

Spin-Statistics from Ribbons

Co-adjoint orbits and the geometric phase

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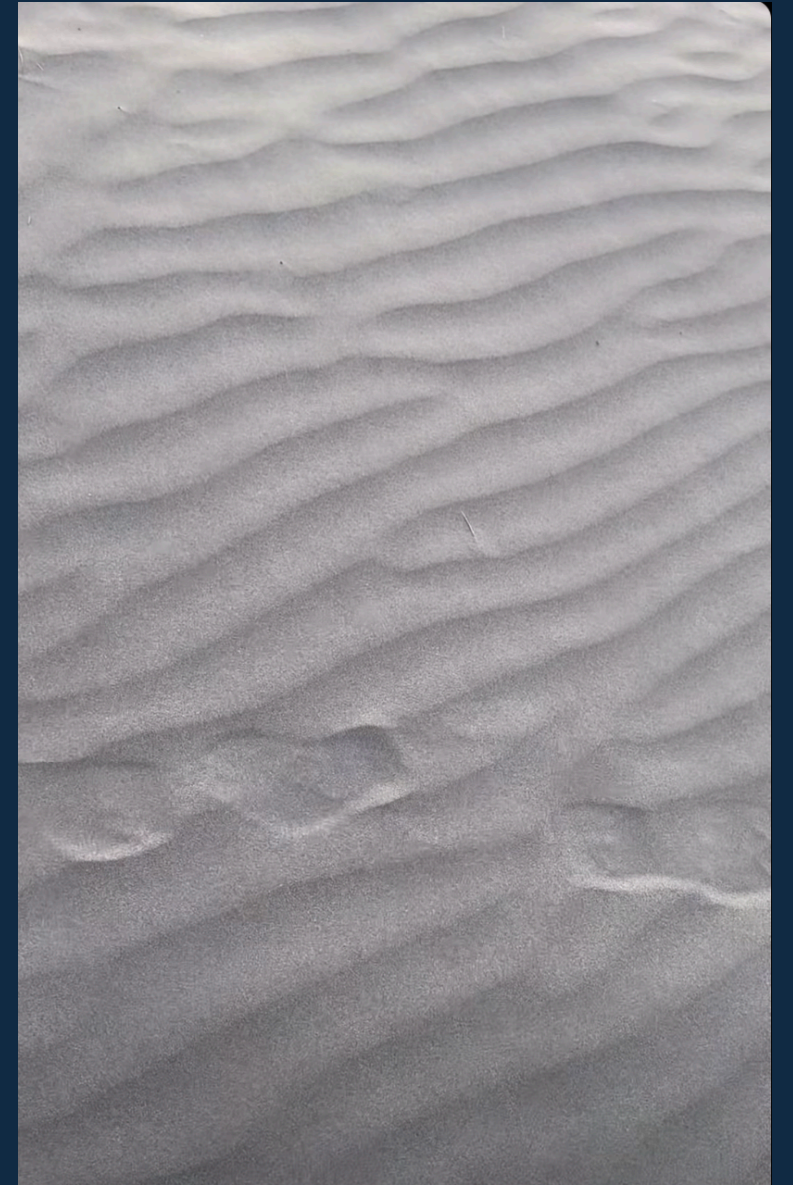


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Plan of talk:

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Konstant-Kirillov-Souriau: Symmetries and

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Introduction 10

half integral spin Fermi, integral Bose

Pauli: relativistic proof

Feynman: conceptual core of the theorem not fully understood.

Neuenschwander: Is there a simple intuitive proof?

Duck, Sudarshan: a critical summary of SST,

Q1: why should spin (1 particle property) be connected to statistics (2 particle)?

Spin and Statistics observable in NR lab,

Q2: Do we really need relativity to understand the SS theorem? Berry, Robbins answer No, NR proof

Introduction 10

Simple visualisable topological proof, quantisation on multiply connected spaces,

Spin and Stat both arise as quantisation ambiguities, one ambiguity not two

Key idea: Histories of spinning particles are Ribbons,
exchange of two identical ribbons
equivalent to 2π twist of a single ribbon

Introduction 10

WHY RIBBONS?

