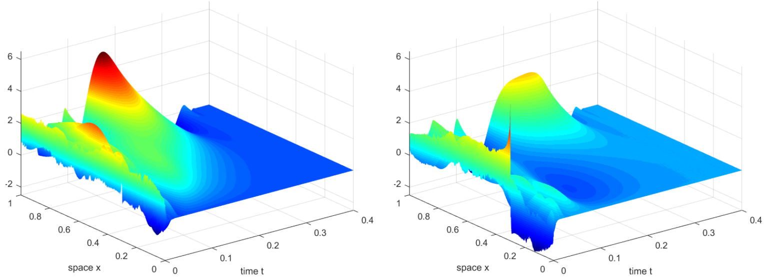
$$\theta \in C([0,T], L^2(0,1)) \cap C^\infty((0,T] \times [0,1]).$$

Numerical simulations for Schrödinger: initial data



Initial condition θ_0 (red) and odd extension (blue);
real parts (top), imaginary parts (bottom)

Numerical simulations for Schrödinger



Evolution of θ: real part (left), imaginary part (right)

Computations by Philippe Martin

2.4-2.5 Null controllability of parabolic equations

Variable coefficients

- Consider now the equation

$$(a(x)\theta_x)_x + b(x)\theta_x + c(x)\theta - \rho(x)\theta_t = 0$$

  where $a, b, c, \rho \in L^1(0,1)$.

- **Alessandrini-Escauriaza (2006)** proved the null controllability of this equation (with internal or Dirichlet boundary control) when $a, b, c, \rho \in L^\infty(0,1)$ with

$$a(x) > K > 0 \quad \text{and } \rho(x) > K > 0 \quad \text{a.e. in } (0,1)$$

  Method of proof: Lebeau-Robbiano strategy + complex variable analysis.

- We shall see that this result can be **extended** to parabolic equations with singular or degenerate coefficients by using the flatness approach.
  (for degenerate eq., we refer to **Cannarsa-Martinez-Vancostenoble** 2004,...)

2.4 Null controllability of (weakly degenerate) parabolic equations

P. Martin, L. Rosier, P. Rouchon, *Null controllability of one-dimensional parabolic equations by the flatness approach*, SIAM J. Control Optim. Vol. 54 (2016), No. 1, pp. 198–220

Main result

Let a, b, c, ρ with

$$\begin{aligned} & a(x) > 0 \text{ and } \rho(x) > 0 \quad \text{for a.e. } x \in (0,1) \\ & (\frac{1}{a}, \frac{b}{a}, c, \rho) \in [L^1(0,1)]^4 \\ & \exists K \geq 0,\ \frac{c(x)}{\rho(x)} \leq K \quad \text{for a.e. } x \in (0,1) \\ & \exists p \in (1, \infty],\ \ a^{1-\frac{1}{p}} \rho \in L^p(0,1). \end{aligned}$$

Theorem

Let (a, b, c, ρ) be as above, and $(\alpha_0, \beta_0) \neq (0,0)$, $(\alpha_1, \beta_1) \neq (0,0)$. Let $\theta_0 \in L^1_{\rho(x)dx}(0,1)$ and $T > 0$. Pick $\tau \in (0, T)$ and $s \in (1, 2 - p^{-1})$. Then there exists a control $h = h(t)$ Gevrey of order s on $[0, T]$ such that the solution θ of

$$\begin{aligned} (a(x)\theta_x)_x + b(x)\theta_x + c(x)\theta - \rho(x)\theta_t &= 0, \qquad x \in (0,1) \\ \alpha_0\theta(t,0) + \beta_0(a\theta_x)(t,0) &= 0, \\ \alpha_1\theta(t,1) + \beta_1(a\theta_x)(t,1) &= h(t), \\ \theta(0,x) &= \theta_0(x) \end{aligned}$$

satisfies $\theta(T,.) = 0$.

Examples

- $(a(x)\theta_x)_x - \theta_t = 0$, with $a(x) > 0$ a.e. and

$$a, 1/a \in L^1(0,1)$$

  Possible: $a(x) \sim (x - x_0)^r$ with
  - $-1 < r < 0$ (**singular**) or
  - $0 < r < 1$ (**weakly degenerate**)

  Degeneracies can occur
  - at a **single point** $x_0 \in [0,1]$ (ex $a(x) = x^r$),
  - at a **sequence** of points. Ex:

$$a(x) = |\sin(x^{-1})|^r, \quad -1 < r < 1$$

  The **strongly degenerate** case $1 \leq r < 2$ has been treated with the flatness approach by Ivan Moyano, 2016.

- $\theta_{xx} + \frac{\mu}{x^2}\theta - \theta_t = 0$, $\mu \leq 1/4$ (no need of Carleman or Hardy inequal.).
- Transmission problem (a and ρ discontinuous)

Sketch of the proof

Step 1

Using changes of variables, we can put the system in the **canonical form**

$$\begin{aligned}
\theta_{xx} - \rho(x)\theta_t &= 0, \qquad x \in (0,1)\\
\alpha_0\theta(t,0) + \beta_0\theta_x(t,0) &= 0,\\
\alpha_1\theta(t,1) + \beta_1\theta_x(t,1) &= h(t),\\
\theta(0,x) &= \theta_0(x)
\end{aligned}$$

where $\rho \in L^p(0,1)$, $1 < p \leq \infty$.

More details about the changes of variables

$$\begin{aligned} B(x) &:= \int_0^x \frac{b(s)}{a(s)}\,ds, \\ \tilde{a}(x) &:= a(x)e^{B(x)} \\ \tilde{c}(x) &:= (K\rho(x) - c(x))e^{B(x)}. \end{aligned}$$

Then $B \in W^{1,1}(0,1)$, $\tilde{c} \in L^1(0,1)$, and

$$\tilde{a}(x) > 0 \text{ and } \tilde{c}(x) \geq 0 \text{ for a.e. } x \in (0,1).$$

We introduce the solution v to the elliptic boundary value problem

$$\begin{aligned} -(\tilde{a}v_x)_x + \tilde{c}v &= 0, \quad x \in (0,1), \\ v(0) = v(1) &= 1, \end{aligned}$$

and set

$$u_1(x,t) := e^{-Kt}u(x,t), \quad u_2(x,t) := \frac{u_1(x,t)}{v(x)}.$$

Finally, let

$$L := \int_0^1 (a(s)v^2(s)e^{B(s)})^{-1}ds, \quad y(x) := \frac{1}{L}\int_0^x (a(s)v^2(s)e^{B(s)})^{-1}ds$$

Let finally

$$\hat{u}(y,t) := u_2(x,t), \qquad \hat{\rho}(y) := L^2 a(x) v^4(x) e^{2B(x)} \rho(x)$$

for $0 < t < T$, $y = y(x)$ with $x \in [0,1]$. Then the following result holds.

Proposition

(i) $v \in W^{1,1}(0,1)$ *and* $0 < v(x) \leq 1 \ \forall x \in [0,1]$;

(i) $y : [0,1] \to [0,1]$ *is an increasing bijection with* $y, y^{-1} \in W^{1,1}(0,1)$;

(iii) $\hat{\rho}(y) > 0$ *for a.e.* $y \in (0,1)$, *and* $\hat{\rho} \in L^p(0,1)$;

(iv) $\hat{u}$ *solves the system*

$$\hat{u}_{yy} - \hat{\rho}\hat{u}_t = 0, \quad y \in (0,1),\ t \in (0,T) \ + \text{b.c. and i.c.}$$

Step 2.

In the time interval $(0, \tau)$, we apply a **null control to smooth out the state**, while in the interval (τ, T) we apply a **non-trivial control to reach 0** at time $t = T$. The trajectory will be written as

$$\begin{aligned} \theta(x,t) &= \sum_{n\geq 0} e^{-\lambda_n t} e_n(x), \quad x \in (0,1),\ t \in [0,\tau], \\ \theta(x,t) &= \sum_{i\geq 0} y^{(i)}(t) g_i(x), \quad x \in (0,1),\ t \in [\tau, T]. \end{aligned}$$

Sketch of the proof (2)

For $t \in (0,\tau)$, $\theta(x,t) = \sum_{n\geq 0} e^{-\lambda_n t} e_n(x)$.
(e_n, λ_n) is the nth pair of eigenfunction/eigenvalue for

$$\begin{aligned} -e_n'' &= \lambda_n \rho\, e_n, \quad x \in (0,1) \\ \alpha_0 e_n(0) + \beta_0 e_n'(0) &= 0, \\ \alpha_1 e_n(1) + \beta_1 e_n'(1) &= 0, \end{aligned}$$

We proved that $\lambda_n \geq Cn$ by using a Prüfer substitution

$$e' = r\cos\theta, \tag{2.10}$$
$$e = r\sin\theta \tag{2.11}$$

We can improve the estimate in $\lambda_n \geq Cn^2$ for pure Dirichlet b.c. or Neumann b.c.

Generating functions

For $t \in (\tau, T)$, we have

$$\theta(x,t) = \sum_{i \geq 0} y^{(i)}(t) g_i(x) \tag{2.12}$$

where the **generating function** g_i is defined inductively as follows:

$$\begin{aligned} g_0'' &= 0, \quad x \in (0,1) \\ \alpha_0 g_0(0) + \beta_0 g_0'(0) &= 0, \\ \beta_0 g_0(0) - \alpha_0 g_0'(0) &= 1 \end{aligned}$$

for $i = 0$, and g_i, for $i \geq 1$, is the solution to the **Cauchy problem**

$$\begin{aligned} g_i'' &= \rho\, g_{i-1}, \quad x \in (0,1) \\ g_i(0) &= 0, \\ g_i'(0) &= 0 \end{aligned}$$

We can prove

$$||g_i||_{W^{2,p}(0,1)} \leq \frac{C}{R^i (i!)^{2-\frac{1}{p}}}$$

which allows to prove the convergence of the series in (2.12) if y is Gevrey of order $s \in (1, 2-1/p)$.

