Chat Time Sam!

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Exploratory Sheets for the Math Circles Session

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Have Fun!

Domino Tilings

Suppose, one tiles a rectangular $m \times n$ grid using dominos. For a $2 \times n$ grid, it is very easy to determine the number c_n of domino tilings.

Question 0. Find c_n.

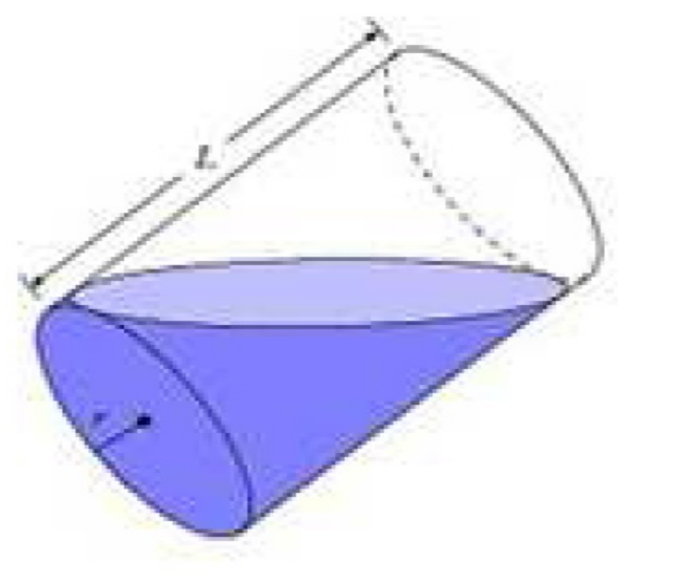
Question 1. For a $3 \times 2n$ grid, can you find a recursive formula in terms of n for the number of domino tilings?

Question 2. What about an $4 \times n$ grid, for general n?

Question 3. Finally, what about $m \times n$ grid for general m, n with mn even?

Filling Glasses

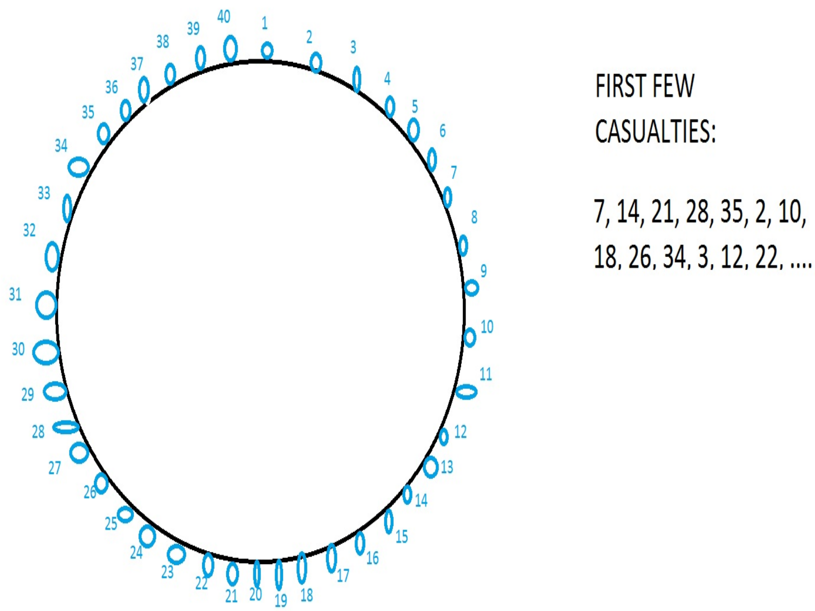
If we have a perfectly cylindrical glass vessel with some amount of liquid but no measuring aid, how do we know if the glass is more than half filled or not? That is easy:



Consider now the following problem where we have two glass vessels of equal size. Suppose, we can always check if the two glasses have equal amounts of liquids by placing them side by side and checking the levels. Given an unlimited quantity of water, what proportions of glass vessels can be measured just using these two glasses? Can you also provide an algorithm that carries out the process of filling the exact amounts which are possible to measure and fill precisely?

If you can't beat them, join them

Flavius Josephus and 39 of his comrades were surrounded when holding a revolt against the Romans during the 1st century A.D. Rather than become slaves, they decided to kill themselves by their own comrades. They arranged themselves along a circle. Starting somewhere, they went clockwise around the circle and every 7th person was eliminated. This continued with the 7th among the surviving ones being killed at each step. Apparently, Josephus was a clever mathematician and arranged himself in such a position that he would be the last survivor. The story goes that he did not kill himself but came and joined the Romans!
We need to find out Josephus's position. The general problem is of n people, designated by $1, 2, \cdots, n$ in clockwise order, say, and every d-th person is eliminated going around in the clockwise direction. Can we describe somehow the position of the last survivor?



When $d = 2$, there is a beautiful formula in binary digits to find the position of the last person alive; can you find it?

A puzzling scenario

Two brilliant scientists (**A** and **B**) have made a brilliant discovery, a mind-swapping machine! Yes, you heard that right. They created a machine that can swap the mind of the two people who are using it. Excited, they use it on themselves immediately. After some time, they decide to switch back to their original bodies. But, it doesn't work. The scientists then realize that due to an unexpected side effect of the machine, a pair of bodies can not have their mind swapped more than once!

(I) An assistant (**C**) walks into the lab and professor **A** (in **B**s body) decides to use this new person to swap their minds back. Can it work?

(II) In a rushed attempt, **A** (in **B**s body) swaps mind with **C**, then the ones in bodies **A** and **C** swap minds. Who is in whose body now? If another assistant **D** walks in at this point, can they unscramble everyone's mind using these four people?

(III) Can you think of a way to unscramble everyone's mind if we are allowed as many volunteers as we want?

(IV) What is the minimum number of volunteers that will be required?

The more the merrier

We will ask three questions to generalize the result.

(I) If we start with 5 people, and scramble their minds in however ways we want, how many volunteers will be needed to be sure to unscramble them?

(II) What is the minimum number of fresh volunteers that will be enough to unscramble n people's minds regardless of how they were swapped beforehand?

(III) If we start with n people, is it possible to achieve all $n!$ possible arrangements of their minds using the mind swapping machine? [Without involving any volunteers.]

Geometry and primes

Let us start with the following simple question:
On the unit circle, take n points dividing the circumference into n equal parts. From one of these n points, draw the $n-1$ chords joining it to the other points. What is the product of the lengths of these chords?.

A more difficult problem is to start from one of the points and - go in one direction (say, the anticlockwise direction) - and draw the chords joining it to the k-th point from it for each k relatively prime to n. What is the product of the lengths of these chords in this case?