Spacetime symmetry is described by a group

Bacry symmetry group is at centrestage (uses math work by KKS),

A group naturally defines a dynamical system,

Group $\rightarrow$ Lie Algebra, Lie brackets $\rightarrow$ dual $\rightarrow$ Poisson Brackets

For spacetime groups (Poincare, Galilei, Aristotle)

This implies a ribbon structure (2 Casimirs, orbital motion, spin)

Elementary particle carries a Unitary IRREP of the symmetry group

Ribbons first assume, then prove; restrict to massive free particles.

Introduction 10

Groups of interest: Poincare, Galilei, Aristotle

Elementary particle is one that carries a IRREP of the symmetry group

classical: transitive action on the phase space;

quantum: projective IRREP of the group on the Hilbert space

All retract (DR) to the Euclidean group $ISO(3)$

Unified view of all these groups

Proof based on the quantisation ambiguity in multiply connected spaces

Introduction 10

Ribbons have been used to model DNA elasticity

Link Twist and Writhe gives us the geometric phase.

Poster: Molecular Elasticity and the Geometric Phase

ACK: Supurna Sinha, Abhishek Dhar, Rajesh Gopakumar,
Rajaram Nityananda, Subhro Bhattacharya

Quantisation on Multiply Connected Spaces10

Multiply connected spaces show a quantisation ambiguity

Simplest example the Aharonov Bohm effect
(waterwave ab) flux or nontrivial topology.

Abstract point of view.

can replace the punctured plane with a circle

Introduce deformation Retract (DR)

$$X = \{ \vec{r} = (x, y) \mid \vec{r} \cdot \vec{r} \neq 0 \} \quad \text{Punctured plane}$$

$$A = \{ \vec{r} = (x, y) \mid \vec{r} \cdot \vec{r} = 1 \} \quad \text{Unit circle}$$

$$F_t(\vec{r}) = (1 - t)\vec{r} + t(\hat{r}) \quad \text{DR}$$

X, A of the same homotopy type; dimensions different

Quantisation on Multiply Connected Spaces10

Both spin and statistics arise from multiply connected spaces

$$\pi_1(SO(3)) = Z_2$$

Multiply connected configuration space Q .

$\pi_1(Q)$ fundamental group. Assign a phase to each element so that multiplication works. Homomorphism

$$\pi_1(Q) \rightarrow U(1)$$

Topological phase

Quantisation on Multiply Connected Spaces10

Canonical approach canonical commutation relations CCR

$$q \rightarrow \hat{q}; \quad p \rightarrow \hat{p} = \frac{\hbar}{i} \nabla . \text{ Can add a gradient } \nabla_q \chi(q) \text{ (Dirac)}$$

Can remove this gradient by a rephasing of wave function.

In Multiply Connected Space add a flat connection without spoiling CCR

Anholonomy of the flat connection gives the phases and
Quantisation Ambiguity

$$q \rightarrow \hat{q}; \quad p \rightarrow \hat{p} = \frac{\hbar}{i} \nabla + A$$

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Path integral approach. Propagator sum over histories

$$K = \sum_a \alpha_a \int d\mathcal{H}_a \exp iS_{cl}(\mathcal{H}_a) \text{ where } S_{cl}(\mathcal{H}_a) \text{ is the classical}$$

Action and α_a is a phase factor assigned to the Homotopy class a .

homotopy classes can be assigned phases. Choose a reference class $\mathcal{H}_0$ and assign the phase 1. rest of the phases determined

Closed Path History (CPH) $\mathcal{H}_a \circ \overline{\mathcal{H}_0}$ Probabilities CPH

$$\alpha_a \alpha_b = \alpha_{a \circ b} \quad \text{Homomorphism}$$

Consistency of dynamical evolution

In canonical approach CCR violated

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Phase space vs configuration space

In usual situations topology same for phase and configuration space, since Cotangent space retracts (DR) to Configuration space. This is not true in our situation

We directly get a phase space formulation from KKS. The phase spaces that emerge can be compact (like S^2) and may not have an underlying configuration space.

Quantisation on Multiply Connected Spaces10

Spin:

Rotation group is multiply connected; Fundamental Group Z_2

$$\mathcal{R}^2 = 1, \alpha(\mathcal{R}) = \pm 1 \quad \text{cube* ??}$$

4π rotation homotopic to no rotation. $\alpha(\mathcal{R})$ spin phase factor.

Statistics: Mod out by Exchange $\mathcal{E}^2 = 1; \alpha(\mathcal{E}) = \pm 1$

Statistics phase factor.