Sketch of the proof (3)

- To ensure that the two expressions of θ agree at $t = \tau$, we have to relate the eigenfunctions e_n to the generating functions g_i.
- We have

$$e_n(x) = \zeta_n \sum_{i \geq 0} (-\lambda_n)^i g_i(x) \qquad (*)$$

  with $\zeta_n \in \mathbb{R}$.
  [similar to $\cos(n\pi x) = \sum_{i\geq 0}(-n^2\pi^2)^i x^{2i}/(2i!)$]
- Thus, the generating function $g_i \in W^{2,p}(0,1)$ replaces the function $x^{2i}/(2i)!$ we had for the heat equation with Neumann b.c.

Transmission pb. (piecewise constant coef.)

$$\begin{aligned}
\rho_0\theta_t &= a_0\theta_{xx}, \quad 0 < x < X \\
\rho_1\theta_t &= a_1\theta_{xx}, \quad X < x < 1 \\
\theta(t, X^-) &= \theta(t, X^+) \\
a_0\theta_x(t, X^-) &= a_1\theta_x(t, X^+)
\end{aligned}$$

Parameters: $X = 1/2$, $(a_0, \rho_0, a_1, \rho_1) = (10/19, 15/8, 10, 1/8)$

Numerical simulations (N=1, transm. pb) Trajectory

Initial state: $\theta_0 := \frac{1}{2}1_{(1/2,1)}(x) - \frac{1}{2}1_{(0,1/2)}(x)$

Parameters: $\tau = 0.3$, $T = 0.35$, $s = 1.6$, $(a_0, \rho_0, a_1, \rho_1) = (10/19, 15/8, 10, 1/8)$

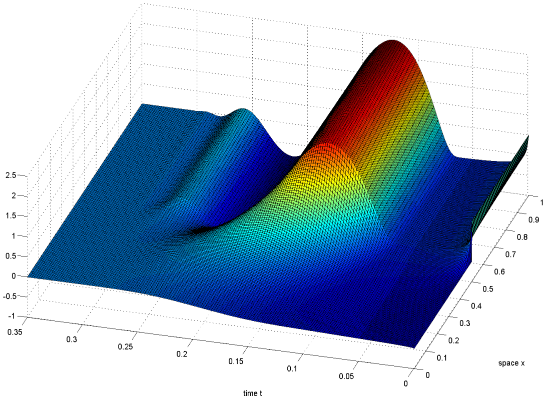


Fig.1. $\theta(t, x)$

Computations by Philippe Martin

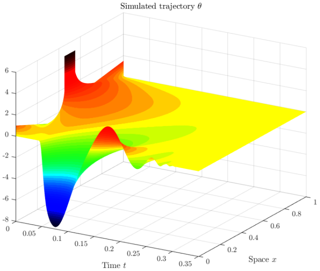
Degenerate heat equation

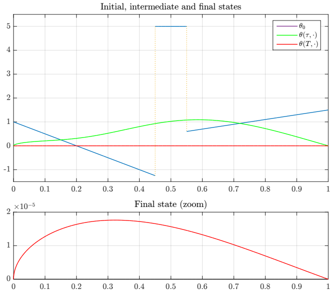
We consider a weakly degenerate ($0 < \gamma < 1$) heat equation with a control applied at the point ($x = 0$) where the equation is degenerate.

$$\begin{aligned}
\theta_t - (x^\gamma \theta_x)_x &= 0, \quad 0 < x < 1 \\
\alpha_0 \theta(0,t) + \beta_0 (x^\gamma \theta_x(x,t))_{|x=0} &= u(t) \\
\alpha_1 \theta(1,t) + \beta_1 \theta_x(1,t) &= 0 \\
\theta(x,0) &= \theta_0(x), \quad 0 < x < 1.
\end{aligned}$$

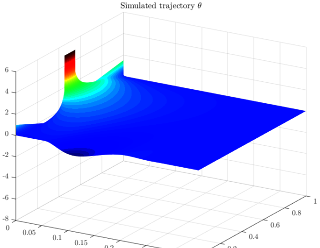
Degenerate heat equation

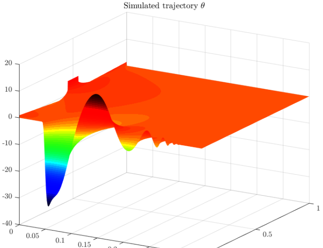
$\gamma = 1/2$





$\gamma = 0$ (left) $\gamma = 2/3$ (right)





2.5 Null controllability of (strongly degenerate) parabolic equations

A. Benoit, R. Loyer, L. Rosier, *Null controllability of strongly degenerate parabolic equations*, ESAIM Control Optim. Calc. Var. 29 (2023), Paper No. 48, 36 pp.

- We consider the control system:

$$\begin{aligned}
(a(x)u_x)_x + q(x)u = \rho(x)u_t, \qquad & x \in (0,1),\ t \in (0,T),\\
(au_x)(0,t) = 0,\ t \in (0,T), &\\
\alpha u(1,t) + \beta(au_x)(1,t) = h(t), &\\
u(x,0) = u_0(x), \qquad & x \in (0,1)
\end{aligned}$$

  where $(\alpha,\beta) \in \mathbb{R}_+^2 \setminus \{(0,0)\}$, $T>0$, $u_0 \in L^2(0,1)$ and $h \in L^2(0,T)$.
- Goal: $u(x,T) = 0 \quad \forall x \in [0,1]$.
- Difficulties:
  - a may be strongly degenerate (e.g. $a(x) = x^{2-\varepsilon}$, $0<\varepsilon<1$)
  - q may be singular (e.g. $q(x) = x^{-\varepsilon}$)

Previous results

- $a(x) = x^{2-\varepsilon}$, $1 < \varepsilon < 2$ (weak), $0 < \varepsilon < 1$ (strong): Cannarsa-Martinez-Vancostenoble (2008-2020)
- $a \in W^{1,1}(0,1)$, $x \to a(x)/x^\gamma$ nondecreasing: Fragnelli-Mugnai (2016-2021)
- $a \in L^\infty(0,1)$, $0 < \alpha < a(x)$ a.e.: Alessandrini-Escauriaza (2008)
- $a, 1/a \in L^1(0,1)$: Martin-LR-Rouchon (2016) by the flatness approach
- $a(x) = x^{2-\varepsilon}$, $0 < \varepsilon < 1$, $q(x) = 0$: Moyano (2016) by the flatness approach

Assumption

$$a(x) > 0 \text{ and } \rho(x) > 0 \ \text{ for a.e. } x \in (0,1),$$

$$a \in L^1_{loc}(0,1), \quad \left(x \to \frac{x}{a(x)}\right) \in L^p(0,1),$$

$$\rho \in L^r(0,1), \quad \limsup_{x\to 0^+} \rho(x) < \infty,$$

$$\lim_{x\to 0^+} a(x)^{-1} \left(\int_x^1 \frac{ds}{a(s)} \right)^{-2} = +\infty,$$

$$(H_q) \qquad \exists v \in W^{1,1}(0,1) \text{ s.t. } \begin{cases} v(x) > 0 \text{ for all } x \in [0,1], \\ (av_x)_x + qv = 0 \text{ in } (0,1), \\ (av_x)(0) = 0, \end{cases}$$

with

$$p \in (1,+\infty], \ \ r \in (p',+\infty], \ \ p' := \frac{p}{p-1}.$$

(H_q) holds e.g. if $\int_0^1 a(x)^{-1}(\int_0^x |q(s)|ds)dx < 1$ or if $q(x) \le C\rho(x)$ a.e.

Theorem

Let the functions $a, q, \rho, v : (0,1) \to \mathbb{R}$ *satisfy the above assumptions for some* $p \in (1, +\infty]$ *and* $r \in (p', +\infty]$*. Let* $(\alpha, \beta) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$ *and* $T > 0$*. Pick any* $u_0 \in L^2_\rho$ *and any* $s \in (1, 1 + \frac{1}{p'} - \frac{1}{r})$*. Then there exists a function* $h \in G^s([0,T])$ *such that the solution* u *of*

$$\begin{aligned}
(a(x)u_x)_x + q(x)u &= \rho(x)u_t, \qquad x \in (0,1),\ t \in (0,T),\\
(au_x)(0,t) &= 0,\ t \in (0,T),\\
\alpha u(1,t) + \beta(au_x)(1,t) &= h(t),\\
u(x,0) &= u_0(x), \qquad x \in (0,1)
\end{aligned}$$

satisfies

$$u(x,T) = 0 \quad \forall x \in [0,1].$$

Reductions

- By (H_q) there exists $v \in W^{1,1}(0,1)$ s.t.

$$\begin{cases} v(x) > 0 \text{ for all } x \in [0,1], \\ (av_x)_x + qv = 0 \text{ in } (0,1), \\ (av_x)(0) = 0 \end{cases}$$

- Setting $\hat{u}(x,t) := u(x,t)/v(x)$, $\hat{a}(x) := v(x)^2 a(x)$, $\hat{\rho}(x) = v(x)^2 \rho(x)$, we obtain

$$\begin{aligned} (\hat{a}\hat{u}_x)_x &= \hat{\rho}\hat{u}_t, \quad x \in (0,1),\ t \in (0,T), \\ (\hat{a}\hat{u}_x)(0) &= 0,\ t \in (0,T). \end{aligned}$$