SST: Two ambiguities become one $\alpha(\mathcal{R}) = \alpha(\mathcal{E})$

Ribbon Histories and the Spin-Statistics Theorem

Assume ribbon

What is a ribbon? a framed space curve together

With a “ribbon vector” $\{\vec{x}(s), \hat{e}_1(s)\}$ so that $\hat{t}(s) \cdot \hat{e}_1(s) = 0$

where $\hat{t}(s) = \frac{d\vec{x}(s)}{ds}$ is the unit tangent vector to the curve

$\vec{x}(s)$ parametrised by arc length s . Defines a framed curve via

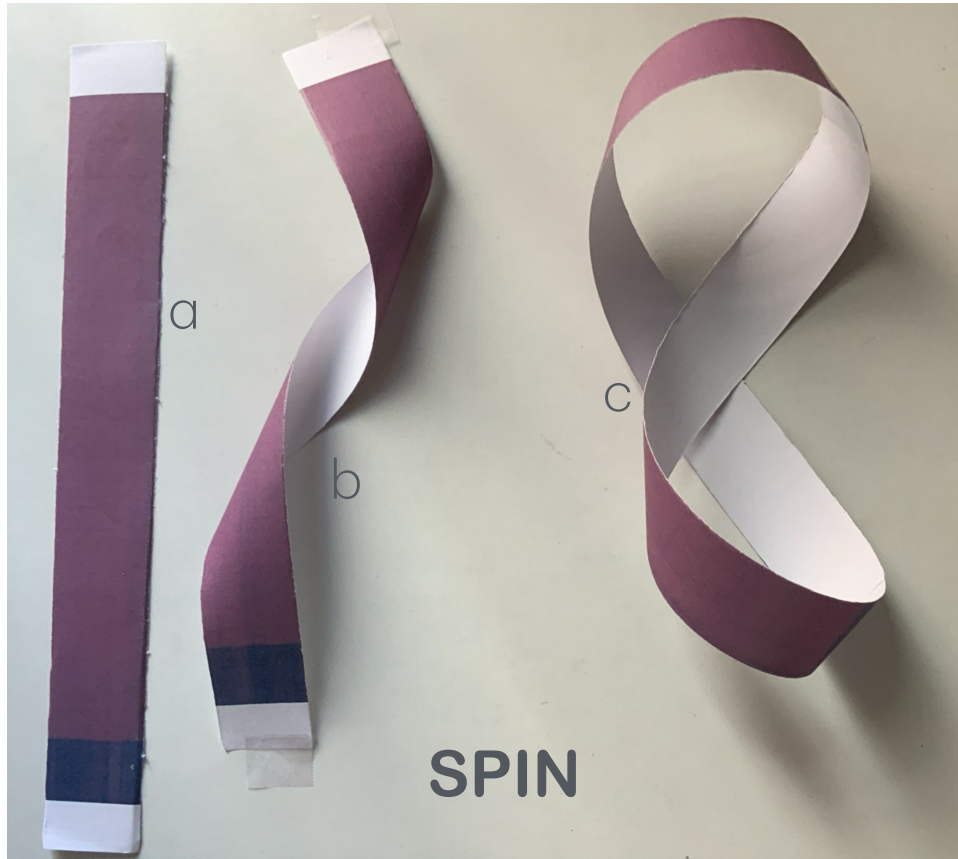
$$\hat{e}_3(s) = \hat{t}(s); \quad \hat{e}_2 = \hat{e}_3 \times \hat{e}_1$$

Conversely, a framed curve in space can be continuously

Deformed to a ribbon

SINGLE PARTICLE HISTORY

TWO PARTICLE HISTORY



- a) Direct process
- b) Twisted process
- c) Their join



- d) direct process
- e) exchange process
- f) their join

Visual Proof

suspenders*

?



Ribbon Histories and the Spin-Statistics Theorem

Spin phase factor: interference between a straight ribbon and a twisted ribbon

Statistics phase factor: interference between a direct path and an exchange path (Two particle property is homotopic to a one particle property) Hanbury Brown Twiss Intensity interferometry

Visual Proof: $\alpha(\mathcal{R}) = \alpha(\mathcal{E})$

By homotopy ($\sqrt{\text{belt}}$)



Ribbon Histories and the Spin-Statistics Theorem

Proves SST, intuitive, topological, but all based on the assumption of a ribbon

ribbon idea is based on a sound mathematical foundation.

Ribbons have been used before: Finkelstein-Rubinstein extended objects in GR (1969)

Berry-Robbins used them as a book keeping device

Guerra-Marra, Gould, Hartung (1979)

Early work possibly overwhelmed by the prejudice that spin is a relativistic effect. In fact it comes from the topology of $SO(3)$.

The idea here is that spinning particle histories ARE ribbons in a topological sense.

KKS: Symmetries, Groups and Dynamics

We are used to thinking of groups as dynamical systems rigid body $SO(3)$ configuration, need moment of inertia etc

KKS: somewhat different Group defines a dynamical system and directly gives us a phase space, and dynamics.

For spacetime groups describes particles with momentum and spin

Two Casimirs of group describe orbital motion and spin.

G group, $\mathcal{G}$ Lie algebra tgt space at id. Lie algebra has a Lie bracket $[X_i, X_j] = C_{ij}^k X_k$ satisfying Jacobi

$\mathcal{G}^*$ dual space linear functions on $\mathcal{G}$. Vice versa.

X_i are functions on $\mathcal{G}^*$, and satisfy PB relations.

KKS: Symmetries, Groups and Dynamics

Extend P.B to ALL functions on $\mathcal{G}^*$ PB is a derivation

Casimir Invariant of Group generates trivial transformations

Restrict to Co-adjoint orbits to get a classical phase space.

Quantise using Path integrals or geometric quantisation.

Examples: Rotation group $\vec{J}$, standard P.B; one Casimir $C_1 = \vec{J} \cdot \vec{J}$

Coadjoint orbit is a sphere of fixed radius. A compact phase space!

Just the structure we have in the geometric phase

$$U(1) \hookrightarrow S^3 \rightarrow S^2$$

Poincare Group (Two Casimirs) square of 4-momentum and square of Pauli Lubanski tensor.

KKS: Symmetries, Groups and Dynamics

LAGRANGE: G group $\xi_0 \in \mathcal{G}^*$

Maurer-Cartan Form $\Theta = g^{-1}dg$ Lie algebra valued 1-form

Use ξ_0 to turn make a real valued 1-form $\theta = \langle \xi_0, \Theta \rangle$ living on

G . Define $\omega = d\theta$ an exact (and therefore closed) 2-form on G

ω is closed but could be degenerate. Null vector fields on G form

An integrable subspace and generate the isotropy subgroup H the

“Gauge group” of the theory. The co-adjoint action of G on ξ_0

defines an orbit $\mathcal{O}$, which is the phase space of the theory,

(symplectic manifold) where the symmetry group G is realised

transitively as canonical transformations.

The isotropy subgroup of the coadjoint action on ξ_0 is H

And $\mathcal{O} = G/H$. The group plays the role of the extended phase space, the arena for dynamics of the system.

The physical phase space is 8 dimensional and represents

The space of initial conditions. The dynamics is generated by

The little group generators a la Dirac's theory of constraints

The orbit is a homogeneous space.

KKS: Symmetries, Groups and Dynamics

All Groups of interest (Poincare, Galilei, Aristotle)
can be retracted to $E(3)$ the Euclidean group

In three dimensions. Lie algebra $\vec{J}, \vec{P}$; Casimirs

$$C_1 = \vec{P} \cdot \vec{P}; C_2 = \vec{J} \cdot \vec{P} \text{ Helicity}$$

Given a closed loop in $E(3)$, $g(\tau)$ based at the identity,

Consider a small piece of it and choose $\xi_0 = \{ \vec{P}, \vec{J} \}$

In the form $\vec{P} = \{0,0,p\}; \vec{J} = \{0,0,s\}$

Dynamics generated by $v_3 P_3 + \Omega_3 J_3$ with motion along

The curve and rotation around the tangent vector: ribbon

Ribbons, DNA Elasticity and the Geometric Phase

More mundane use of Ribbons in physics is in the study of Molecular Elasticity (Poster)

DNA is an example of a Twist Storing Polymer. At length scale around 50 nm, acts like a bit of stiff cord.

It is possible to perform experiments on Single molecules of DNA and these experiments are

best interpreted using a Ribbon model (the WLC Worm Like Chain).

Ribbons, DNA Elasticity and the Geometric Phase

Link, Twist and Writhe. Tube, Wool

Fuller-Calugereanu-White $Lk = Tw + Wr$. Relation to geometric phase.