  We can thus assume that $q = 0$.
- We first investigate the elliptic problem:

$$\begin{aligned} -(au')' &= \rho f \qquad \text{in } (0,1), \\ (au')(0) &= 0, \\ \alpha u(1) + \beta (au')(1) &= 0. \end{aligned}$$

Generalized Hardy inequality

Introduce the space

$$H_a := \{u \in W^{1,1}_{loc}(0,1);\ \sqrt{a}\, u' \in L^2(0,1) \text{ and } u(1) = 0\}$$

endowed with the norm

$$\|u\|_{H_a} := \left(\int_0^1 a(x) u'(x)^2 dx \right)^{\frac{1}{2}}.$$

Extend a to $(0, +\infty)$ by setting $a(x) = x^2$ for $x \geq 1$, and set

$$b(x) := a(x)^{-1} \left(\int_x^\infty \frac{ds}{a(s)} \right)^{-2}, \quad x \in (0, +\infty).$$

Then $\lim_{x\to 0^+} b(x) = +\infty$ by our assumption, and we have the following

Lemma (Generalized Hardy inequality)

$$\int_0^1 b(x) u(x)^2 dx \leq 4 \int_0^1 a(x) u'(x)^2 dx, \quad \forall u \in H_a$$

Spectral problem (1)

Introduce the spaces

$$\begin{aligned} L^2_\rho &:= \{f:(0,1)\to\mathbb{R};\ \int_0^1 f(x)^2\rho(x)dx<\infty\} \\ H_{a,\rho} &:= \{u\in W^{1,1}_{loc}(0,1);\ \sqrt{a}\,u'\in L^2(0,1) \text{ and } \sqrt{\rho}\,u\in L^2(0,1)\} \end{aligned}$$

endowed respectively with the norms

$$\|f\|_{L^2_\rho} := \left(\int_0^1 f(x)^2\rho(x)dx\right)^{\frac{1}{2}},\quad \|u\|_{H_{a,\rho}} := \left(\int_0^1 [a(x)u'(x)^2+\rho(x)u(x)^2]dx\right)^{\frac{1}{2}}.$$

Then the embeddings $H_{a,\rho}\subset L^2(0,1)$ and $H_{a,\rho}\subset L^2_\rho$ are compact.

Theorem

Let a, ρ and (α,β) be as above. Then there are a sequence $(e_n)_{n\ge 0}$ in L^2_ρ and a nondecreasing sequence $(\lambda_n)_{n\ge 0}$ in $(0,+\infty)$ such that

1. *$(e_n)_{n\ge 0}$ is an orthonormal basis in L^2_ρ;*
2. *for all $n\ge 0$, $e_n\in H_{a,\rho}$, $ae_n'\in W^{1,\min(2,r)}(0,1)$, and e_n solves*

$$\begin{aligned} -(ae_n')' &= \lambda_n\,\rho\,e_n \ \text{ in } (0,1), \\ (ae_n')(0) &= 0, \\ \alpha e_n(1)+\beta(ae_n')(1) &= 0. \end{aligned}$$

Spectral problem (2)

Theorem

Let $a, \rho, (\alpha, \beta)$ and the sequences $(e_n)_{n\geq 0}$, $(\lambda_n)_{n\geq 0}$ be as before. Then

1. $e_n \in W^{1,1}(0,1)$ *and* $ae_n' \in W^{1,r}(0,1)$ *for all* $n \geq 0$*;*
2. *there exists some constant* $C_1 > 0$ *such that*

$$\|e_n\|_{L^\infty(0,1)} \leq C_1 \lambda_n^{\frac{3}{4}(1+\frac{p' r}{r-p'})} \quad \text{if } \lambda_n > 0;$$

(For $r = \infty$, $\frac{p' r}{r-p'} = p'$.)

3. *let* $\kappa := [\frac{1}{2} + \frac{1}{p}(\frac{p' r}{r-p'})]^{-1} > 0$ *if* $p < \infty$ *and pick any* $\kappa < 2$ *if* $p = \infty$*. Then there exists some constant* $C_2 > 0$ *such that*

$$\lambda_n \geq C_2 n^\kappa \quad \forall n \geq 0.$$

Proof of the lower bound for λ_n

We used a modified Prüfer transform:

$$\begin{aligned} a e_n' &= \lambda_n^{\frac{1}{4}} R_n \cos\theta_n \\ e_n &= \lambda_n^{-\frac{1}{4}} R_n \sin\theta_n \end{aligned}$$

Then θ_n solves the Cauchy problem

$$\begin{aligned} \theta_n' &= \lambda_n^{\frac{1}{2}} (\rho \sin^2\theta_n + \frac{1}{a}\cos^2\theta_n), \\ \theta_n(0) &= \frac{\pi}{2} \end{aligned}$$

Integrating over $(0,1)$ yields

$$\theta_n(1) - \frac{\pi}{2} = \lambda_n^{\frac{1}{2}} \int_0^1 \rho \sin^2\theta_n dx + \lambda_n^{\frac{1}{2}} \int_0^1 \frac{\cos\theta_n}{a} dx$$

Difficulty: $1/a \notin L^1(0,1)$

Solution: for the 2nd integral term, write $\int_0^1 = \int_0^{A_n} + \int_{A_n}^1$ with $A_n := (2C\lambda_n)^{\frac{p'r}{p'-r}}$.

Generating functions (1)

We consider the simplified system

$$\begin{aligned} (au_x)_x &= \rho u_t \\ (au_x)(0,t) &= 0 \end{aligned}$$

and we seek a solution in the form

$$u(x,t) = \sum_{i=0}^{\infty} y^{(i)}(t) g_i(x)$$

where y is the flat output and the g_i's are the generating functions, defined as

$$\begin{aligned} (ag_{0,x})_x &= 0 \\ (ag_{i,x})_x &= \rho g_{i-1}, \ \forall i \geq 1, \\ (ag_{i,x})(0) &= 0 \qquad \forall i \geq 0 \end{aligned}$$

Generating functions (2)

Proposition

There are some constants $C, R > 0$ such that

$$\|g_i\|_{W^{1,1}(0,1)} + \|ag_{i,x}\|_{W^{1,r}(0,1)} \le \frac{C}{R^i (i!)^{1+\frac{1}{p'}-\frac{1}{r}}} \quad \forall i \in \mathbb{N}$$

Proposition

Let (e_n, λ_n) be a pair of eigenfunction/eigenvalue for some $n \in \mathbb{N}$. Then

$$e_n = e_n(0) \sum_{i\ge 0} (-\lambda_n)^i g_i \quad \textit{in } W^{1,1}(0,1).$$

Flatness approach

Let $u_0 \in L^2_\rho$. Expand u_0 as

$$u_0 = \sum_{n\geq 0}^{\infty} c_n e_n \quad \in L^2_\rho$$

Pick $s \in (1, 1 + \frac{1}{p'} - \frac{1}{r})$ and $\varphi \in G^s([0,T])$ with $\varphi(t) = \begin{cases} 1 & \text{if } t \leq \frac{T}{3} \\ 0 & \text{if } t \geq \frac{2T}{3} \end{cases}$

Set

$$y(t) = \varphi(t) \sum_{n=0}^{\infty} c_n e_n(0) e^{-\lambda_n t}, \quad 0 < t < T$$

and

$$u(x,t) = \begin{cases} u_0(x) & \text{if } t = 0 \\ \sum_{i=0}^{\infty} y^{(i)}(t) g_i(x) & \text{if } 0 < t \leq T. \end{cases}$$

We can see that $u(x,t) = \sum_{n=0}^{\infty} c_n e^{-\lambda t} e_n(x)$ for $t < T/3$ (free evolution)
The control input is taken as a trace:

$$h(t) = \sum_{n=0}^{\infty} y^{(i)}(t) \left(\alpha g_i(1) + \beta (a g_{i,x})(1) \right).$$

3. Reachable states

3.1 Reachable states for the boundary control of the 1D heat equation

P. Martin, L. Rosier, P. Rouchon, *On the reachable states for the boundary control of the heat equation*, Appl. Math. Res. Express. AMRX 2016, no. 2, 181–216.

Reachable states

- A state θ_1 is said to be reachable (from 0 in time T) for the heat equation if there exist some control inputs $h_0, h_1 \in L^2(0,T)$ so that the solution of

$$\begin{aligned} \theta_t - \theta_{xx} &= 0, \qquad x \in (0,1),\ t \in (0,T) \\ \theta(t,0) = h_0(t),\ \theta(t,1) &= h_1(t), \qquad t \in (0,T) \\ \theta(0,x) &= 0 \end{aligned}$$

satisfies

$$\theta(T,x) = \theta_1(x)$$

- $\theta_1(x) = \sum_{n\geq 1} c_n \sin(n\pi x)$ is a **reachable state** if

$$\exists \varepsilon > 0 \text{ s.t.} \qquad \sum_{n=1}^{\infty} |c_n| n^{-1} e^{(1+\varepsilon)n\pi} < \infty \qquad \text{Fattorini-Russell (FR), 1971}$$

$$\sum_{n=1}^{\infty} |c_n|^2 n e^{2n\pi} < \infty \qquad \text{Ervedoza-Zuazua (EZ), 2011}$$

- Both (FR) and (EZ) imply that the reachable state θ_1 has to satisfy the condition

$$\theta_1^{(2p)}(0) = \theta_1^{(2p)}(1) = 0 \quad \forall p \in \mathbb{N}.$$

  Very conservative!! (no nontrivial polynomial function concerned!!)
- In what follows, we consider a control problem in $(-1, 1)$:

$$\begin{aligned} \theta_t - \theta_{xx} &= 0, \qquad x \in (-1,1),\ t \in (0,T) \\ \theta(t,-1) = h_0(t),\ \theta(t,1) &= h_1(t), \qquad t \in (0,T) \\ \theta(0,x) &= 0 \end{aligned}$$

  for simplify the exposition.

Two controls: $\theta(t,-1) = h_0(t), \theta(t,1) = h_1(t)$

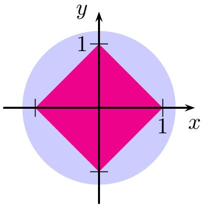
Such states should be **complex analytic** in the square $\{z = x + iy;\ |x| + |y| < 1\}$ (Gevrey 1926)

Notation: $Hol(\Omega)$ denotes the set of (complex) analytic functions in the domain $\Omega \subset \mathbb{C}$.

Theorem (Martin-R-Rouchon)

1. *If $\theta_1 \in Hol(\{z;\ |z| < R\})$ with $R > R_0 := e^{(2e)^{-1}} \sim 1.2$, then θ_1 is reachable from 0 in any time $T > 0$.*
2. *Conversely, any* **reachable** *state belongs to*

$$Hol(\{z = x + iy;\ |x| + |y| < 1\})$$



One control: $\theta(t,0)=0, \theta(t,1)=h_1(t)$

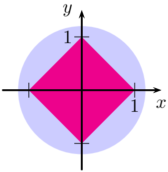
$$\begin{aligned} \theta_t - \theta_{xx} &= 0, \quad x\in(0,1),\ t\in(0,T) \\ \theta(t,0)=0,\ \theta(t,1) &= h_1(t), \quad t\in(0,T) \\ \theta(0,x) &= 0 \end{aligned}$$

Reachable states need also to be **odd** !

Theorem (Martin-R-Rouchon)

1. *If* $\theta_1 \in Hol(\{z;\ |z|<R\})$ *with* $R > R_0 := e^{(2e)^{-1}} \sim 1.2$ *and* θ_1 *is odd, then* θ_1 *is reachable from* 0 *in any time* $T>0$.
2. *Conversely, any* **reachable** *state is odd and it belongs to*

$$Hol(\{z=x+iy;\ |x|+|y|<1\})$$



Remarks about the reachable states with 2 controls h_0, h_1

- Any polynomial function is reachable!!

Def. A function $y = y(t)$, $t \in [0, T]$ is Gevrey of order $s \geq 0$ if there exist $C, R > 0$ s.t.

$$|y^{(n)}(t)| \leq C\frac{(n!)^s}{R^n}, \quad \forall t \in [0, T], \ \forall n \in \mathbb{N}.$$

- For the sufficient part, the control input driving the state to the target function can be chosen Gevrey of order 2.
- Result much better than the classical controllability to the trajectories. Indeed, the **controllability to the trajectories** involves states of Gevrey order $1/2$ (like $\exp(cx^2)$), while the reachable states are solely complex analytic, that is Gevrey of order 1, with possible poles.

- **Main tools in the proof: flatness approach** + a **Borel-Ritt thm**

Flatness property: the limit case $s = 2$

Proposition

Assume that for some constants $M > 0$, $R > 1$ we have

$$|y^{(i)}(t)| \leq M \frac{(2i)!}{R^{2i}} \quad \forall i \geq 0,\ \forall t \in [0, T]$$

Then the function

$$\theta(t, x) = \sum_{i \geq 0} \frac{x^{2i}}{(2i)!} y^{(i)}(t)$$

is well-defined in $[0, 1] \times [0, T]$, Gevrey of order 1 in x and 2 in t on $[0, 1] \times [0, T]$, and it is the solution of the ill-posed problem

$$\begin{aligned} \theta_t - \theta_{xx} &= 0, && t \in (0, T),\ x \in (0, 1), \\ \theta(0, t) &= y(t), && t \in (0, T), \\ \theta_x(0, t) &= 0, && t \in (0, T). \end{aligned}$$

Borel-Ritt theorem

Theorem

For any $R > 1$ and any sequence $(a_n)_{n\geq 0}$ of real numbers satisfying

$$|a_n| \leq C\frac{(2n)!}{R^{2n}}$$

one can find a function $f : \mathbb{R} \to \mathbb{R}$ with compact support in $[-1,1]$ and Gevrey of order 2, such that

$$f^{(n)}(0) = a_n \quad \forall n \geq 0,$$

$$|f^{(n)}(t)| \leq C' \frac{(2n)!}{(R/R_0)^{2n}} \quad \forall t \in [-1,1]$$

where $R_0 = e^{(2e)^{-1}} \sim 1.2$.

Inspired by **Petzsche 1988**

For Borel-Ritt theorem, see:

- **Ramis** (1978)
- **Chaumat-Cholet** (1994)
- **Thilliez** (2003) (complex analysis)
- **Petzsche** (1998) (real analysis)

However, in all these references, the issue of determining the greatest lower bound of R_0 was never considered.

Recent improvements for the domain of analyticity

Theorem (Dardé-Ervedoza, SICON 2018)

If $\theta_1 \in Hol(\{z = x + iy;\ |x| + |y| < 1 + \varepsilon\})$ with $\varepsilon > 0$, then θ_1 is reachable from 0 with two controls in $L^2(0, T)$.

$$\Omega = \{z = x + iy \in \mathbb{C};\ |x| + |y| < 1\}$$

$$A^2(\Omega) = Hol(\Omega) \cap L^2(\Omega) \quad (\text{ Bergman space })$$

$$E^2(\Omega) = \{\theta \in A^2(\Omega);\ \theta \in L^2(\partial\Omega) \text{ and } \int_{\partial\Omega} z^n \theta(z) dz = 0\ \forall n \in \mathbb{N}\} \subset A^2(\Omega)$$

(Hardy-Smirnov space)

Theorem (Hartmann-Kellay-Tucsnak, JEMS 2018)

1. *If θ_1 is reachable with two controls in $L^2(0, T)$, then $\theta_1 \in A^2(\Omega)$.*
2. *If $\theta_1 \in E^2(\Omega)$, then θ_1 is reachable with two controls in $L^2(0, T)$.*

The sharp result

Consider again the system

$$\begin{aligned} \theta_t - \theta_{xx} &= 0, \qquad x \in (-1,1),\ t \in (0,T) \\ \theta(t,-1) = h_0(t),\ \theta(t,1) &= h_1(t), \qquad t \in (0,T) \\ \theta(0,x) &= 0 \end{aligned}$$

Let $\Omega = \{x + iy;\ |x| + |y| < 1\}$. Assume that $h_0, h_1 \in L^2(0,T)$.

Theorem (Hartmann-Orsoni, J. Funct. Anal. 2021)

The reachable space is the Bergman space $A^2(\Omega) = Hol(\Omega) \cap L^2(\Omega)$.

Tools in the proof: Reproducing Kernel Hilbert space + separation of singularities in Bergman spaces

A similar result was also given with only one control.

3.2 Reachable states for the distributed control of the heat equation.

M. Chen, L. Rosier, *Reachable states for the distributed control of the heat equation*, C. R. Math. Acad. Sci. Paris 360 (2022), 627–639.

Distributed control

Let $0 < l_1 < l_2 < 1$. We are concerned with the reachable states for the control problem

$$\begin{aligned} y_t = y_{xx} + 1_{(l_1,l_2)} u(x,t), &\qquad x \in (0,1),\ t \in (0,T), \\ y(0,t) = y(1,t) = 0, &\qquad t \in (0,T), \\ y(x,0) = 0, &\qquad x \in (0,1). \end{aligned}$$

where $u \in L^2(0,T,L^2(0,1))$.
For any $L > 0$, we introduce the set

$$\mathcal{S}(L) = \{x + iy \in \mathbb{C};\ |x| + |y| < L\},$$

and the space

$$\mathcal{H}(L) = \{f \in H^1(0,L);\ f \text{ can be extended as an odd analytic function on } \mathcal{S}(L)\}.$$

Reachable states

Theorem

Let $T > 0$ and $0 < l_1 < l_2 < 1$. Then

- (i) *for any $u \in L^2(0, T; L^2(0,1))$, the solution y of the control system satisfies $y(\cdot, T) \in H_0^1(0,1)$, $y(\cdot, T) \in \mathcal{H}(l_1)$ and $y(1-\cdot, T) \in \mathcal{H}(1-l_2)$;*
- (ii) *for any $0 < \varepsilon < (l_2 - l_1)/2$, for any $y_T \in H_0^1(0,1)$ with $y_T \in \mathcal{H}(l_1+\varepsilon)$ and $y_T(1-\cdot) \in \mathcal{H}(1-l_2+\varepsilon)$, there exists a control function $u \in L^2(0, T; L^2(0,1))$ such that the solution y of the control system satisfies $y(\cdot, T) = y_T$ in $(0,1)$.*

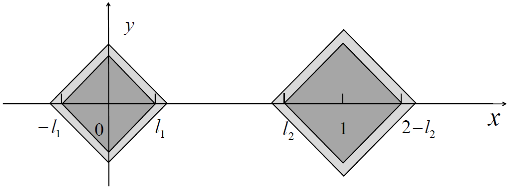


Figure: Reachable states for homogeneous Dirichlet boundary conditions.

Ideas in the proof

- In the limit case $(l_1, l_2) = (0, 1)$ (control distributed everywhere), the reachable space is nothing but $H_0^1(0, 1)$.
  This is proved using series of sinus.
- Using a partition of unity, we can use the characterization of the reachable states corresponding to the boundary control.