Lk topological invariant. In the WLC, the ribbon can pass through itself. Belt trick *.

and release two units of Link. Brown and Cozzarelli

Topoisomerase II. Link is not a topological invariant.

But its parity is . Can still have $(-1)^{Lk} = (-1)^{Wr} (-1)^{Tw}$
pass to phase factors rather than phases.

Given any closed space curve (not necessarily simple)

Define the Wreath $w = -\exp i\Omega/2$, where Ω is the solid

Angle enclosed by the tangent vector.

Ribbons, DNA Elasticity and the Geometric Phase

Wreath of a space curve ; well defined on all curves

In the spin case, the Lk is entirely twist mod 2. In statistics, it is entirely writhe mod 2.

spin phase = Exchange phase (SST) Any other choice violates CCR

Hanbury Brown and Twiss intensity interferometry

two photon process measures statistics phase factor

one photon process measures spin phase factor

The two are homotopic. This answers why two particle and one

Particle properties are related.

Conclusion

MAIN CONCLUSION: SST HAS NOTHING TO DO WITH RELATIVITY EITHER SPECIAL OR GALILEAN. IT ALSO APPLIES TO ARISTOTELIAN PHYSICS. DEPENDS ONLY ON STRUCTURE OF $E(3)$.

Conclusion Counter example spinless fermions

Pauli proof, Berry-Robbins proof spin phase factor=1;
Actually an inconsistent quantisation (AB effect) CCR
Hanbury Brown Twiss Effect would show the opposite factor
Coincidence inhibition rather than enhancement

KKS is essentially mathematical. Not all mathematical
Possibilities are realised in Nature. Eg. Tachyons ($m^2 < 0$)
Continuous spin representations for massless particles
Little group is $E(2)$
On the other hand $m^2 > 0$, $m^2 = 0$ and $s = \text{half-integer}$ are
Realised and interesting.

$\xi_0 = 0$ gives vacuum 0 dim phase space

Conclusion

Massive case $m^2 > 0$, phase space

Has 8 degrees of freedom. 6 for orbital 2 for spin.

Massless case $m^2 = 0$. Find only 6 degrees of freedom.

The null analogue of Fermi Transport is Integrable
(Transport along null curves Samuel and Nityananda
J.phys A 2000). There still exist topological phases
but no geometric phase.

$\{x^i, x^j\} = \epsilon^{ijk} s^k$ positions don't commute (Rytov).

This implies no position wave function for the photon
Argument first given by Mukunda (Bo Skagerstam)

Ribbons, DNA Elasticity and the Geometric Phase

Feynman monopole Duck Sudarshan Berry
Finkelstein-Rubinstein (extended objects)
same idea to elementary particles without assuming
a finite spatial extent.

Have talked only about massive spinning particles
Massless case similar analysis $m^2=0$. Six dim phase space

But Fermi transport
Is locally integrable. Transport along null curves

Ribbons, DNA Elasticity and the Geometric Phase

Composite particles: all that matters is low energies

Geometric Phase and Dynamical Phase

System with reparametrisation invariance

Geometric phase is the whole story

Interactions: topology is expected to be
robust against them

THANK YOU

GEOMETRY & GROUPS

$\mu = \cos \theta$

G (Lie group)	$R \times SO(3)$ Rotation, Time Translation	ISO(3,1) Poincare	ISO(3) Euclidean
H (Isotropy)	$R \times SO(2)$	$R \times SO(2)$	$R \times SO(2)$
$\pi_1(G)$ Fundamental Group	Z_2	Z_2	Z_2
$\mathcal{O}$ (Orbit)	S^2	$T^*\mathcal{H} \times S^2$	T^*S^2
Dimension of orbit	2	8	4
Symplectic Structure	$d\mu \wedge d\phi$	$dx^i \wedge dp_i + d\mu \wedge d\phi$	$dq^i \wedge dp_i$
Casimirs	$\mathcal{H} \times J^2$	P^2, W^2 $W^\mu = \epsilon^{\mu\nu\alpha\beta} P_\nu J_{\alpha\beta}$	$P^2, He = \vec{P} \cdot \vec{J}$

DYNAMICS

PATHS IN G FALL INTO TWO HOMOTOPY CLASSES; QUANTISATION AMBIGUITY
SPINORIAL AND TENSORIAL STATES;