A new proof of Dardé-Ervedoza theorem

$$\mathcal{S}(L) = \{x + iy \in \mathbb{C};\ |x| + |y| < L\}$$

Theorem

Let $L > 1$, $T > 0$, and $\psi \in Hol(S(L))$. Then there exist $h_{-1}, h_1 \in G^2([0,T])$ such that the solution $w = w(x,t)$ of the control system

$$\begin{aligned} w_t - w_{xx} &= 0, && (x,t) \in (-1,1) \times (0,T), \\ w(-1,t) = h_{-1}(t),\ w(1,t) &= h_1(t), && t \in (0,T), \\ w(x,0) &= 0, && x \in (-1,1), \end{aligned}$$

satisfies $w \in C^\infty([-1,1] \times [0,T])$ and $w(x,T) = \psi(x)$ for $x \in [-1,1]$.
If, in addition, ψ is odd, then we can require that $w(.,t)$ be odd for all $t \in [0,T]$, so that $h_{-1}(t) = -h_1(t)$ and $w(0,t) = 0$ for all $t \in [0,T]$.

Step 1: Separation of singularities

$$\mathcal{S}(L) = \{x + iy \in \mathbb{C};\ |x| + |y| < L\}, \qquad \Omega(\theta, R) := \{z \in \mathbb{C};\ \mathrm{dist}(z, e^{i\theta}\mathbb{R}) < R\}$$

Lemma

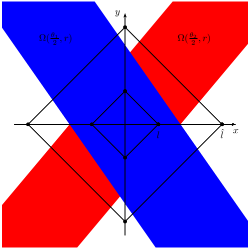
Let $1 < l < L$ *and* $\psi \in Hol(S(L))$. *Then there exist* $\theta_1 \in (\pi, \frac{3\pi}{2})$, $\theta_2 \in (\frac{\pi}{2}, \pi)$, $r \in (\frac{1}{\sqrt{2}}, +\infty)$, $\psi_1 \in Hol(\Omega(\frac{\theta_1}{2}, r))$ *and* $\psi_2 \in Hol(\Omega(\frac{\theta_2}{2}, r))$ *such that*

$$\overline{S(l)} \subset \Omega(\frac{\theta_1}{2}, r) \cap \Omega(\frac{\theta_2}{2}, r),$$

$$\partial_z^j \psi_i \in L^\infty(\Omega(\frac{\theta_i}{2}, r)), \quad i = 1, 2,\ j \in \mathbb{N},$$

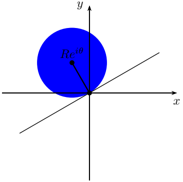
$$\psi = \psi_1 + \psi_2 \qquad \text{in } S(l).$$

Proof: From Cauchy formula, $\psi(z) = (2\pi i)^{-1} \int_\gamma \psi(\zeta)(\zeta - z)^{-1} d\zeta$. Next split γ.



Step 2: Integration of the heat kernel along an oblique line

$$\mathcal{O}(\theta, R) := \{z \in \mathbb{C};\ |z - Re^{i\theta}| < R\}$$



Lemma

Let $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$, $r > 1/\sqrt{2}$, and $\psi \in Hol\big(\Omega(\frac{\theta}{2}, r)\big) \cap L^\infty\big(\Omega(\frac{\theta}{2}, r)\big)$. Then the function

$$v(z, \tau) := \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} e^{-\frac{\zeta^2}{4\tau}} \psi(z - \zeta)\, d\zeta$$

is well-defined and analytic in z and τ for $z \in \Omega(\frac{\theta}{2}, r)$ and $\tau \in \mathcal{O}(\theta, R)$ for any $R > 0$. Furthermore, v satisfies

$$\begin{aligned} v_\tau - v_{zz} &= 0, \quad z \in \Omega(\frac{\theta}{2}, r),\ \tau \in \mathcal{O}(\theta, R), \\ \lim_{\tau \to 0^-} v(z, \tau) &= \psi(z), \quad z \in \Omega(\frac{\theta}{2}, r). \end{aligned}$$

3.3 Reachable states for the linear Korteweg-de Vries equation

P. Martin, I. Rivas, L. Rosier, P. Rouchon, *Exact controllability of a linear Korteweg-de Vries equation by the flatness approach*, SIAM J. Control Optim. **57** (2019), no.4, 2467–2486.

Korteweg-de Vries equation

- The Korteweg-de Vries (KdV) equation was introduced by Boussinesq (1877) and by Korteweg and de Vries (1895) as a model for water waves:

$$y_t + y_{xxx} + y\, y_x + y_x = 0$$

  where $y_t = \partial y / \partial_t$, $y_x = \partial y / \partial_x$, etc.
- Well-posedness of KdV studied by R. Temam, J.-C. Saut, T. Kato, C. Kenig - G. Ponce - L. Vega, J. Bourgain, F. Linares, T. Tao and many others...
- Control of KdV first considered by D. Russell and B.-Y. Zhang in 1996 (1993 for the linear KdV)

$$y_t + y_{xxx} + y\, y_x = Gh, \quad x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}.$$

Boundary control of KdV

The KdV equation

$$y_t + y_{xxx} + y\, y_x + y_x = 0$$

is supplemented with three boundary conditions

$$y(0,t) = u(t), \quad y(L,t) = v(t), \quad y_x(L,t) = w(t),$$

and an initial condition

$$y(x,0) = y_0(x).$$

Definition

We say that the equation is exactly controllable (resp. null controllable) if for any y_0 and for any y_1 (resp. for $y_1 = 0$), one can pick some boundary controls among u, v, w s.t.

$$y(x,T) = y_1(x).$$

Boundary control of KdV (2)

$$\begin{aligned} y_t + y_{xxx} + y\, y_x + y_x = 0 \\ y(0,t) = u(t), \quad y(L,t) = v(t), \quad \partial_x y(L,t) = w(t) \\ y(x,0) = y_0(x) \end{aligned}$$

- With w as only control ($u = v = 0$), system exact control. for $L \neq 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$ (LR, 1997)
- With u as only control ($v = w = 0$), system null control. (LR 2004, Glass-Guerrero 2008) Some Carleman inequality needed. Also controllability to the trajectories
- With v as only control ($u = w = 0$), system exact control. for L not critical (Glass-Guerrero, 2010)

Carleman inequality

For $q = q(x,t)$ with
$$q(0,t) = q(L,t) = q_x(L,t) = 0$$
we have
$$\int_0^T \int_0^L \{(s\varphi)^5 |q|^2 + (s\varphi)^3 |q_x|^2 + (s\varphi)|q_{xx}|^2\} e^{-2s\varphi}\, dxdt$$
$$\leq C \left(\int_0^T \int_0^L |q_t + q_{xxx}|^2 e^{-2s\varphi}\, dxdt + \int_0^T \left[s\varphi |q_{xx}|^2 e^{-2s\varphi} \right]_{x=L} dt \right)$$
where $\varphi(x,t) = \frac{\psi(x)}{t(T-t)}$ for some $\psi = \psi(x) > 0$, $C > 0$ and $s \geq s_0$.

Reachable states

We are now interested in describing precisely what are “all” the states that are indeed reachable with the only control $y(0, t) = u(t)$.

Reachable states for the Korteweg-de Vries equation

In order to to make the exposition of our results easier, we assume that the space domain is $(-1, 0)$ instead of $(0, L)$ and we consider the control problem

$$\begin{aligned} & y_t + y_{xxx} + a\, y_x = 0, \quad x \in (-1, 0) \\ & y(-1, t) = u(t), \quad y(0, t) = y_x(0, t) = 0, \\ & y(x, 0) = y_0(x) \end{aligned}$$

where $a \geq 0$ is a coefficient. (In practice, $a = 0$ or $a = 1$.)

Null controllability

Theorem (Thm 1)

Let $y_0 \in L^2(-1,0)$, $T > 0$, and $s \in [\frac{3}{2}, 3)$. Then there exists a control input $u \in G^s([0,T])$ such that the solution y of

$$
\begin{aligned}
&y_t + y_{xxx} + a\,y_x = 0, \quad x \in (-1,0)\\
&y(-1,t) = u(t), \quad y(0,t) = y_x(0,t) = 0,\\
&y(x,0) = y_0(x)
\end{aligned}
$$

satisfies $y(.,T) = 0$. Furthermore, it holds that

$$y \in C([0,T], L^2(-1,0)) \cap G^{\frac{s}{3},s}([-1,0] \times [\varepsilon, T]) \quad \forall \varepsilon \in (0,T).$$

Recall that a function $y \in G^{s_1,s_2}([x_1,x_2] \times [t_1,t_2])$ if there exist some constants $C, R_1, R_2 > 0$ such that

$$|\partial_x^{n_1} \partial_t^{n_2} y(x,t)| \le C \frac{(n_1!)^{s_1} (n_2!)^{s_2}}{R_1^{n_1} R_2^{n_2}} \quad \forall n_1, n_2 \in \mathbb{N},\ \forall (x,t) \in [x_1,x_2] \times [t_1,t_2].$$

Reachable space

Let

$$P := \partial_x^3 + a\partial_x$$

so that KdV can be written $\partial_t y + Py = 0$, and, for any $R > 1$, let

$$\mathcal{R}_R := \{y \in C^0([-1,0]);\ \exists z \in H(D(0,R)),\ y = z_{|[-1,0]},\ \text{and}$$
$$(P^n y)(0) = \partial_x (P^n y)(0) = 0 \ \ \forall n \geq 0\}$$

Theorem (Thm 2)

Let $a \in \mathbb{R}_+$, $T > 0$, and $R > R_0(a) := e^{(3e)^{-1}}(1+a)^{\frac{1}{3}} > 1$. Pick any $y_1 \in \mathcal{R}_R$. Then there exists a control input $u \in G^3([0,T])$ such that the solution y of

$$\begin{aligned} & y_t + y_{xxx} + a\, y_x = 0, \quad x \in (-1,0) \\ & y(-1,t) = u(t), \quad y(0,t) = \partial_x y(0,t) = 0, \\ & y(x,0) = 0 \end{aligned}$$

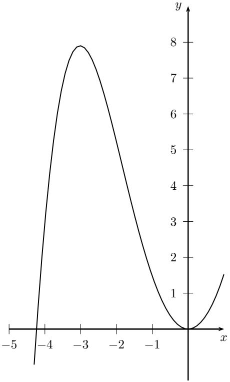
satisfies $y(.,T) = y_1$. Furthermore, $y \in G^{1,3}([-1,0]\times[0,T])$.

Examples of functions in $\mathcal{R}_R$

1. The polynomial functions of the form $y(x) = \sum_{n=0}^{N} a_n x^{3n+2}$
2. The entire function

$$y(x) = e^x + je^{jx} + j^2 e^{j^2 x}$$

where $j := e^{i\frac{2\pi}{3}}$. Note that y is real-valued and $y(-1) > 0$



Exact controllability

Combining Theorem 1 and Theorem 2, we obtain the following result which implies the exact controllability of system

$$
\begin{aligned}
&y_t + y_{xxx} + a\, y_x = 0, \quad x \in (-1, 0) \\
&y(-1, t) = u(t), \quad y(0, t) = y_x(0, t) = 0, \\
&y(x, 0) = y_0(x)
\end{aligned}
$$

in $\mathcal{R}_R$ for $R > R_0$.

Corollary

Let $a \in \mathbb{R}_+$, $T > 0$, $R > R_0(a)$, $y_0 \in L^2(-1, 0)$ and $y_1 \in \mathcal{R}_R$. Then there exists $u \in G^3([0, T])$ such that the solution of the above system satisfies $y(., T) = y_1$.

Sketch of the proof of Thm 1

We need first to investigate the ill-posed problem

$$y_t + y_{xxx} + ay_x = 0, \qquad x \in (-1, 0),\ t \in (0, T), \tag{3.1}$$

$$y(0, t) = y_x(0, t) = 0, \qquad t \in (0, T), \tag{3.2}$$

$$y_{xx}(0, t) = z(t), \qquad t \in (0, T). \tag{3.3}$$

Proposition (Flatness property)

Let $s \in [1, 3)$, $z \in G^s([0, T])$. Then system (3.1)-(3.3) admits a solution $y \in G^{\frac{s}{3}, s}([-1, 0] \times [0, T])$.

Expression of the solution of the ill-posed problem

The solution of the ill-posed reads

$$y(x,t) = \sum_{i\geq 0} g_i(x) z^{(i)}(t),$$

where the generating functions g_i are defined as follows.
g_0 is the solution of the Cauchy problem ($' = d/dx$)

$$\begin{aligned} g_0'''(x) + a g_0'(x) &= 0, \qquad x \in (-1,0), \\ g_0(0) = g_0'(0) &= 0, \\ g_0''(0) &= 1 \end{aligned}$$

g_i for $i \geq 1$ is defined inductively as the solution of the Cauchy problem

$$\begin{aligned} g_i'''(x) + a g_i'(x) &= -g_{i-1}(x), \qquad x \in (-1,0), \\ g_i(0) = g_i'(0) = g_i''(0) &= 0. \end{aligned}$$

We can prove the

Lemma

Let $a \in \mathbb{R}_+$. Then for all $i \geq 0$

$$|g_i(x)| \leq \frac{|x|^{3i+2}}{(3i+2)!} \quad \forall x \in [-1,0].$$

Smoothing effect

Consider the free evolution

$$y_t + y_{xxx} + a y_x = 0, \qquad x \in (-1,0),\ t \in (0,T), \tag{3.4}$$
$$y(-1,t) = y(0,t) = y_x(0,t) = 0, \qquad t \in (0,T), \tag{3.5}$$
$$y(x,0) = y_0(x), \qquad x \in (-1,0), \tag{3.6}$$

Then the following smoothing effect holds.

Proposition

Let $a \geq 0$ and $y_0 \in L^2(-1,0)$. Then the solution y of (3.4)-(3.6) satisfies $y \in G^{\frac{1}{2},\frac{3}{2}}([-1,0]\times[\varepsilon,T])$ for all $0<\varepsilon<T<\infty$. More precisely, there exist some positive constant K, R_1, R_2 such that

$$|\partial_t^n \partial_x^p y(x,t)| \leq K t^{-\frac{3n+p+3}{2}} \frac{n!^{\frac{3}{2}}}{R_1^n} \frac{p!^{\frac{1}{2}}}{R_2^p} \quad \forall p,n \in \mathbb{N},\ \forall t \in (0,T],\ \forall x \in [-1,0].$$

Sketch of the proof of the smoothing effect

- Start from the global Kato smoothing effect (assuming $T = 1$ w.l.g.):

$$\int_0^1 \|\partial_x y(.,t)\|_{L^2}^2 \, dt \le \frac{1}{3}(a+1)\|y_0\|_{L^2}^2$$

- Combining with energy estimate, we get

$$\|y(.,t)\|_{H^1} \le \frac{C}{\sqrt{t}}\|y_0\|_{L^2} \quad \forall t \in (0,1].$$

- Applying this estimate to y_t and using interpolation yields successively ($P = \partial_x^3 + a\partial_x$)

$$\|y(.,t)\|_{H^{p+1}} \le \frac{C}{\sqrt{t}}\|y_0\|_{H^p}, \quad \text{for } p \in \{0,1,2,3\},\ y \in X_p,\ t \in (0,1]$$

$$\|Py(t)\|_{L^2} \le \frac{C'}{t^{\frac{3}{2}}}\|y_0\|_{L^2}, \quad \text{for } y_0 \in L^2(-1,0),\ t \in (0,1].$$

- Splitting $[0,t]$ into $[0,\frac{t}{n}] \cup [\frac{t}{n},\frac{2t}{n}] \cup \cdots \cup [\frac{n-1}{n}t, t]$, we obtain

$$\|P^n y(.,t)\|_{L^2} \le \frac{C'}{(\frac{t}{n})^{\frac{3}{2}}}\|P^{n-1}y(\frac{n-1}{n}t)\|_{L^2} \le \cdots \le \frac{C'^n}{t^{\frac{3n}{2}}} n^{\frac{3n}{2}} \|y_0\|_{L^2}.$$

Sharp smoothing effect on the line

We guess that for $y_0 \in L^2(-1,0)$, we have that

$$y \in G^{\frac{1}{3},1}([-1,0]\times[\varepsilon,T]) \qquad \forall\, 0<\varepsilon<T<\infty.$$

How to prove it??

The smoothing effect from L^2 to $G^{1/3}$ is much easy to establish on $\mathbb{R}$ for data with compact support.

Proposition

Let $y_0 \in L^2(\mathbb{R})$ be such that $y_0(x)=0$ for a.e. $x\in\mathbb{R}\setminus[-L,L]$ for some $L>0$. Let $y=y(x,t)$ denote the solution of the Cauchy problem

$$\begin{array}{rcl} \partial_t y + \partial_x^3 y &=& 0, \quad t>0,\ x\in\mathbb{R}, \\ y(x,0) &=& y_0(x), \quad x\in\mathbb{R}. \end{array}$$

Then $y \in G^{\frac{1}{3},1}([-l,l]\times[\varepsilon,T])$ for all $l>0$ and all $0<\varepsilon<T$.

Some technical facts in the proof of Thm 2

The flatness property has to be extended to the limit case $s = 3$.

Proposition

Assume that $z \in G^{3}([0,T])$ *with*

$$|z^{(j)}(t)| \leq M \frac{(3j)!}{R^{3j}} \quad \forall j \geq 0,\ \forall t \in [0,T]$$

where $R > 1$*, and let* $y = y(x,t) = \sum_{i \geq 0} g_i(x) z^{(i)}(t)$.

Then $y \in G^{1,3}([-1,0] \times [0,T])$ *and it solves the ill-posed problem*

$$\begin{aligned} y_t + y_{xxx} + a y_x &= 0, && x \in (-1,0),\ t \in (0,T),\\ y(0,t) = y_x(0,t) &= 0, && t \in (0,T),\\ y_{xx}(0,t) &= z(t), && t \in (0,T). \end{aligned}$$

Borel-Ritt theorem

(inspired by Petzsche, 1998)

Proposition (AMRX, 2016)

Let $(d_q)_{q\geq 0}$ be a sequence of real numbers such that

$$|d_q| \leq CR^q(3q)! \quad \forall q \geq 0$$

for some $R > 0$ and $C > 0$. Then for all $\rho > e^{e^{-1}} R$, there exists a function $f \in C^\infty(\mathbb{R})$ such that

$$\begin{aligned} f^{(q)}(0) &= d_q \quad \forall q \geq 0, \\ |f^{(q)}(x)| &\leq C\rho^q(3q)! \quad \forall q \geq 0,\ \forall x \in \mathbb{R}. \end{aligned}$$

3.4 Reachable states for the linear Zakharov-Kuznetsov equation

M. Chen, L. Rosier, *Exact controllability of the linear Zakharov-Kuznetsov equation*, Discrete Cont. Dyn. Syst. Ser. B 25 (2020), no. 10, 3889–3916.

Zakharov-Kuznetsov equation

- The Zakharov-Kuznetsov (ZK) equation

$$u_t + au_x + \Delta u_x + uu_x = 0,$$

provides a model for the propagation of nonlinear ionic-sonic waves in a plasma.
- $\Delta u = \partial^2 u/\partial x^2 + \sum_{i=1}^{d} \partial^2 u/\partial y_i^2$ where $x, t \in \mathbb{R}$ and $y \in \mathbb{R}^d$ (with $d \in \{1, 2\}$)
- The constant $a > 0$ is the sound velocity
- Here, we consider only the case $d = 1$ (for the sake of simplicity) and the linearized ZK equation (we remove the nonlinear term uu_x).
- We take $\Omega := (-1, 0)_x \times (0, 1)_y$ as spatial domain.
- As for KdV, exact controllability results can be proved with a control on u_x for $x = 0$, $y \in (0, 1)$ (see Doronin-Larkin (2015))

Control system

Set $\Omega = (-1,0) \times (0,1)$.

We are concerned with the control properties of the system:

$$
\begin{aligned}
u_t + u_{xxx} + u_{xyy} + a u_x &= 0, && (x,y) \in \Omega,\ t \in (0,T),\\
u(0,y,t) = u_x(0,y,t) &= 0, && y \in (0,1),\ t \in (0,T),\\
u(-1,y,t) &= h(y,t), && y \in (0,1),\ t \in (0,T),\\
u(x,0,t) = u(x,1,t) &= 0, && x \in (-1,0),\ t \in (0,T),\\
u(x,y,0) &= u_0(x,y), && (x,y) \in \Omega,
\end{aligned}
$$

where $u_0 = u_0(x,y)$ is the initial data and $h = h(y,t)$ is the control input.

Null controllability

Theorem

Let $u_0 \in L^2(\Omega)$ *and* $s \in [\frac{3}{2}, 2)$. *Then there exists a control input*

$$h \in G^{\frac{s}{2}, s}([0,1] \times [0,T])$$

such that the solution u *satisfies* $u(\cdot, \cdot, T)=0$. *Furthermore, it holds that*

$$u \in C([0,T]; L^2(\Omega)) \cap G^{\frac{s}{2}, \frac{s}{2}, s}([-1,0]_x \times [0,1]_y \times [\varepsilon, T]_t), \quad \forall\, \varepsilon \in (0,T).$$

Reachable states

Introduce the differential operator

$$Pu := \triangle u_x + a u_x$$

and the following space

$$\mathcal{R}_{R_1,R_2} := \{u \in C^\infty(\overline{\Omega});\ \exists C > 0,\ |\partial_x^p \partial_y^q u(x,y)| \le C \frac{(p!)^{\frac{2}{3}}(q!)^{\frac{2}{3}}}{R_1^p R_2^q} \quad \forall p,q \in \mathbb{N},\ \forall (x,y) \in \overline{\Omega},$$

$$\text{and } P^n u(0,y) = \partial_x P^n u(0,y) = P^n u(x,0) = P^n u(x,1) = 0,\ \ \forall n \in \mathbb{N},\ \forall (x,y) \in \overline{\Omega}\}.$$

Theorem

Let $R_0 := \sqrt[3]{9(a+2)} e^{(3e)^{-1}}$, and let $R_1, R_2 \in (R_0, +\infty)$. Then for any $u_1 \in \mathcal{R}_{R_1,R_2}$, there exists a control input $h \in G^{1,2}([0,1] \times [0,T])$ such that the solution u of with $u_0 = 0$ satisfies $u(\cdot,\cdot,T) = u_1$. Furthermore, $u \in G^{1,1,2}([-1,0] \times [0,1] \times [0,T])$, and the trajectory $u = u(x,y,t)$ and the control $h = h(y,t)$ can be expanded as series:

$$u(x,y,t) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} g_{i,j}(x) z_j^{(i)}(t) e_j(y),$$

$$h(y,t) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} g_{i,j}(-1) z_j^{(i)}(t) e_j(y).$$

4. Exact controllability of nonlinear PDE

4.1 Exact controllability of semi-linear heat equations

C. Laurent, L. Rosier, *Exact controllability of semi-linear heat equations in spaces of analytic functions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **37** (2020), no. 4, 1047-1073.

Semi-linear heat equation

- Our aim is to derive in a space of analytic functions the (local) **exact controllability** of

$$\begin{aligned} \partial_t y = \partial_x^2 y + f(x, y, \partial_x y), &\qquad x \in [-1, 1],\ t \in [0, T], \\ y(-1, t) = h_{-1}(t), &\qquad t \in [0, T], \\ y(1, t) = h_1(t), &\qquad t \in [0, T], \\ y(x, 0) = y_0(x), &\qquad x \in [-1, 1], \end{aligned}$$

  where $f : \mathbb{R}^3 \to \mathbb{R}$ is an analytic function in a neighbourhood of $(0, 0, 0)$.

- Classical examples:
  1. $\partial_t y = \partial_x^2 y + \phi(x) y$ (linear heat eq. with an analytic potential function ϕ)
  2. $\partial_t y = \partial_x^2 y - y \partial_x y$ (viscous Burgers' eq.)
  3. $\partial_t y = \partial_x^2 y + y - y^3$ (Allen-Cahn eq.)

Assumptions about the nonlinear term

We assume that

$$f(x,0,0) = 0 \quad \forall x \in (-4,4)$$

and that

$$f(x,y_0,y_1) = \sum_{(p,q,r)\in\mathbb{N}^3} a_{p,q,r}(y_0)^p(y_1)^q x^r \quad \forall (x,y_0,y_1) \in (-4,4)^3$$

with

$$|a_{p,q,r}| \le \frac{M}{b_0^p\, b_1^q\, b_2^r}$$

where $M > 0$, $b_0 > 4$, $b_1 > 4$ and $b_2 > 4e^{(2e)^{-1}} \approx 4.81$.

Main result (Exact controllability of the nonlinear heat equation)

For given $R > 1$ and $C > 0$, we denote by $\mathcal{R}_{R,C}$ the set

$$\mathcal{R}_{R,C} := \{y : [-1,1] \to \mathbb{R};\ \exists (a_n)_{n\geq 0} \in \mathbb{R}^{\mathbb{N}},\ |a_n| \leq C\frac{n!}{R^n}\ \ \forall n \geq 0 \text{ and } y(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}\ \ \forall x \in [-1,1]\}.$$

Theorem

Let $R > R_0 := 4e^{(2e)^{-1}} \approx 4.81$, let $b_2 > R_0$ and let $T > 0$. Then there exists some number $C > 0$ such that for all $y_0, y_1 \in \mathcal{R}_{R,C}$, there exist $h_{-1}, h_1 \in G^2([0,T])$ such that the solution y of the nonlinear heat eq.

$$\partial_t y = \partial_x^2 y + f(x, y, \partial_x y)$$

with initial data y_0 and boundary data h_{-1}, h_1 is defined for all $t \in [0,T]$ and it satisfies

$$y(x,T) = y_1(x) \quad x \in [-1,1].$$

Method of proof

1. **Study of a Cauchy problem in x:**

$$\begin{aligned} &\partial_x^2 y = \partial_t y - f(x, y, \partial_x y), \quad x \in [-1, 1],\ t \in [0, T] \\ &y(0, t) = g_0(t), \\ &\partial_x y(0, t) = g_1(t) \end{aligned}$$

   See also Nirenberg (1972), Nishida (1977), Guo-Littman (1995).

2. **Jet analysis** (replacing fixed-point argument):

   Study of the relationship between
   the jets $\{\partial_t^n y(0, T)\}_{n\geq 0} \cup \{\partial_x \partial_t^n y(0, T)\}_{n\geq 0}$ and the jet $\{\partial_x^n y(0, T)\}_{n\geq 0}$

Step 1: Cauchy problem in x

We are concerned with the wellposedness of the following Cauchy problem in the x variable:

$$\partial_x^2 y = \partial_t y - f(x, y, \partial_x y), \qquad x \in [-1, 1],\ t \in [t_1, t_2] \tag{4.1}$$

$$y(0, t) = g_0(t), \qquad t \in [t_1, t_2], \tag{4.2}$$

$$\partial_x y(0, t) = g_1(t), \qquad t \in [t_1, t_2]. \tag{4.3}$$

Theorem

Let f be as in the main result, $-\infty < t_1 < t_2 < \infty$ and $R > 4$. Then there exists some number $C > 0$ s.t. for $g_0, g_1 \in G^2([0, T])$ with

$$|g_i^{(n)}(t)| \le C \frac{(n!)^2}{R^n}, \quad i = 0, 1,\ n \ge 0,\ t \in [t_1, t_2]$$

there exists a solution y of (4.1)-(4.3) defined for $x \in [-1, 1],\ t \in [t_1, t_2]$. Furthermore $y \in G^{1,2}([-1, 1] \times [t_1, t_2])$.

Proof: fixed-point in a scale of Banach spaces of Gevrey functions:
We consider a family of Banach spaces $(X_s)_{s \in [0,1]}$ satisfying for $0 \le s' \le s \le 1$

$$X_s \subset X_{s'}$$

$$\|f\|_{X_{s'}} \le \|f\|_{X_s} \quad \forall f \in X_s.$$

Abstract existence theorem

We are concerned with the abstract Cauchy problem:

$$\partial_x U(x) = T(x)U(x), \qquad -1 \leq x \leq 1,$$
$$U(0) = U^0$$

where $U^0 \in X_1$ and $(T(x))_{x\in[-1,1]}$ is a family of (nonlinear) operators with possible loss of derivatives. The following result is inspired by Niremberg (1972), Nishida (1977).

Theorem (Global well-posedness result)

For any $\varepsilon \in (0,1/4)$, there exists $D > 0$ such that for any family $(T(x))_{x\in[-1,1]}$ of nonlinear maps from X_s to $X_{s'}$ for $0 \leq s' < s \leq 1$ satisfying

$$\|T(x)U\|_{X_{s'}} \leq \frac{\varepsilon}{s-s'}\|U\|_{X_s},$$
$$\|T(x)U - T(x)V\|_{X_{s'}} \leq \frac{\varepsilon}{s-s'}\|U-V\|_{X_s}$$

for $0 \leq s' < s \leq 1$, $x \in [-1,1]$ and $U, V \in X_s$ with $\|U\|_{X_s} \leq D$, $\|V\|_{X_s} \leq D$, there exists $\eta > 0$ such that for any $U^0 \in X_1$ with $\|X^0\|_{X_1} \leq \eta$, there exists a solution $U \in C([-1,1], X_{s_0})$ for some $s_0 \in (0,1)$ to the integral equation

$$U(x) = U^0 + \int_0^x T(\tau)U(\tau)\,d\tau.$$

Step 2. Jet analysis

- For linear heat eq $\partial_t y = \partial_x^2 y$, we have

$$\partial_t^n y = \partial_x^{2n} y \quad \text{and} \quad \partial_x \partial_t^n y = \partial_x^{2n+1} y.$$

- For the nonlinear heat eq. $\partial_t y = \partial_x^2 y - f(x, y, \partial_x y)$, there is still a one-to-one correspondance between
the jets $\{\partial_t^n y(0,T)\}_{n\ge 0} \cup \{\partial_x \partial_t^n y(0,T)\}_{n\ge 0}$ and the jet $\{\partial_x^n y(0,T)\}_{n\ge 0}$

Step 2. Jet analysis (Quantification)

Let f be as in the main result. Pick any solution $y \in C^\infty([-1,1] \times [t_1, t_2])$ of

$$\partial_t y = \partial_x^2 y - f(x, y, \partial_x y)$$

if $|\partial_x^n y(0,T)| \leq Cn!/R^n$ with $R > 4$ and C small enough, then

$$|\partial_t^n y(0,T)| + |\partial_x \partial_t^n y(0,T)| \leq C'(2n)!/R'^{2n}$$

for some $R' \in (4, R)$ and $C' > 0$ with $C' \to 0$ as $C \to 0$.

4.2 Exact controllability of anisotropic 1D equations

C. Laurent, I. Rivas, L. Rosier, *Exact controllability of anisotropic 1D equations equations in spaces of analytic functions*, in progress.

Anisotropic equations

We consider PDEs with **more** derivatives in space than in time

$$\partial_t^N y = \sum_{j=0}^{M} \zeta_j \partial_x^j y + f(x, y, .., \partial_x^{M-1} y), \text{ where } N < M.$$

Classical examples:

1. $\partial_t y + \partial_x^3 y + \partial_x y + y\partial_x y = 0$ (KdV)
2. $\partial_t^2 y = \pm\partial_x^4 y + \partial_x^2 y - \partial_x^2(y^2)$ (good(-)/bad(+) Boussinesq equation)
3. $\partial_t y = e^{i\theta}\partial_x^2 y + e^{i\varphi}|y|^2 y$ (Ginzburg-Landau)
4. $\partial_t y + \partial_x^4 y + \partial_x^2 y + y\partial_x y = 0$ (Kuramoto-Sivashinsky)

PDEs that are ill-posed (forward in time) are still concerned!!
Examples:

1. $(\partial_t + \partial_x^2)y = 0$ (backward heat equation);
2. $(\partial_t + \partial_x^2)(\partial_t - \partial_x^2)y = \partial_x^2 y + \partial_x^2(y^2)$ (bad Boussinesq equation)

Space of analytic functions

For given $R > 1$ and $C > 0$, we denote by $\mathcal{N}_{R,C}$ and $\mathcal{R}_{R,C}$ the sets

$$\mathcal{N}_{R,C} \quad := \quad \{(\alpha_n)_{n\geq 0} \in \mathbb{R}^{\mathbb{N}},\ |\alpha_n| \leq C\frac{n!}{R^n} \ \ \forall n \geq 0\} \subset \mathbb{R}^{\mathbb{N}}.$$

$$\mathcal{R}_{R,C} \quad := \quad \{y : [-1,1] \to \mathbb{R};\ \exists (\alpha_n)_{n\geq 0} \in \mathcal{N}_{R,C} \text{ with } y(x) = \sum_{n=0}^{\infty} \alpha_n \frac{x^n}{n!} \ \ \forall x \in [-1,1]\}$$

As for the semi-linear heat equation, our result will be stated in the space $\mathcal{R}_{R,C}$.

Rmk. The result we shall obtain can be seen as a local exact controllability in some **Hardy space**.

Let us introduce the Hardy space $H^\infty_R = Hol(B(0,R)) \cap L^\infty(B(0,R))$, which is a Banach space for the norm $\|\cdot\|_{L^\infty(B(0,R))}$. Let

$$\mathcal{B}_{R,C} = \{y : [-1,1] \to \mathbb{R};\ \exists f \in H^\infty_R,\ \|f\|_{L^\infty(B(0,R))} \leq C,\ f_{|[-1,1]} = y\}.$$

Then

$$\mathcal{B}_{R,C} \subset \mathcal{R}_{R,C} \subset \mathcal{B}_{r,C(1-\frac{r}{R})^{-1}}$$

for $1 < r < R$ and $C > 0$.

Boundary conditions

Denote $Py := \sum_{j=0}^{M} \zeta_j \partial_x^j y$.

Introduce the vectors ("partial jets")

$$\begin{aligned} Y^x(x,t) &= (y(x,t), \partial_x y(x,t), ..., \partial_x^{M-1} y(x,t)) \\ Y^t(x,t) &= (y(x,t), \partial_t y(x,t), ..., \partial_t^{N-1} y(x,t)) \end{aligned}$$

The PDE

$$\partial_t y = Py + f(x, y, \partial_x y, ..., \partial_x^{M-1} y)$$

is supplemented with the homogeneous boundary condition at $x = 0$

$$BY^x(0,t) = 0$$

where $B \in \mathbb{R}^{\nu, M}$ is given, $\nu \in \mathbb{N}$ being the numbers of boundary conditions at $x = 0$ (without any control).
The boundary controls are some traces of the state function y at $x = 1$ (to be chosen as desired). Their number is $M - \nu$.

Compatibility set

From $BY^x(0,t) = 0$ we infer that $\partial_t^l BY^x(0,t) = 0$ for all $t \in [0,T]$ and all $l \in \mathbb{N}$. Then

Lemma

Let $Py := \sum_{j=0}^{M} \zeta_j \partial_x^j y$. For any $l \in \mathbb{N}$, there exist $m = m(l) \in \mathbb{N}$ and a smooth application $J_l : [-1,1] \times (\mathbb{R}^N)^{m+1} \to \mathbb{R}^M$ such that for any $y \in C^\infty([-1,1] \times [0,T])$ solution of

$$\partial_t^N y = Py + f(x, y, \partial_x y, ..., \partial_x^{M-1} y)$$

we have

$$\partial_t^l Y^x = J_l(x, Y^t, \partial_x Y^t, ..., \partial_x^m Y^t) \quad \textit{in } [-1,1] \times [0,T]$$

We are in a position to define the compatibility set

$$\mathcal{C} = \{Y_0 \in C^\infty([0,1])^N;\ BJ_l(0, Y_0, \partial_x Y_0, ..., \partial_x^{m(l)} Y_0)_{|x=0} = 0 \quad \forall l \in \mathbb{N}\}$$

Exact controllability of the anisotropic 1D equation

Let

$$\lambda := \frac{M}{N} > 1$$

be the Gevrey regularity.

Theorem

Let $\hat{R} := 4N\lambda e^{(\lambda e)^{-1}}$. Then for f with $b_2 > \hat{R}$ and for $R > \hat{R}$ and $T > 0$, there exists some constant $C > 0$ such that for all $Y^0, Y^1 \in (\mathcal{R}_{R,C})^N \cap \mathcal{C}$, there exists a solution of the system

$$\begin{aligned} \partial_t^N y &= Py + f(x, y, \partial_x y, ..., \partial_x^{M-1} y) \\ BY^x(0,t) &= 0 \\ Y^t(x,0) &= Y^0(x) \\ Y^t(x,T) &= Y^1(x) \end{aligned}$$

Furthermore, we have $y \in G^{1,\lambda}([0,1] \times [0,T])$.

Controllability of (good or bad) Boussinesq equation

Consider the control system

$$\begin{aligned}
\partial_t^2 y = \pm\partial_x^4 y + \partial_x^2 y - \partial_x^2(y^2), &\qquad x \in (0,1),\ t \in (0,T),\\
\partial_x y(0,t) = 0, &\qquad t \in (0,T),\\
\partial_x^3(0,t) = 0, &\qquad t \in (0,T),\\
\partial_x y(1,t) = v(t), &\qquad t \in (0,T),\\
\partial_x^3 y(1,t) = w(t), &\qquad t \in (0,T),\\
y(x,0) = y^0(x), &\qquad x \in (0,1),\\
y_t(x,0) = y^1(x), &\qquad x \in (0,1).
\end{aligned}$$

Recall that the bad Boussinesq equation (+) is severely ill-posed, even for the linear part.

Theorem

Let $R > \hat{R}$ and $T > 0$. Then there exists some number $\hat{C} > 0$ such that for all pair of functions $(y^0, y^1), (\tilde{y}^0, \tilde{y}^1) \in (\mathcal{R}_{R,\hat{C}})^2$ which are even with respect to 0, there exist $y \in G^{1,2}([0,1] \times [0,T])$ and $v, w \in G^2([0,T])$ satisfying the system above together with

$$y(x,T) = \tilde{y}^0(x),\ y_t(x,T) = \tilde{y}^1(x), \quad \forall x \in [0,1].$$

Conclusion

- The flatness approach is a robust method to derive almost sharp sets of reachable states for parabolic-like equations, including the heat and the KdV equation. Also useful for anisotropic 1D equations.
- First instance of an exact controllability result for a semilinear parabolic equation.
- Also effective for 1D anisotropic PDE, even if they are NOT well-posed
- Approach also useful to design efficient numerical schemes

Thank you!